Faculty of Sciences and Mathematics, University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Filomat **24:1** (2010), 87–95 DOI: 10.2298/FIL1001087U

WARPED PRODUCT CR-SUBMANIFOLDS OF LP-COSYMPLECTIC MANIFOLDS

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Abstract

In this paper, we study warped product CR-submanifolds of LP-cosymplectic manifolds. We have shown that the warped product of the type $M = N_T \times {}_fN_\perp$ does not exist, where N_T and $N_\perp$ are invariant and anti-invariant submanifolds of an LP-cosymplectic manifold $\bar{M}$, respectively. Also, we have obtained a characterization result for a CR-submanifold to be locally a CR-warped product.

1 Introduction

The geometry of warped product was introduced by Bishop and O'Neill [1]. These manifolds appear in differential geometric studies in natural way and these are generalization of Riemannian product manifolds and then it was studied by many geometers in different known spaces [2, 5]. Recently, B.Y. Chen has introduced the notion of CR-warped product in Kaehler manifolds and showed that there exist no proper warped product CR-submanifolds in the form $M = N_\perp \times {}_fN_T$ in a Kaehler manifold [3]. Later on, Hasegawa and Mihai proved that warped product CR-submanifolds $N_\perp \times {}_fN_T$ in Sasakian manifolds are trivial where N_T and $N_\perp$ are $\phi-$invariant and anti-invariant submanifolds of Sasakian manifold respectively [5].

Matsumoto [7] introduced the notion of a Lorentzian almost paracontact manifold. Then Mihai and Rosca [8] introduced the same notion and obtained several results in this manifold. Submanifolds of a Lorentzian almost paracontact manifold have been studied by Prasad and Ojha and defined a class of Lorentzian almost paracontact manifold as an LP-cosymplectic manifold in [9].

In view of the physical applications of these manifolds, the question of existence or non existence of warped product submanifolds assumes significance. In the

2010 *Mathematics Subject Classifications.* 53C15, 53C40, 53C42.
Key words and Phrases. warped product, doubly warped product, contact CR-warped product, LP-cosymplectic manifold.
Received: August 17, 2009
Communicated by Ljubisa Kocinac
The author partially supported by the University of Malaya, 50603 Kuala Lumpur Malaysia.

present paper, we have shown that the warped product in the form $M = N_T \times_f N_\perp$ is trivial where N_T is an invariant submanifold tangent to ξ and $N_\perp$ is an anti-invariant submanifold of an LP-cosymplectic manifold $\bar{M}$. On the other hand we have obtained a characterization result for the warped product of the type $M = N_\perp \times_f N_T$ when ξ is tangent to $N_\perp$. Also, we have shown that there is no warped product $M = N_1 \times_f N_2$ when ξ is tangent to N_2, where N_1 and N_2 are submanifolds of an LP-cosymplectic manifold.

2 Preliminaries

Let $\bar{M}$ be a n-dimensional Lorentzian almost paracontact manifold with the almost paracontact metric structure (ϕ, ξ, η, g), that is, ϕ is a $(1,1)$ tensor field, ξ is a contravariant vector field, η is a $1-$form and g is a Lorentzian metric with signature $(-,+,+,\cdots,+)$ on $\bar{M}$, satisfying [7]:

$$\phi^2 = X + \eta(X)\xi, \quad \eta(\xi) = -1, \quad \phi\xi = 0, \quad \eta \circ \phi = 0, \quad \text{rank}(\phi) = n-1 \tag{2.1}$$

$$g(\phi X, \phi Y) = g(X,Y) + \eta(X)\eta(Y), \quad \eta(X) = g(X, \xi), \tag{2.2}$$

$$\Phi(X,Y) = g(\phi X, Y) = g(X, \phi Y) = \Phi(Y,X), \tag{2.3}$$

for all $X, Y \in T\bar{M}$, where Φ is the fundamental $2-$form defined as above.

A Lorentzian almost contact metric structure on $\bar{M}$ is called a *Lorentzian para-cosymplectic structure* if $\bar{\nabla}\phi = 0$, where $\bar{\nabla}$ denotes the Riemannian connection with respect to g. The manifold $\bar{M}$ in this case is called a *Lorentzian para-cosymplectic* (in brief, an *LP-cosymplectic*) manifold. From formula $\bar{\nabla}\phi = 0$, it follows that $\bar{\nabla}_X \xi = 0$.

Let M be a submanifold of a Lorentzian almost paracontact manifold $\bar{M}$ with Lorentzian almost paracontact structure (ϕ, ξ, η, g). Let the induced metric on M also be denoted by g. Then Gauss and Weingarten formulae are given by

$$\bar{\nabla}_X Y = \nabla_X Y + h(X,Y) \tag{2.4}$$

$$\bar{\nabla}_X N = -A_N X + \nabla^\perp_X N, \tag{2.5}$$

for any X, Y in TM and N in $T^\perp M$, where TM is the Lie algebra of vector field in M and $T^\perp M$ is the set of all vector fields normal to M. $\nabla^\perp$ is the connection in the normal bundle, h the second fundamental form and A_N is the Weingarten endomorphism associated with N. It is easy to see that

$$g(A_N X, Y) = g(h(X,Y), N). \tag{2.6}$$

For any $X \in TM$, we write

$$\phi X = PX + FX, \tag{2.7}$$

where PX is the tangential component and FX is the normal component of ϕX. Similarly for $N \in T^\perp M$, we write

$$\phi N = BN + CN, \tag{2.8}$$

where BN is the tangential component and CN is the normal component of ϕN.

The covariant derivatives of the tensor fields ϕ, P and F are defined as

$$(\bar{\nabla}_X \phi)Y = \bar{\nabla}_X \phi Y - \phi \bar{\nabla}_X Y, \qquad \forall\ X, Y \in T\bar{M} \tag{2.9}$$

$$(\bar{\nabla}_X P)Y = \nabla_X PY - P\nabla_X Y, \qquad \forall\ X, Y \in TM \tag{2.10}$$

$$(\bar{\nabla}_X F)Y = \nabla^{\perp}_X FY - F\nabla_X Y, \qquad \forall\ X, Y \in TM. \tag{2.11}$$

Moreover, for an LP-cosymplectic manifold we have

$$(\bar{\nabla}_X P)Y = A_{FY}X + Bh(X, Y), \tag{2.12}$$

$$(\bar{\nabla}_X F)Y = Ch(X, Y) - h(X, PY). \tag{2.13}$$

For submanifolds tangent to the structure vector field ξ, there are different classes of submanifolds. We mention the following.

(i) A submanifold M tangent to ξ is called an *invariant* submanifold if F is identically zero, that is, $\phi X \in TM$ for any $X \in TM$. On the other hand M is said to be an *anti-invariant* submanifold if P is identically zero, that is, $\phi X \in T^{\perp}M$, for any $X \in TM$.

(ii) A submanifold M tangent to ξ is called a *contact CR-submanifold* if it admits a pair of differentiable distributions $\mathcal{D}$ and $\mathcal{D}^{\perp}$ such that $\mathcal{D}$ is invariant and its orthogonal complementary distribution $\mathcal{D}^{\perp}$ is anti-invariant i.e., $TM = \mathcal{D} \oplus \mathcal{D}^{\perp} \oplus \langle \xi \rangle$ with $\phi(\mathcal{D}_x) \subseteq \mathcal{D}_x$ and $\phi(\mathcal{D}^{\perp}_x) \subset T^{\perp}_x M$, for every $x \in M$.

Let M be an $m-$dimensional CR-submanifold of an LP-cosymplectic manifold $\bar{M}$. Then, $F(T_x M)$ is a subspace of $T^{\perp}_x M$. Thus it follows that $T_x M \oplus F(T_x M)$ is invariant with respect to ϕ. Then for every $x \in M$, there exists an invariant subspace ν_x of $T_x\bar{M}$ such that

$$T_x\bar{M} = T_x M \oplus F(T_x M) \oplus \nu_x.$$

3 Warped and Doubly Warped Product Submanifolds

Let (N_1, g_1) and (N_2, g_2) be two semi-Riemannian manifolds and f, a positive differentiable function on N_1. The warped product of N_1 and N_2 is the manifold $N_1 \times_f N_2 = (N_1 \times N_2,\ g)$, where

$$g = g_1 + f^2 g_2. \tag{3.1}$$

We recall the following general formula on a warped product [1].

$$\nabla_X V = \nabla_V X = (X \ln f)V, \tag{3.2}$$

where X is tangent to N_1 and V is tangent to N_2.

Let $M = N_1 \times {}_f N_2$ be a warped product manifold, this means that N_1 is totally geodesic and N_2 is totally umbilical submanifold of M, respectively.

Doubly warped product manifolds were introduced as a generalization of warped product manifolds by B. Ünal [10]. A *doubly warped product manifold* of N_1 and N_2, denoted as ${}_{f_2}N_1 \times {}_{f_1}N_2$ is endowed with a metric g defined as

$$g = f_2^2 g_1 + f_1^2 g_2 \tag{3.3}$$

where f_1 and f_2 are positive differentiable functions on N_1 and N_2 respectively.

In this case formula (3.2) is generalized as

$$\nabla_X Z = (X \ln f_1)Z + (Z \ln f_2)X \tag{3.4}$$

for each $X \in TN_1$ and $Z \in TN_2$ [10].

If neither f_1 nor f_2 is constant we have a non trivial doubly warped product $M = {}_{f_2}N_1 \times {}_{f_1}N_2$. Obviously in this case both N_1 and N_2 are totally umbilical submanifolds of M.

We now consider a doubly warped product of two semi-Riemannian manifolds N_1 and N_2 embedded into an LP-cosymplectic manifold $\bar{M}$ such that the structure vector field ξ is tangential to the submanifold $M = {}_{f_2}N_1 \times {}_{f_1}N_2$.

Theorem 3.1. *There does not exist a proper doubly warped product submanifold in LP-cosymplectic manifolds.*

Proof. Let $M = {}_{f_2}N_1 \times {}_{f_1}N_2$ be a doubly warped product submanifold of an LP-cosymplectic manifold $\bar{M}$, where N_1 and N_2 are submanifolds of $\bar{M}$. We have using Gauss formula and the fact that $\bar{M}$ is LP-cosymplectic, for any $U \in TM$

$$\nabla_U \xi = 0. \tag{3.5}$$

Thus in case $\xi \in TN_1$ and $U \in TN_2$ equation (3.4) and (3.5) imply that $(\xi \ln f_1)U + (U \ln f_2)\xi = 0$, which shows that f_2 is constant. Similarly, for $\xi \in TN_2$ and $U \in TN_1$, we have $(\xi \ln f_2)U + (U \ln f_1)\xi = 0$, showing that f_1 is constant. This completes the proof. □

In above theorem we see that f_2 is constant if the structure vector field ξ is tangent to N_1 and f_1 is constant if the structure vector field ξ is tangent to N_2. The following corollary is an immediate consequence of the above theorem.

Corollary 3.1. *There does not exist a warped product submanifold $N_1 \times {}_f N_2$ of an LP-cosymplectic manifold $\bar{M}$ such that ξ is tangent to N_2.*

Thus the only remaining case to study is the warped product submanifold $N_1 \times {}_f N_2$ with structure vector field ξ tangential to N_1, we first obtain some useful formulae for later use.

Lemma 3.1. *Let $M = N_1 \times {}_f N_2$ be a proper warped product submanifold of an LP-cosymplectic manifold $\bar{M}$ such that ξ is tangent to N_1, where N_1 and N_2 are submanifolds of $\bar{M}$. Then*

(i) $\xi \ln f = 0$,

(ii) $A_{FZ}X = -Bh(X, Z)$,

(iii) $g(h(X, Y), FZ) = -g(h(X, Z), FY)$,

(iv) $g(h(X, Z), FW) = -g(h(X, W), FZ)$

for any $X,\ Y \in TN_1$ *and* $Z,\ W \in TN_2$.

Proof. The first part of the lemma is an immediate consequence of the fact that $\bar{\nabla}_U \xi = 0$, for $U \in TM$ and using formula (2.4) and separating the tangential and normal parts. Now, for any $X \in TN_1$ and $Z \in TN_2$, then formula (2.12) gives

$$(\bar{\nabla}_X P)Z = A_{FZ}X + Bh(X, Z). \tag{3.6}$$

Also, we have

$$(\bar{\nabla}_X P)Z = \nabla_X PZ - P\nabla_X Z = (X \ln f)PZ - P(X \ln f)Z = 0, \tag{3.7}$$

for any $X \in TN_1$ and $Z \in TN_2$. Part (ii) follows by equations (3.6) and (3.7). Parts (iii) and (iv) follow by taking the product in (ii) by Y and W respectively. □

4 CR-Warped Product Submanifolds

Throughout this section the structure vector field ξ is either tangent to the invariant submanifold N_T or tangent to the anti-invariant submanifold $N_\perp$. There are two types of warped product in an LP-cosymplectic manifold $\bar{M}$, namely $N_T \times_f N_\perp$ and $N_\perp \times_f N_T$ are called *CR-warped product* submanifolds with ξ tangent to N_T and $N_\perp$, respectively. The following theorem is dealt with the case when ξ is tangent to N_T.

Theorem 4.1. *There does not exist a proper warped product submanifold $N_T \times_f N_\perp$ where N_T is an invariant and $N_\perp$ is an anti-invariant submanifolds of an LP-cosymplectic manifold $\bar{M}$ such that ξ is tangent to N_T.*

Proof. Let $M = N_T \times_f N_\perp$ be a warped product CR-submanifold of an LP-cosymplectic manifold $\bar{M}$ with $\xi \in TN_T$ then from equations (2.2), (2.4) and the fact that $\bar{M}$ is an LP-cosymplectic, we have

$$g(\nabla_X Z, W) = g(\nabla_Z X, W) = g(\bar{\nabla}_Z X, W) = g(\phi\bar{\nabla}_Z X, \phi W)$$

for any $X \in TN_T$ and $Z \in TN_\perp$. Using (3.2), we get

$$(X \ln f)g(Z, W) = g(\bar{\nabla}_Z \phi X, \phi W) = g(\nabla_Z \phi X + h(Z, \phi X), \phi W),$$

or

$$(X \ln f)g(Z, W) = g(h(Z, \phi X), \phi W) + (\phi X \ln f)g(Z, \phi W) = g(h(Z, \phi X), \phi W).$$

That is,

$$(X \ln f)g(Z,W) = g(h(Z,\phi X),\phi W). \tag{4.1}$$

Again, we have

$$g(h(Z,\phi X),\phi W) = g(\bar{\nabla}_{\phi X} Z,\phi W). \tag{4.2}$$

Making use of equations (2.3), (2.5), (2.6) and (2.10) we deduce from (4.2) that

$$g(h(Z,\phi X),\phi W) = -g(h(\phi X,W),\phi Z). \tag{4.3}$$

Interchanging Z and W in (4.1) and then adding the resulting equation in (4.1), we get

$$2(X \ln f)g(Z,W) = g(h(Z,\phi X),\phi W) + g(h(\phi X,W),\phi Z).$$

Using (4.3), we obtain

$$(X \ln f)g(Z,W) = 0, \tag{4.4}$$

for all $X \in TN_T$ and $Z,\ W \in TN_\perp$. As $N_\perp \neq \{0\}$ anti-invariant submanifold then equation (4.4) and Lemma 3.1 (i) imply that f is constant on N_T, proving the result. □

Now, the other case i.e., $N_\perp \times {}_f N_T$ with ξ is tangent to $N_\perp$.

Lemma 4.1. *Let $M = N_\perp \times {}_f N_T$ be a warped product submanifold of an LP-cosymplectic manifold $\bar{M}$. Then*

$$g(h(X,\phi Y),\phi Z) = -(Z \ln f)g(X,Y), \tag{4.5}$$

for any $X,\ Y \in TN_T$ and $Z \in TN_\perp$.

Proof. For any $X, Y \in TN_T$ and $Z \in TN_\perp$, by formula (3.2) we have

$$g(\bar{\nabla}_X Y,Z) = g(\nabla_X Y,Z) = -g(\nabla_X Z,Y) = -(Z \ln f)g(X,Y). \tag{4.6}$$

Now, for any $X, Y \in TN_T$ and $Z \in TN_\perp$, consider

$$\begin{aligned} g(\bar{\nabla}_X Y,Z) &= g(\phi\bar{\nabla}_X Y,\phi Z) \\ &= g(\bar{\nabla}_X \phi Y,\phi Z) \\ &= g(h(X,\phi Y),\phi Z), \end{aligned}$$

i.e,

$$g(\bar{\nabla}_X Y,Z) = g(h(X,\phi Y),\phi Z). \tag{4.7}$$

Thus equation (4.5) follows by (4.6) and (4.7). This completes the proof of the lemma. □

Theorem 4.2. *Let M be a CR-submanifold of an LP-cosymplectic manifold $\bar{M}$. Then M is locally a contact CR-warped product if and only if*

$$A_{\phi Z} X = -Z(\mu)\phi X, \quad X \in \mathcal{D}, \quad Z \in \mathcal{D}^\perp \oplus \langle\xi\rangle \tag{4.8}$$

for some function μ on M satisfying $W'(\mu) = 0$ for each $W' \in \mathcal{D}$.

Proof. If $M = N_\perp \times {}_f N_T$ is CR-warped product submanifold, then on applying Lemma 4.1, we obtain (4.8). In this case $\mu = \ln f$.

Conversely, suppose M is CR-submanifold of $\bar{M}$ and satisfying

$$A_{\phi Z} X = -Z(\mu)\phi X,$$

then

$$g(h(X,X), \phi Z) = g(A_{\phi Z}X, X) = -Z(\mu)g(\phi X, X) = 0$$

i.e., $h(X,Y) \in \nu$ the orthogonal complementary distribution of $\phi(\mathcal{D}^\perp \oplus \langle\xi\rangle)$. On the other hand, for any $X \in TN_T$ and $Z,\ W \in TN_\perp$ we have

$$g(\nabla_W Z, \phi X) = g(\bar{\nabla}_W Z, \phi X).$$

As g is Lorentzian and $\bar{M}$ is LP-cosymplectic, the above equation takes the form

$$g(\nabla_W Z, \phi X) = -g(\bar{\nabla}_W \phi Z,\ X).$$

Thus, on using (2.5) and (2.6) we get

$$g(\nabla_W Z, \phi X) = g(A_{\phi Z}W,\ X) = g(h(X,\ W),\ \phi Z).$$

Also, by (2.4) we have

$$\begin{aligned} g(h(X,\ W),\ \phi Z) &= g(\bar{\nabla}_X W, \phi Z) \\ &= -g(\bar{\nabla}_X \phi Z,\ W) \\ &= g(A_{\phi Z}X, W). \end{aligned}$$

Using (4.8) in above, we get

$$g(\nabla_W Z, \phi X) = -(Z\mu)g(\phi X,\ W) = 0.$$

This means that $\mathcal{D}^\perp \oplus \langle\xi\rangle$ is integrable and its leaves are totally geodesic in M. Also, we have

$$\begin{aligned} g(\nabla_X Y, Z) &= g(\bar{\nabla}_X Y, Z) = -g(\bar{\nabla}_X Z, Y) = -g(\bar{\nabla}_X \phi Z, \phi Y) \\ &= g(A_{\phi Z}X, \phi Y) = -Z(\mu)g(\phi X, \phi Y) = -Z(\mu)g(X,Y) \end{aligned}$$

i.e.,

$$g(\nabla_X Y, Z) = -Z(\mu)g(X,Y) \tag{4.9}$$

for any $X, Y \in \mathcal{D}$ and $Z \in \mathcal{D}^\perp \oplus \langle\xi\rangle$. Now, by Gauss formula

$$g(h'(X,Y), Z) = g(\nabla_X Y, Z)$$

where h' denotes the second fundamental form of the immersion of N_T into M. On using (4.9), the last equation gives

$$g(h'(X,Y),Z) = -Z(\mu)g(X,Y)$$

which shows that each leaf of N_T of $\mathcal{D}$ is totally umbilical in M. Moreover the fact that $W'\mu = 0$ for all $W' \in \mathcal{D}$, implies that the mean curvature vector on N_T is parallel along N_T i.e., each leaf of $\mathcal{D}$ is an extrinsic sphere in M. Hence by virtue of a result in [6] which states that -*"If the tangent bundle of a Riemannian manifold M splits into an orthogonal sum $TM = E_0 \oplus E_1$ of non trivial vector sub bundles such that E_1 is spherical and its orthogonal complement E_0 is auto parallel, then the manifold M is locally isometric to a warped product $M_0 \times_f M_1$"*, we get that, M is locally a warped $N_\perp \times_f N_T$ of a holomorphic submanifold N_T and a totally real submanifold $N_\perp$ of M. Here N_T is a leaf of $\mathcal{D}$ and $N_\perp$ is a leaf of $\mathcal{D}^\perp \oplus \langle\xi\rangle$ and f is a warping function. □

Acknowledgement. The author is grateful to referees and professor Angelina Chin Yan Mui for their valuable suggestions and comments.

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