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WIENER-TYPE INVARIANTS OF SOME GRAPH OPERATIONS*

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Abstract

Let $d(G,k)$ be the number of pairs of vertices of a graph G that are at distance k, λ a real number, and $W_\lambda(G) = \sum_{k\geq 1} d(G,k)k^\lambda$. $W_\lambda(G)$ is called the Wiener-type invariant of G associated to real number λ. In this paper, the Wiener-type invariants of some graph operations are computed. As immediate consequences, the formulae for reciprocal Wiener index, Harary index, hyper-Wiener index and Tratch-Stankevich-Zefirov index are calculated. Some upper and lower bounds are also presented.

1 Introduction

In this paper, we only consider simple graphs. Suppose G is a simple graph. As usual, the distance between the vertices u and v of G is denoted by $d_G(u,v)$ ($d(u,v)$ for short). It is defined as the length of a minimum path connecting them. Let $d(G,k)$ be the number of pairs of vertices of a graph G that are at distance k, λ a real number, and $W_\lambda(G) = \sum_{k\geq 1} d(G,k)k^\lambda$. $W_\lambda(G)$ is called the Wiener-type invariant of G associated to real number λ, see [3] for details. Note that $d(G,0)$ and $d(G,1)$ represent the number of vertices and edges, respectively. The case of $\lambda = 1$, -1 and -2 are called the classical Wiener index [21], reciprocal Wiener index [16] and Harary index [2] and [17], respectively. The quantities $WW = \frac{1}{2}[W_1 + W_2]$ and $TSZ = \frac{1}{6}W_3 + \frac{1}{2}W_2 + \frac{1}{3}W_1$ are the so-called hyper-Wiener index and Tratch−Stankevich−Zefirov index [4].

The Cartesian product $G\times H$ of graphs G and H is a graph such that $V(G\times H) = V(G)\times V(H)$, and any two vertices (a,b) and (u,v) are adjacent in $G\times H$ if and only if either $a = u$ and b is adjacent with v, or $b = v$ and a is adjacent with u, see [7] for details. The join $G = G_1 + G_2$ of graphs G_1 and G_2 with disjoint vertex sets V_1 and V_2 and edge sets E_1 and E_2 is the graph union $G_1 \cup G_2$ together with all the

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edges joining V_1 and V_2. The composition $G = G_1[G_2]$ of graphs G_1 and G_2 with disjoint vertex sets V_1 and V_2 and edge sets E_1 and E_2 is the graph with vertex set $V_1 \times V_2$ and $u = (u_1, v_1)$ is adjacent with $v = (u_2, v_2)$ whenever (u_1 is adjacent with u_2) or ($u_1 = u_2$ and v_1 is adjacent with v_2),[7, p. 185]. The disjunction $G \vee H$ of graphs G and H is the graph with vertex set $V(G) \times V(H)$ and (u_1, v_1) is adjacent with (u_2, v_2) whenever $u_1u_2 \in E(G)$ or $v_1v_2 \in E(H)$. The symmetric difference $G \oplus H$ of two graphs G and H is the graph with vertex set $V(G) \times V(H)$ and $E(G \oplus H) = \{(u_1, u_2)(v_1, v_2) \mid u_1v_1 \in E(G) \textit{ or } u_2v_2 \in E(H) \textit{ but not both}\}$.

The first Zagreb index was originally defined as $M_1(G) = \sum_{u \in V(G)} d(u)^2$. The first Zagreb index can be also expressed as a sum over edges of G, $M_1(G) = \sum_{uv \in E(G)}[d(u) + d(v)]$, where $d(x)$ denotes the degree of vertex x in G. We refer the reader to [18] for the proof of this fact and for more information on Zagreb index.

Throughout this paper, C_n, P_n, K_n and S_n denote the cycle, path, complete and star graphs on n vertices. Also, $K_{m,n}$ denotes the complete bipartite graph. The complement of a graph G is a graph H on the same vertices such that two vertices of H are adjacent if and only if they are not adjacent in G. The graph H is usually denoted by $\bar{G}$. Our other notations are standard and taken mainly from [1, 6, 20].

2 Main Results

In this section, some exact formulae for the Wiener-type invariants of the Cartesian product, composition, join, disjunction and symmetric difference of graphs are presented. We begin with the following crucial lemma related to distance properties of some graph operations.

Lemma 2.1. Let G and H be graphs. Then we have:

$$\begin{aligned}
(a)\ |V(G \times H)| &= |V(G \vee H)| = |V(G[H])| = |V(G \oplus H)| = |V(G)| \cdot |V(H)|, \\
|E(G \times H)| &= |E(G)| \cdot |V(H)| + |V(G)| \cdot |E(H)|, \\
|E(G + H)| &= |E(G)| + |E(H)| + |V(G)| \cdot |V(H)|, \\
|E(G[H])| &= |E(G)| \cdot |V(H)|^2 + |E(H)| \cdot |V(G)|, \\
|E(G \vee H)| &= |E(G)| \cdot |V(H)|^2 + |E(H)| \cdot |V(G)|^2 - 2|E(G)| \cdot |E(H)|, \\
|E(G \oplus H)| &= |E(G)| \cdot |V(H)|^2 + |E(H)| \cdot |V(G)|^2 - 4|E(G)| \cdot |E(H)|.
\end{aligned}$$

(b) $G \times H$ is connected if and only if G and H are connected.

(c) If (a, c) and (b, d) are vertices of $G \times H$ then $d_{G \times H}((a, c), (b, d)) = d_G(a, b) + d_H(c, d)$.

(d) The Cartesian product, join, composition, disjunction and symmetric difference of graphs are associative and all of them are commutative except from composition.

(e) $d_{G+H}(u,v) = \begin{cases} 0 & u = v \\ 1 & uv \in E(G) \ or \ uv \in E(H) \ or \ (u \in V(G) \ v \in V(H)) \\ 2 & otherwise \end{cases}$.

(f) $d_{G[H]}((a,b),(c,d)) = \begin{cases} d_G(a,c) & a \neq c \\ 0 & a = c \ \& \ b = d \\ 1 & a = c \ \& \ bd \in E(H) \\ 2 & a = c \ \& \ bd \notin E(H) \end{cases}$.

(g) $d_{G\vee H}((a,b),(c,d)) = \begin{cases} 0 & a = c \ \& \ b = d \\ 1 & ac \in E(G) \ or \ bd \in E(H) \\ 2 & otherwise \end{cases}$.

(h) $d_{G\oplus H}((a,b),(c,d)) = \begin{cases} 0 & a = c \ \& \ b = d \\ 1 & ac \in E(G) \ or \ bd \in E(H) \ but \ not \ both \\ 2 & otherwise \end{cases}$.

Proof. The parts (a−e) are consequences of definitions and some well-known results of the book of Imrich and Klavžar, [7]. For the proof of (f−h) we refer to [11]. ∎

The Wiener index of the Cartesian product graphs was studied in [5, 19]. In [14], Klavžar, Rajapakse and Gutman computed the Szeged index of the Cartesian product graphs. The present authors, [8, 9, 10, 11, 12, 13, 22], computed some exact formulae for the hyper-Wiener, vertex PI, edge PI, the first Zagreb, the second Zagreb, the edge Wiener and the edge Szeged indices of some graph operations. The aim of this section is to continue this program for computing the Wiener-type invariants for five graph operations.

It is easy to see that $W_\lambda(G) = \sum_{k=1}^{l} d(G,k)k^\lambda$, where $l = \mathrm{diam}(G)$ denotes diameter of the graph G. Define $\alpha(k,r) = \sum_{i=1}^{k} i^r$. In the following simple lemma, we compute W_λ for five classes of known graphs.

Lemma 2.2. The following statements are hold:
1) $W_\lambda(P_n) = n\alpha(n-1,\lambda) - \alpha(n-1,\lambda+1)$,
2) $W_\lambda(K_n) = \binom{n}{2}$,
3) $W_\lambda(S_n) = (n-1) + \binom{n-1}{2}2^\lambda$,
4) $W_\lambda(K_{m,n}) = mn + 2^\lambda(\binom{m}{2} + \binom{n}{2})$,
5) $W_\lambda(C_n) = \begin{cases} n\alpha(\frac{n}{2}-1,\lambda) + (\frac{n}{2})^\lambda \frac{n}{2} & n \ is \ even \\ n\alpha(\frac{n-1}{2},\lambda) & n \ is \ odd \end{cases}$.

Proposition 2.3. Let G and H be connected graphs. Then $W_\lambda(G+H) = |E(G)| + |E(H)| + |V(G)||V(H)| + 2^\lambda(|E(\bar{G})| + |E(\bar{H})|)$.

Proof. By Lemma 2.1, we have:

$$\begin{aligned} W_\lambda(G+H) &= \sum_{k=1}^{2} d(G+H,k)k^\lambda \\ &= d(G+H,1) + d(G+H,2)2^\lambda \\ &= |E(G)| + |E(H)| + |V(G)||V(H)| + 2^\lambda\left(|E(\bar{G})| + |E(\bar{H})|\right), \end{aligned}$$

proving the result. ∎

Corollary 2.4. The reciprocal Wiener index of $G+H$ is computed as follows:

$$W_{-1}(G+H) \quad = \quad |E(G)|+|E(H)|+|V(G)||V(H)|+\frac{1}{2}\left(|E(\bar{G})|+|E(\bar{H})|\right).$$

Corollary 2.5. The Harary index of $G+H$ is computed as follows:

$$W_{-2}(G+H) \quad = \quad |E(G)|+|E(H)|+|V(G)||V(H)|+\frac{1}{4}\left(|E(\bar{G})|+|E(\bar{H})|\right).$$

Corollary 2.6. The Tratch-Stankevich-Zefirov index of $G+H$ is computed as follows:

$$TSZ(G+H)=|E(G)|+|E(H)|+|V(G)||V(H)|+4\left(|E(\bar{G})|+|E(\bar{H})|\right).$$

Corollary 2.7.(See [11, Theorem 2]) Let G and H be graphs. Then $WW(G+H) = \frac{3}{2}|V(G)|^2+\frac{3}{2}|V(H)|^2-2|E(H)|-2|E(G)|-\frac{3}{2}|V(G)|-\frac{3}{2}|V(H)|+|V(G)||V(H)|$.

Corollary 2.8.(see [11, Corollary of Theorem 2].) Suppose G_1, G_2, ..., G_n are graphs. Then

$$W_\lambda(G_1+G_2+\cdots+G_n)=(1-2^\lambda)\sum_{i=1}^{n}|E(G_i)|+\sum_{1\le i<j\le n}|V(G_i)||V(G_j)|+2^\lambda\sum_{i=1}^{n}\binom{|V(G_i)|}{2}$$

In particular,

$$WW(G_1+...+G_n)=\sum_{i=1}^{n}\left(3\binom{|V_i|}{2}-2|E_i|\right)+\frac{1}{2}\sum_{i\neq j,i,j=1}^{n}|V_i||V_j|$$

and $WW(nG)=\frac{1}{2}(n^2+2n)|V(G)|^2-2n|E(G)|-\frac{3n}{2}|V(G)|$, where nG denotes the join of n copy of G.

Proof. By a simple calculation,

$$W_\lambda(G_1+G_2+\cdots+G_n)=\sum_{i=1}^{n}|E(G_i)|+\sum_{1\le i<j\le n}|V(G_i)||V(G_j)|+2^\lambda\left[\sum_{i=1}^{n}|E(\bar{G}_i)|\right]$$

$$=(1-2^\lambda)\sum_{i=1}^{n}|E(G_i)|+\sum_{1\le i<j\le n}|V(G_i)||V(G_j)|+2^\lambda\sum_{i=1}^{n}\binom{|V(G_i)|}{2},$$

as desired. ∎

Consider a complete n-partite graph $G=K_{m_1,m_2,\ldots,m_n}$ containing $v=|V(G)|$ vertices, Figure 1. By definition, in this graph the set of vertices can be partitioned into subsets $V_1,V_2,...,V_n$ of V such that for every i, $1\le i\le n$, there is no edge between the vertices of V_i.

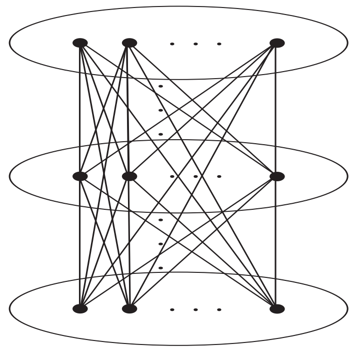

Figure 1. The Complete n-Partite Graph.

By the previous corollary, one can see that

$$W_\lambda(K_{m_1,m_2,\ldots,m_n}) = \sum_{1\le i<j\le n} m_i m_j + 2^\lambda \sum_{i=1}^{n} \binom{m_i}{2}.$$

Proposition 2.9. Let G and H be graphs. Then $W_\lambda(G \vee H) = |E(G)||V(H)|^2 + |E(H)||V(G)|^2 - 2|E(G)||E(H)| + 2^\lambda \left(|V(G)||E(\bar{H})| + |V(H)||E(\bar{G})| + 2|E(\bar{G})||E(\bar{H})|\right)$.

Proof. By Lemma 2.1 and definition of disjunction,

$$\begin{aligned} W_\lambda(G \vee H) &= \sum_{k=1}^{2} d(G \vee H, k) k^\lambda \\ &= d(G \vee H, 1) + d(G \vee H, 2) 2^\lambda \\ &= |E(G)||V(H)|^2 + |E(H)||V(G)|^2 - 2|E(G)||E(H)| \\ &+ 2^\lambda \left(|V(G)||E(\bar{H})| + |V(H)||E(\bar{G})| + 2|E(\bar{G})||E(\bar{H})|\right), \end{aligned}$$

proving the result. ∎

Corollary 2.10. The reciprocal Wiener index of $G \vee H$ is computed as follows:

$$\begin{aligned} W_{-1}(G \vee H) &= |E(G)||V(H)|^2 + |E(H)||V(G)|^2 - 2|E(G)||E(H)| \\ &+ \frac{1}{2} \left(|V(G)||E(\bar{H})| + |V(H)||E(\bar{G})| + 2|E(\bar{G})||E(\bar{H})|\right). \end{aligned}$$

Corollary 2.11. The Harary index of $G \vee H$ is computed as follows:

$$\begin{aligned} W_{-2}(G \vee H) &= |E(G)||V(H)|^2 + |E(H)||V(G)|^2 - 2|E(G)||E(H)| \\ &+ \frac{1}{4} \left(|V(G)||E(\bar{H})| + |V(H)||E(\bar{G})| + 2|E(\bar{G})||E(\bar{H})|\right). \end{aligned}$$

Corollary 2.12. The Tratch-Stankevich-Zefirov index of $G \vee H$ is computed as follows:

$$\begin{aligned} TSZ(G \vee H) &= |E(G)||V(H)|^2 + |E(H)||V(G)|^2 \\ &- 2|E(G)||E(H)| + 4(|V(G)||E(\bar{H})| \\ &+ |V(H)||E(\bar{G})| + 2|E(\bar{G})||E(\bar{H})|). \end{aligned}$$

Corollary 2.13.(See [11, Theorem 4].) Let G and H be graphs. Then $WW(G \vee H) = 3\binom{|V(G)||V(H)|}{2} + 4|E(G)||E(H)| - 2|V(H)|^2|E(G)| - 2|V(G)|^2|E(H)|$.

Proposition 2.14. Let G and H be graphs. Then the Wiener-type invariant of the symmetric difference of G and H is: $W_\lambda(G \oplus H) = 2^\lambda(2|E(G)||E(H)| + |V(H)||E(\bar{G})| + |V(G)||E(\bar{H})| + 2|E(\bar{H})||E(\bar{G})|) + |E(G)||V(H)|^2 + |E(H)||V(G)|^2 - 4|E(G)||E(H)|$.

Proof. By Lemma 2.1 and definition of symmetric difference of two graphs, we have:

$$\begin{aligned} W_\lambda(G \oplus H) &= \sum_{k=1}^{2} d(G \oplus H, k)k^\lambda \\ &= d(G \oplus H, 1) + d(G \oplus H, 2)2^\lambda \\ &= |E(G)||V(H)|^2 + |E(H)||V(G)|^2 - 4|E(G)||E(H)| \\ &+ 2^\lambda \left(2|E(G)||E(H)| + |V(H)||E(\bar{G})| + |V(G)||E(\bar{H})| \right. \\ &+ \left. 2|E(\bar{H})||E(\bar{G})|\right), \end{aligned}$$

proving the result. ∎

Corollary 2.15. The reciprocal Wiener index of $G \oplus H$ is computed as follows:

$$\begin{aligned} W_{-1}(G \oplus H) &= |E(G)||V(H)|^2 + |E(H)||V(G)|^2 - 4|E(G)||E(H)| \\ &+ \frac{1}{2}(2|E(G)||E(H)| + |V(H)||E(\bar{G})| + |V(G)||E(\bar{H})| \\ &+ 2|E(\bar{H})||E(\bar{G})|). \end{aligned}$$

Corollary 2.16. The Harary index of $G \oplus H$ is computed as follows:

$$\begin{aligned} W_{-2}(G \oplus H) &= |E(G)||V(H)|^2 + |E(H)||V(G)|^2 - 4|E(G)||E(H)| \\ &+ \frac{1}{4}\left(2|E(G)||E(H)| + |V(H)||E(\bar{G})| + |V(G)||E(\bar{H})| \right. \\ &+ \left. 2|E(\bar{H})||E(\bar{G})|\right) \end{aligned}$$

Corollary 2.17. The Tratch-Stankevich-Zefirov index of $G \oplus H$ is computed as follows:

$$\frac{1}{6}W_3(G \oplus H) + \frac{1}{2}W_2(G \oplus H) + \frac{1}{3}W_1(G \oplus H) =$$

$$= |E(G)||V(H)|^2 + |E(H)||V(G)|^2 - 4|E(G)||E(H)| + 4(2|E(G)||E(H)| + |V(H)||E(\bar{G})| + |V(G)||E(\bar{H})| + 2|E(\bar{H})||E(\bar{G})|)$$

Corollary 2.18.(See [11, Theorem 5].) Let G and H be connected graphs. Then $WW(G\oplus H) = 3\binom{|V(G)||V(H)|}{2} + 8|E(G)||E(H)| - 2|V(H)|^2|E(G)| - 2|V(G)|^2|E(H)|$.

Proposition 2.19. Let G and H be graphs. Then $W_\lambda(G[H]) = |V(H)|^2 W_\lambda(G) + |V(G)||E(H)| + 2^\lambda |V(G)||E(\bar{H})|$.

Proof. Suppose $d_1(G[H], k)$ and $d_2(G[H], k)$ denote the number of 2-subsets $\{(a, b), (x, y)\}$ such that $d_{G[H]}((a, b), (x, y)) = k$, $a = x$ and $a \neq x$, respectively. In the first case, by Lemma 2.1 $d_{G[H]}((a, b), (x, y)) \leq 2$ and so,

$$\begin{aligned} W_\lambda(G[H]) &= \sum_{k=1}^{2} d_1(G[H], k)k^\lambda + \sum_{k=1}^{d_G} d_2(G[H], k)k^\lambda \\ &= d_1(G[H], 1) + 2^\lambda d_1(G[H], 2) + |V(H)|^2 W_\lambda(G) \\ &= |V(G)||E(H)| + 2^\lambda |V(G)||E(\bar{H})| + |V(H)|^2 W_\lambda(G), \end{aligned}$$

proving the result. ∎

Corollary 2.20. The reciprocal Wiener index of $G[H]$ is computed as follows:

$$W_{-1}(G[H]) = |V(H)|^2 W_{-1}(G) + |V(G)||E(H)| + \frac{1}{2}|V(G)||E(\bar{H})|.$$

Corollary 2.21. The Harary index of $G[H]$ is computed as follows:

$$W_{-2}(G[H]) = |V(H)|^2 W_{-2}(G) + |V(G)||E(H)| + \frac{1}{4}|V(G)||E(\bar{H})|.$$

Corollary 2.22. The Tratch-Stankevich-Zefirov index of $G[H]$ is computed as $TSZ(G[H]) = |V(H)|^2 TSZ(G) + |V(G)||E(H)| + 4|V(G)||E(\bar{H})|$.

Corollary 2.23.(See [11, Theorem 3].) Let G and H be graphs and G be connected. Then $WW(G[H]) = |V(H)|^2 WW(G) + \frac{|V(G)|}{2}(WW(2H) - |V(H)|^2)$.

3 Bounds on Wiener-Type Invariants of Graphs

In this section some bounds on Wiener-type invariants of graphs are computed. We also find necessary and sufficient conditions for sharpness of these bounds.

Proposition 3.1. Let G be a graph with exactly n vertices and m edges. If $\lambda \geq 0$ then

$$2^{\lambda-1}M_1 + m(1-2^\lambda) + l^\lambda\left(\binom{n}{2} - \frac{1}{2}M_1\right) \geq W_\lambda(G) \geq 2^{\lambda-1}M_1 + m(1-2^\lambda) + 3^\lambda\left(\binom{n}{2} - \frac{1}{2}M_1\right)$$

and if $\lambda \leq 0$ then

$$2^{\lambda-1}M_1 + m(1-2^{\lambda}) + l^{\lambda}\left(\binom{n}{2} - \frac{1}{2}M_1\right) \leq W_{\lambda}(G) \leq 2^{\lambda-1}M_1 + m(1-2^{\lambda}) + 3^{\lambda}\left(\binom{n}{2} - \frac{1}{2}M_1\right).$$

Moreover, if G is triangle and quadrangle free then the equalities hold if and only if $\mathrm{diam}(G) = 3$.

Proof. Suppose $\lambda \geq 0$. By definition and [23, Eqs. 4 and 5],

$$\begin{aligned} W_{\lambda}(G) &= \sum_{k=1}^{l} d(G,k)k^{\lambda} \\ &= d(G,1) + 2^{\lambda}d(G,2) + \sum_{k\geq 3} d(G,k)k^{\lambda} \\ &\geq m + 2^{\lambda}(\frac{1}{2}M_1 - m) + 3^{\lambda}\sum_{k\geq 3} d(G,k) \\ &= 2^{\lambda-1}M_1 + m(1-2^{\lambda}) + 3^{\lambda}\left(\binom{n}{2} - \frac{1}{2}M_1\right), \end{aligned}$$

as desired. On the other hand,

$$\begin{aligned} W_{\lambda}(G) &= \sum_{k=1}^{l} d(G,k)k^{\lambda} \\ &= d(G,1) + 2^{\lambda}d(G,2) + \sum_{k\geq 3} d(G,k)k^{\lambda} \\ &\leq 2^{\lambda-1}M_1 + m(1-2^{\lambda}) + l^{\lambda}\left(\binom{n}{2} - \frac{1}{2}M_1\right). \end{aligned}$$

The second part is an immediate consequence of above inequalities. ∎

Proposition 3.2. Let G and H be graphs and R and S defined as follows:

$$\begin{aligned} R &= \frac{1}{2}|V(G)|M_1(H)(2^{\lambda} - d^{\lambda}) + \frac{1}{2}|V(H)|M_1(G)(2^{\lambda} - d^{\lambda}) \\ &\quad +4|E(G)||E(H)|(2^{\lambda} - d^{\lambda}) + d^{\lambda}\binom{|V(G)||V(H)|}{2} \\ &\quad +(1-2^{\lambda})|V(G)||E(H)| + (1-2^{\lambda})|V(H)||E(G)|, \end{aligned}$$

$$\begin{aligned} S &= \frac{1}{2}|V(G)|M_1(H)(2^{\lambda} - 3^{\lambda}) + \frac{1}{2}|V(H)|M_1(G)(2^{\lambda} - 3^{\lambda}) \\ &\quad +4|E(G)||E(H)|(2^{\lambda} - 3^{\lambda}) + 3^{\lambda}\binom{|V(G)||V(H)|}{2} \\ &\quad +(1-2^{\lambda})|V(G)||E(H)| + (1-2^{\lambda})|V(H)||E(G)|. \end{aligned}$$

If $\lambda \geq 0$ then $R \geq W_\lambda(G\times H) \geq S$ and if $\lambda \leq 0$ then $S \geq W_\lambda(G\times H) \geq R$. Moreover, if $G\times H$ is triangle and quadrangle free then the equalities hold if and only if $\mathrm{diam}(G\times H) = 3$.

Proof. Suppose $\lambda \geq 0$. By Lemma 2.1 and [8, Theorem 1], we have:

$$\begin{aligned}
W_\lambda(G\times H) &\geq 2^{\lambda-1}M_1(G\times H) + 3^\lambda\left(\binom{n_{G\times H}}{2} - \frac{1}{2}M_1(G\times H)\right)\\
&\qquad +(1-2^\lambda)m_{G\times H}\\
&= 2^{\lambda-1}\left(|V(G)|M_1(H) + |V(H)|M_1(G) + 8|E(G)||E(H)|\right)\\
&+ 3^\lambda\left(\binom{|V(G)||V(H)|}{2} - \frac{1}{2}(|V(G)|M_1(H) + 8|E(G)||E(H)|\right)\\
&+ (1-2^\lambda)\left(|V(G)||E(H)| + |V(H)||E(G)|\right) - \frac{1}{2}3^\lambda|V(H)|M_1(G)\\
&= \frac{1}{2}|V(G)|M_1(H)(2^\lambda - 3^\lambda) + \frac{1}{2}|V(H)|M_1(G)(2^\lambda - 3^\lambda)\\
&+ 4|E(G)||E(H)|(2^\lambda - 3^\lambda) + 3^\lambda\binom{|V(G)||V(H)|}{2}\\
&+ (1-2^\lambda)|V(G)||E(H)| + (1-2^\lambda)|V(H)||E(G)|.
\end{aligned}$$

Let d denote the diameter of $G\times H$. Then

$$\begin{aligned}
W_\lambda(G\times H) &\leq \frac{1}{2}|V(G)|M_1(H)(2^\lambda - d^\lambda) + \frac{1}{2}|V(H)|M_1(G)(2^\lambda - d^\lambda)\\
&+ 4|E(G)||E(H)|(2^\lambda - d^\lambda) + d^\lambda\binom{|V(G)||V(H)|}{2}\\
&+ (1-2^\lambda)|V(G)||E(H)| + (1-2^\lambda)|V(H)||E(G)|.
\end{aligned}$$

We now assume that $\lambda \leq 0$. Then,

$$\begin{aligned}
W_\lambda(G\times H) &\leq \frac{1}{2}|V(G)|M_1(H)(2^\lambda - 3^\lambda) + \frac{1}{2}|V(H)|M_1(G)(2^\lambda - 3^\lambda)\\
&+ 4|E(G)||E(H)|(2^\lambda - 3^\lambda) + 3^\lambda\binom{|V(G)||V(H)|}{2}\\
&+ (1-2^\lambda)|V(G)||E(H)| + (1-2^\lambda)|V(H)||E(G)|
\end{aligned}$$

and,

$$\begin{aligned}
W_\lambda(G\times H) &\geq \frac{1}{2}|V(G)|M_1(H)(2^\lambda - d^\lambda) + \frac{1}{2}|V(H)|M_1(G)(2^\lambda - d^\lambda)\\
&+ 4|E(G)||E(H)|(2^\lambda - d^\lambda) + d^\lambda\binom{|V(G)||V(H)|}{2}\\
&+ (1-2^\lambda)|V(G)||E(H)| + (1-2^\lambda)|V(H)||E(G)|,
\end{aligned}$$

which completes our argument. ∎

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References

[1] M. V. Diudea, I. Gutman and L. Jantschi, *Molecular Topology*, Huntington, NY, 2001.

[2] M. V. Diudea, *Indices of reciprocal properties or Harary indices*, J. Chem. Inf. Comput. Sci., **37** (1997), 292−299.

[3] I. Gutman, *A property of the Wiener number and its modifications*, Indian J. Chem., **36A** (1997), 128−132.

[4] I. Gutman, A. A. Dobrynin, S. Klavžar and L. Pavlović, *Wiener-type invariants of trees and their relation*, Bull. Inst. Combin. Appl., **40** (2004), 23−30.

[5] A. Graovac and T. Pisanski, *On the Wiener index of a graph*, J. Math. Chem. **8**(1991), 53−62.

[6] F. Harary, *Graph Theory*, AddisonWesley, Reading, MA, 1969.

[7] W. Imrich and S. Klavžar, *Product graphs: structure and recognition*, John Wiley & Sons, New York, USA, 2000.

[8] M. H. Khalifeh, H. Yousefi-Azari and A. R. Ashrafi, *The first and second Zagreb indices of some graph operations*, Discrete Appl. Math., **157**(4) (2009), 804−811.

[9] M. H. Khalifeh, H. Yousefi-Azari, A. R. Ashrafi, and S. G. Wagner, *Some new results on distance-based graph invariants*, European J. Combin., **30**(5) (2009), 1149−1163.

[10] M. H. Khalifeh, H. Yousefi-Azari, A. R. Ashrafi, *A matrix method for computing Szeged and vertex PI indices of join and composition of graphs*, Linear Algebra Appl., **429**(11-12) (2008), 2702−2709.

[11] M. H. Khalifeh, H. Yousefi-Azari and A. R. Ashrafi, *The hyper-Wiener index of graph operations*, Comput. Math. Appl., **56**(5) (2008), 1402−1407.

[12] M. H. Khalifeh, H. Yousefi-Azari and A. R. Ashrafi, *Vertex and edge PI indices of Cartesian product graphs*, Discrete Appl. Math., **156**(10) (2008), 1780−1789.

[13] M. H. Khalifeh, H. Yousefi-Azari, A. R. Ashrafi, I. Gutman, *The edge szeged index of product graphs*, Croat. Chem. Acta, **81**(2) (2008), 277−281.

[14] S. Klavžar, A. Rajapakse and I. Gutman, *The Szeged and the Wiener index of graphs*, Appl. Math. Lett., **9**(1996), 45−49.

[15] S. Klavžar, *On the PI index: PI-partitions and Cartesian product graphs*, MATCH Commun. Math. Comput. Chem., **57**(3) (2007), 573−586.

[16] S. Klavžar and I. Gutman, *A theorem on Wiener-type invariants for isometric subgraphs of hypercubes*, Appl. Math. Lett., **19** (2006), 1129−1133.

[17] B. Lučić, A. Miličević, S. Nikolić and N. Trinajstić, *Harary index – twelve years later*, Croat. Chem. Acta, **75**(4) (2002), 847-868.

[18] S. Nikolić, G. Kovačević, A. Miličević, N. Trinajstić, *The Zagreb Indices 30 Years After*, Croat. Chem. Acta, **76**(2003), 113−124.

[19] B. E. Sagan, Y.-N. Yeh and P. Zhang, *The Wiener polynomial of a graph*, Int. J. Quant. Chem., **60** (5)(1996), 959−969.

[20] N. Trinajstić, *Chemical Graph Theory*, CRC Press, Boca Raton, FL. 1992.

[21] H. Wiener, *Structural determination of the paraffin boiling points*, J. Am. Chem. Soc., **69**(1947), 17−20.

[22] H. Yousefi-Azari, B. Manoochehrian and A. R. Ashrafi, *The PI index of product graphs*, Appl. Math. Lett., **21**(6) (2008), 624−627.

[23] B. Zhou and I. Gutman, *Relations between Wiener*, hyper-Wiener and Zagreb indices, Chem. Phys. Lett., **394** (2004), 93−95.

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