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SOME GENERALIZATIONS OF THE JACOBSTHAL NUMBERS

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Abstract

The main object of this paper is to introduce and investigate some properties and relations involving sequences of numbers $F_{n,m}(r)$, for $m = 2, 3, 4$, and r is some real number. These sequences are generalizations of the Jacobsthal and Jacobsthal Lucas numbers.

1 Introduction

In [1] we considered the following classes of polynomials: $J_{n,m}(x)$–Jacobsthal polynomials, $j_{n,m}(x)$–Jacobsthal Lucas polynomials, and polynomials $F_{n,m}(x)$ and $f_{n,m}(x)$. These polynomials are given by the following recurrence relations ([1]):

$$J_{n,m}(x) = J_{n-1,m}(x) + 2xJ_{n-m,m}(x), \tag{1}$$

$$(n \geq m;\ n, m \in \mathbb{N};\ J_{0,m}(x) = 0,\ J_{n,m}(x) = 1,\ when\ n = 1, 2, \ldots, m-1);$$

$$j_{n,m}(x) = j_{n-1,m}(x) + 2xj_{n-m,m}(x), \tag{2}$$

$$(n \geq m;\ n, m \in \mathbb{N};\ j_{0,m}(x) = 2,\ j_{n,m}(x) = 1,\ when\ n = 1, 2, \ldots, m-1);$$

$$F_{n,m}(x) = F_{n-1,m}(x) + 2xF_{n-m,m}(x) + 3, \tag{3}$$

$$(n \geq m;\ n, m \in \mathbb{N};\ F_{0,m}(x) = 0,\ F_{n,m}(x) = 1,\ when\ n = 1, 2, \ldots, m-1);$$

$$f_{n,m}(x) = f_{n-1,m}(x) + 2xf_{n-m,m}(x) + 5, \tag{4}$$

$$(n \geq m;\ n, m \in \mathbb{N};\ f_{0,m}(x) = 0,\ f_{n,m}(x) = 1,\ when\ n = 1, 2, \ldots, m-1.)$$

The polynomials $J_{n,2}(x)$, $j_{n,2}(x)$, $F_{n,2}(x)$ and $f_{n,2}(x)$ are considered in [3]. For $x = 1$ and for a some real number r, by (3), we get the following sequences of numbers $\{C_{n,m}(r)\}$:

$$C_{n,m}(r) = C_{n-1,m}(r) + 2C_{n-m,m}(r) + r, \tag{5}$$

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$$(n \geq m;\ n, m \in \mathbb{N};\ \ C_{0,m}(r) = 0,\ C_{n,m}(r) = 1,\ for\, n = 1, 2, \ldots, m-1).$$

Particular cases of these numbers are Jacobsthal numbers J_n and Lucas numbers j_n, which were investigated by Horadam [4].

In this note we consider the sequences $\{C_{n,3}(r)\}$ and $\{C_{n,4}(r)\}$. Namely, for these sequence of numbers we find some interesting relations, which are analogous to those corresponding to generalized Fibonacci numbers [2].

2 The sequence $\{C_{n,3}(r)\}$

For $m = 3$ in (5), we have

$$C_{n,3}(r) = C_{n-1,3}(r) + 2C_{n-3,3}(r) + r, \tag{6}$$

$$(n \geq 3;\ n \in N;\ C_{0,3}(r) = 0,\ C_{1,3}(r) = C_{2,3}(r) = 1).$$

Applying (6), we obtain the first few members of the sequence numbers $\{C_{n,3}(r)\}$:

$$\begin{aligned}
C_{0,3}(r) &= 0, & C_{1,3}(r) &= 1,\\
C_{2,3}(r) &= 1, & C_{3,3}(r) &= 1 + r,\\
C_{4,3}(r) &= 3 + 2r, & C_{5,3}(r) &= 5 + 3r,\\
C_{6,3}(r) &= 7 + 6r, & C_{7,3}(r) &= 13 + 11r,\\
C_{8,3}(r) &= 23 + 18r, & C_{9,3}(r) &= 63 + 54r.
\end{aligned}$$

First of all, we introduce the following operators which will be needed in our proposed investigation. Hence, I is the identity operator, E_i is the "the coordinate" operator $(i = 1, 2, 3)$, E is the shift operator.

Furthermore, we consider the following operators Δ_i for $i = 1, 2, 3$, and ∇_i, for $i = 1, \ldots, 5$, as well as operators Δ_i^n, $(i = 1, 2, 3)$, $n \in \mathbb{N}$, and ∇_i^n, $(i = 1, \ldots, 5)$, $n \in N$.

$$\begin{aligned}
\Delta_1 &= -4I + E_1 + 4E_2, & \Delta_1^n &= \sum_{i+j=n} \binom{n}{i,j} (-1)^{n-i-j} 4^{n-i} E_1^i E_2^j,\\
\Delta_2 &= 4I + E_1 + E_2, & \Delta_2^n &= \sum_{i+j=n} \binom{n}{i,j} 4^{n-i-j} E_1^i E_2^j,\\
\Delta_3 &= 4I + 4E_1 - E_2, & \Delta_3^n &= \sum_{i+j=n} \binom{n}{i,j} (-1)^{j} 4^{n-j} E_1^i E_2^j,
\end{aligned}$$

where $\binom{n}{i,j} = \dfrac{n!}{i!j!(n-i-j)!}$, $n \in \mathbb{N}$.

$$\begin{aligned}
\nabla_1 &= -4I + 4E_1 + 2E_2 + E_3, \quad \nabla_1^n = \sum_{i+j+k=n} \binom{n}{i,j,k} (-4)^{n-i-j-k} 4^i 2^j E_1^i E_2^j E_3^k,\\
\nabla_2 &= 2E_1 - 4I + 4E_2 + E_3, \quad \nabla_2^n = \sum_{i+j+k=n} \binom{n}{i,j,k} (-4)^{n-i-j-k} 2^i 4^j E_1^i E_2^j E_3^k,\\
\nabla_3 &= -4I + E_1 + 4E_2 + 2E_3, \quad \nabla_3^n = \sum_{i+j+k=n} \binom{n}{i,j,k} (-4)^{n-i-j-k} 4^j 2^k E_1^i E_2^j E_3^k,\\
\nabla_4 &= -4I + E_1 + 2E_2 + 4E_3, \quad \nabla_4^n = \sum_{i+j+k=n} \binom{n}{i,j,k} (-4)^{n-i-j-k} 2^j 4^k E_1^i E_2^j E_3^k,\\
\nabla_5 &= -4I + 4E_1 + 2E_2 - 3E_3, \quad \nabla_5^n = \sum_{i+j+k=n} \binom{n}{i,j,k} (-4)^{n-i-j-k} 4^i 2^j (-3)^k E_1^i E_2^j E_3^k,
\end{aligned}$$

where

$$\binom{n}{i,j,k} = \frac{n!}{i!j!k!(n-i-j-k)!},$$

Applying operators Δ_1^n, Δ_2^n and Δ_3^n to the function $f(i,j)$, (see also [5]), we find the following functions

$$g(n,k) = \Delta_i^n f(0,k), \;\; n = 1,2,3; \;\; n \in \mathbb{N}.$$

Applying ∇_i^n, $(i = 1,\dots,5)$, to the function $f(i,j,k)$, we get

$$g_p f(n,0,m) = \nabla_p^n f(0,0,m), \;\; p = 1,\dots,5, \;\; n \in \mathbb{N}.$$

We prove the following two statement.

Lemma 2.1. *For a nonnegative integer k, the following relation holds*

$$4C_{k,3}(r) - 4C_{k+3,3}(r) + C_{k+6,3}(r) = C_{k+4,3}(r). \tag{7}$$

Proof. Using (6), we get

$$\begin{aligned}
&4C_{k,3}(r) - 4C_{k+3,3}(r) + C_{k+6,3}(r)\\
&= 2(C_{k+3,3}(r) - C_{k+2,3}(r) - r) - 4C_{k+3,3}(r) + C_{k+5,3}(r) + 2C_{k+3,3}(r) + r\\
&= C_{k+5,3}(r) - 2C_{k+2,3}(r) - r\\
&= C_{k+4,3}(r) + 2C_{k+2,3}(r) + r - 2C_{k+2,3}(r) - r = C_{k+4,3}(r).
\end{aligned}$$

□

Theorem 2.1. *Let $n \in \mathbb{N}$ and k be nonnegative integer. Then the following hold:*

$$C_{6n+3k,3}(r) = \sum_{i+j=n} \binom{n}{i,j} (-1)^{n-i-j} 4^{n-1} C_{4i+3(j+k),3}(r); \tag{8}$$

$$C_{6n+4k,3}(r) = \sum_{i+j=n} \binom{n}{i,j} (-1)^{n+j} 4^{n-j} C_{3i+4j+k,3}(r); \tag{9}$$

$$C_{4n+3k,3}(r) = \sum_{i+j=n} \binom{n}{i,j} (-1)^{n+j} 4^{n-i} C_{6i+3(j+k),}(r); \tag{10}$$

$$C_{3n+4k,3}(r) = \sum_{i+j=n} \binom{n}{i,j} 4^{-i-j} (-1)^{j} C_{6i+4(j+k),3}(r); \tag{11}$$

$$C_{3n+6k,3}(r) = \sum_{i+j=n} \binom{n}{i,j} 4^{-i-j} C_{4i+6(j+k),3}(r). \tag{12}$$

Proof. We apply Δ_1 to $f(i,j) = C_{4i+3j,3}(r)$, and obtain

$$\begin{aligned} \Delta_1 f(i,j) &= -4C_{4i+3j,3}(r) + C_{4i+4+3j,3}(r) + 4C_{4i+3j+3,3}(r) \\ &= C_{4i+3j+6,3}(r) = E_2^2 f(i,j). \end{aligned}$$

Now, (8) follows:

$$\Delta_1^n f(0,k) = E_2^{2n} f(0,k) = \sum_{i+j=n} \binom{n}{i,j} (-1)^{n-i-j} 4^{n-i} C_{4i+3(j+k),3}(r)$$
$$= C_{3(k+2n),3}(r) = C_{6n+3k,3}(r).$$

Applying Δ_3 to $f(i,j) = (-1)^i C_{3i+4j,3}(r)$, we have

$$\begin{aligned} \Delta_3 f(i,j) &= 4(-1)^i C_{3i+4j,3}(r) + 4(-1)^{i+1} C_{3i+3+4j,3}(r) - (-1)^i C_{3i+4j+4,3}(r) \\ &= (-1)^i \left(4C_{3i+4j,3}(r) - 4C_{3i+4j+3,3}(r) - C_{3i+4j+4,3}(r)\right) \\ &= (-1)^i C_{3i+4j+6,3}(r) = -E_1^2 f(i,j). \end{aligned}$$

Hence

$$\Delta_3^n f(0,k) = (-1)^n E_1^{2n} f(0,k) = (-1)^n \sum_{i+j=n} \binom{n}{i,j} 4^{n-i} (-1)^j C_{3i+4(j+k),3}(r)$$
$$= (-1)^n C_{6n+4k,3}(r).$$

It follows that the relation (9) holds.

Again, applying Δ_1 to $f(i,j) = (-1)^i C_{6i+3j,3}(r)$, we get

$$\begin{aligned} \Delta_1 f(i,j) &= -4(-1)^i C_{6i+3j,3}(r) + (-1)^{i+1} C_{6i+6+3j,3}(r) + 4(-1)^i C_{6i+3j+3,3}(r) \\ &= -(-1)^i \left(4C_{6i+3j,3}(r) + C_{6i+6+3j,3}(r) - 4C_{6i+3j+3,3}(r)\right) \\ &= -(-1)^i C_{6i+3j+4,3}(r) = -E_2^{4/3} f(i,j). \end{aligned}$$

Hence we conclude

$$\Delta_1^n f(0,k) = (-1)^n E_2^{4n/3} f(0,k) = (-1)^n C_{4n+3k,3}(r),$$

It follows that the relation (10) is satisfied.

We apply Δ_2 to $f(i,j) = (-1)^j C_{6i+4j,3}(r)$, and obtain (11):

$$\Delta_2 f(i,j) = 4E_1^{1/2} f(i,j),$$

wherefrom

$$\Delta_2^n f(0,k) = 4^n E_1^{n/2} f(0,k) = 4^n C_{3n+4k,3}(r),$$

Applying Δ_2 to $f(i,j) = (-1)^i C_{4i+6j,3}(r)$, we obtain

$$\begin{aligned} \Delta_2 f(i,j) &= 4(-1)^i C_{4i+6j,3}(r) + (-1)^{i+1} C_{4i+4+6j,3}(r) + (-1)^i C_{4i+6j+6,3}(r) \\ &= (-1)^i \left(4C_{4i+6j,3}(r) - C_{4i+6j+4,3}(r) + C_{4i+6j+6,3}(r)\right) \\ &= (-1)^i C_{4i+6j+3,3}(r) = 4E_2^{1/2} f(i,j). \end{aligned}$$

Thus, we get (12):

$$\Delta_2^n f(0,k) = 4^n E_2^{n/2} f(0,k) = 4^n C_{3n+6k,3}(r).$$

□

As a special case, we obtain the following result.

Corollary 2.1. *For $k = 0$ the relations (8)–(12) become, respectively:*

$$C_{6n,3}(r) = \sum_{i+j=n} \binom{n}{i,j} (-1)^{n-i-j} 4^{n-i} C_{4i+3j,3}(r);$$

$$C_{6n,3}(r) = \sum_{i+j=n} \binom{n}{i,j} (-1)^{n+j} 4^{n-j} C_{3i+4j,3}(r);$$

$$C_{4n,3}(r) = \sum_{i+j=n} \binom{n}{i,j} (-1)^{n+j} 4^{n-i} C_{6i+3j,3}(r);$$

$$C_{3n,3}(r) = \sum_{i+j=n} \binom{n}{i,j} 4^{-i-j} (-1)^j C_{6i+4j,3}(r);$$

$$C_{3n,3}(r) = \sum_{i+j=n} \binom{n}{i,j} 4^{-i-j} C_{4i+6j,3}(r).$$

Lemma 2.2. *If the sequence* $\{X_n\}$ $(n \in \mathbb{N})$ *satisfies the following relation*

$$X_n = X_{n-2} + 4X_{n-3} - 4X_{n-6}, \;\; n \geq 6,$$

then

$$I = E^{-2} + 4E^{-3} - 4E^{-6}.$$

So,

$$I = (I^n) = \sum_{i+j=n} \binom{n}{i,j} (-1)^{n-i-j} 4^{n-i} E^{-6n+4i+3j}. \tag{13}$$

Also, for nonnegative integers n *and* k*, the sequence* $\{X_{6n+k}\}$ *satisfies the following relation*

$$X_{6n+k} = \sum_{i+j=n} \binom{n}{i,j} (-1)^{n-i-j} E_{4i+3j+k}. \tag{14}$$

Proof. Applying the identity operator (13) to the sequence $\{X_{6n+k}\}$, we obtain the relation (14). □

Corollary 2.2. *The following relation holds*

$$C_{6n+k,3}(r) = \sum_{i+j=n} \binom{n}{i,j} (-1)^{n-i-j} 4^{n-i} C_{4i+3j+k,3}(r). \tag{15}$$

Proof. Follows from Lemma 2.1 and Lemma 2.2. □

For $k = 0$ in (15), we get the following result.

Corollary 2.3. *For every nonnegative integer* n*, we get*

$$C_{6n,3}(r) = \sum_{i+j=n} \binom{n}{i,j} (-1)^{n-i-j} 4^{n-i} C_{4i+3j,3}(r).$$

3 The sequence $\{C_{n,4}(r)\}$

From (5), for $m = 4$, we get the sequence of numbers $C_{n,4}(r)$ which satisfy the following recurrence relation

$$C_{n,4}(r) = C_{n-1,4}(r) + 2C_{n-4,4}(r) + r, \tag{16}$$

$$(n \geq 4;\; n \in \mathbb{N};\; C_{0,4}(r) = 0,\; C_{n,4}(r) = 1,\; n = 1, 2, 3.)$$

Hence, using (16), we obtain the some initial values of $C_{n,4}(r)$:

$$\begin{aligned} C_{0,4}(r) &= 0, & C_{1,4}(r) &= 1, \\ C_{2,4}(r) &= 1, & C_{3,4}(r) &= 1, \\ C_{4,4}(r) &= 1+r, & C_{5,4}(r) &= 3+2r, \\ C_{6,4}(r) &= 5+3r, & C_{7,4}(r) &= 7+4r, \\ C_{8,4}(r) &= 9+7r, & C_{9,4}(r) &= 15+12r, \\ C_{10,4}(r) &= 22+19r. \end{aligned}$$

These numbers satisfy the following two statement.

Lemma 3.1. *For a positive integer k the following relation holds*

$$4C_{k+2,4}(r) - 4C_{k,4}(r) + 2C_{k+3,4}(r) + C_{k+9,4}(r) = 3C_{k+7,4}(r). \tag{17}$$

Proof. Using the recurrence relation (16), we get

$$\begin{aligned} &4C_{k+2,4}(r) - 4C_{k,4}(r) + 2C_{k+3,4}(r) + C_{k+9,4}(r) \\ &= 2(C_{k+6,4}(r) - C_{k+5,4}(r) - r) - 2(C_{k+4,4}(r) - C_{k+3,4}(r) - r) \\ &\quad +2C_{k+3,4}(r) + C_{k+8,4}(r) + 2C_{k+5,4}(r) + r \\ &= 2C_{k+6,4}(r) + 4C_{k+3,4}(r) - 2C_{k+4,4}(r) + r \\ &\quad +C_{k+7,4}(r) + 2C_{k+4,4}(r) + r \\ &= 2C_{k+6,4}(r) + C_{k+7,4}(r) + 2(C_{k+7,4}(r) - C_{k+6,4}(r) - r) + 2r \\ &= 3C_{k+7,4}(r). \end{aligned}$$

□

Theorem 3.1. *Let n and k be nonnegative integers. Then the following hold:*

$$3^n C_{7n+9m,4}(r) = \sum_{i+j+k=n} \binom{n}{i,j,k} (-1)^{n-i-j-k} 4^{n-j-k} 2^j A_1; \tag{18}$$

$$3^n C_{7n+3m,4}(r) = \sum_{i+j+k=n} \binom{n}{i,j,k} (-1)^{n-i-j-k} 4^{n-i-k} 2^i A_2; \tag{19}$$

$$3^n C_{7n+2m,4}(r) = \sum_{i+j+k=n} \binom{n}{i,j,k} (-4)^{n-i-j-k} 2^j 4^k A_3; \tag{20}$$

$$C_{9n+7m,4}(r) = \sum_{i+j+k=n} (-1)^{i+j} 4^{n-j-k} 2^j 3^k C_{2i+3j+7(k+m),4}(r), \tag{21}$$

where

$A_1 = C_{2i+3j+4(k+m),4}(r)$, $A_2 = C_{9i+2j+3(k+m),4}(r)$, $A_3 = C_{9i+3j+2(k+m),4}(r)$.

Proof. We apply ∇_1 to $f(i,j,k) = C_{2i+3j+9k,4}(r)$, and we get

$$\begin{aligned}\nabla_1 f(i,j,k) &= -4C_{2i+3j+9k,4}(r) + 4C_{2i+2+3j+9k,4}(r) + 2C_{2i+3j+3+9k,4}(r)\\ &\quad +C_{2i+3j+9k+9,4}(r) = 3C_{2i+3j+9k+7,4}(r)\\ &= \begin{cases} 3E_1^{7/2} f(i,j,k)\\ 3E_2^{7/3} f(i,j,k)\\ 3E_3^{7/9} f(i,j,k).\end{cases}\end{aligned}$$

Hence, we obtain the relation (18) in three ways:

$$\nabla_1^n f(0,0,m) = 3^n E_1^{7n/2} f(0,0,m) = 3^n C_{2(0+7n/2)+9m,4}(r) = 3^n C_{7n+9m,4}(r),$$
$$\nabla_1^n f(0,0,m) = 3^n E_2^{7n/3} f(0,0,m) = 3^n C_{3(0+7n/3)+9m,4}(r) = C_{7n+9m,4}(r),$$
$$\nabla_1^n f(0,0,m) = 3^n E_3^{7n/4} f(0,0,m) = 3^n C_{9(m+7n/9),4}(r) = 3^n C_{7n+9m,4}(r).$$

Furthermore, applying ∇_2 to $f(i,j,k) = C_{3i+2j+9k,4}(r)$, and using (16), we obtain the relation (19):

$$\begin{aligned}\nabla_2 f(i,j,k) &= -4C_{3i+2j+9k,4}(r) + 2C_{3i+3+2j+9k,4}(r) + 4C_{3i+2j+2+9k,4}(r)\\ &\quad +C_{3i+2j+9k+9,4}(r) = 3C_{3i+2j+9k+7,4}(r)\\ &= \begin{cases} 3E_1^{7/3} f(i,j,k)\\ 3E_2^{7/2} f(i,j,k)\\ 3E_3^{7/9} f(i,j,k).\end{cases}\end{aligned}$$

Applying ∇_3 to $f(i,j,k) = C_{9i+2j+3k,4}(r)$, we find that

$$\begin{aligned}\nabla_3 f(i,j,k) &= -4C_{9i+2j+3k,4}(r) + C_{9i+9+2j+3k,4}(r) + 4C_{9i+2j+2+3k,4}(r)\\ &\quad +2C_{9i+2j+3k+3,4}(r)\\ &= 3C_{9i+2j+3k+7,4}(r) = 3E_3^{7/3} f(i,j,k).\end{aligned}$$

So, we obtain (19):

$$\nabla_3^n f(0,0,m) = 3^n E_3^{7n/3} f(0,0,m) = 3^n C_{7n+3m,4}(r).$$

Simialrly, applying ∇_4 to $f(i,j,k) = C_{9i+3j+2k,4}(r)$, we obtain (20):

$$\nabla_4 f(i,j,k) = 3E_1^{7/9} f(i,j,k).$$

Hence, the relation (17) follows:

$$\nabla_4^n f(0,0,m) = 3^n E_1^{7n/9} f(0,0,m) = 3^n C_{9(0+7n/9)+3\cdot 0+2m,4}(r) = 3^n C_{7n+2m,4}(r).$$

Finally, applying ∇_5 to $f(i,j,k) = C_{2i+3j+7k,4}(r)$, we get

$$\begin{aligned}\nabla_5 f(i,j,k) &= -4C_{2i+3j+7k,4}(r) + 4C_{2i+2+3j+7k,4}(r) + 2C_{2i+3j+3+7k,4}(r)\\ &\quad -3C_{2i+3j+7k+7,4}(r)\\ &= -C_{2i+3j+7k+9,4}(r) = -E_2^3 f(i,j,k),\end{aligned}$$

wherefrom the relation (21) follows:

$$\nabla_5^n f(0,0,m) = (-1)^n E_2^{3n} f(0,0,m) = (-1)^n C_{9n+7m,4}(r).$$

□

As a special interesting case, we obtain the following result.

Corollary 3.1. *For* $m = 0$ *the relations (18)–(21) become*

$$3^n C_{7n,4}(r) = \sum_{i+j+k=n} \binom{n}{i,j,k} (-1)^{n-i-j-k} 4^{n-j-k} 2^j C_{2i+3j+4k,4}(r);$$

$$3^n C_{7n,4}(r) = \sum_{i+j+k=n} \binom{n}{i,j,k} (-1)^{n-i-j-k} 4^{n-i-k} 2^i C_{9i+2j+3k,4}(r);$$

$$3^n C_{7n,4}(r) = \sum_{i+j+k=n} \binom{n}{i,j,k} (-4)^{n-i-j-k} 2^j 4^k C_{9i+3j+2k,4}(r);$$

$$C_{9n,4}(r) = \sum_{i+j+k=n} \binom{n}{i,j,k} (-1)^{i+j} 4^{n-j-k} 2^j 3^k C_{2i+3j+7k,4}(r).$$

References

[1] G. B. Djordjević, H. M. Srivastava, *Incomplete generalized Jacobsthal and Jacobsthal–Lucas numbers*, Mathl. Comput. Modelling, 42 (2005), 1049–1056.

[2] G. B. Djordjević, H. M. Srivistava, *Some generalizations of certain sequences associated with the Fibonacci numbers*, J. Indones. Math. Soc. (MIHMI), Vol. 12, No. 1(2006), 99–112.

[3] A. F. Horadam, *Jacobsthal representation polynomials*, Fibonacci Quart. 35 (1997), 137–148.

[4] A. F. Horadam, *Jacobsthal representation numbers*, Fibonacci Quart. 34 (1996), 40–54.

[5] Z. Z. Zhang, *Some properties of the generalized Fibonacci sequences* $C_n = C_{n-1} + C_{n-2} + r$, Fibonacci Quart. 35 (1997), 169–171.

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