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CONNECTEDNESS AND SEPARATION IN THE CATEGORY OF CLOSURE SPACES

Mehmet Baran, Deniz Tokat, and Muammer Kula

Abstract

In previous papers, various notions of (strongly) closed subobject, (strongly) open subobject, connected and T_i, $i = 0, 1, 2$ objects in a topological category were introduced and compared. The main objective of this paper is to characterize each of these classes of objects in the category of closure spaces as well as to examine how these generalizations are related.

1 Introduction

Despite the fact that closure operators had been used in calculus first ([33] and [36]), they have been used in other fields of mathematics such as logic ([26] and [37]), algebra ([12] , [13] and [34]) and topology ([28] and [15]).

In 1940, G. Birkhoff observed that the collection of closed sets of a closure space forms a complete lattice [13]. Since his work, the interrelation between closures and complete lattices has been investigated by many authors and a general treatment of this subject can be found in [23]. Another motivation for considering closures is G. Birkhoff's work on association of closures to binary relations in his book [13]. By using similar ideas, G. Aumann worked on contact relations with application to social sciences [4] or B. Ganter and R. Wille worked on formal contexts with application to data analysis and knowledge representation [24].

In recent years, closure operators are used in quantum logic and representation theory of physical systems [2], [3].

A closure space $(X, \mathcal{C})$ is a pair, where X is a set and $\mathcal{C}$ is a subset of the power set $P(X)$ satisfying the conditions that X and $\emptyset$ belong to $\mathcal{C}$ and that $\mathcal{C}$ is closed for arbitrary unions. A function $f : (X, \mathcal{C}) \to (Y, \mathcal{D})$ between closure spaces $(X, \mathcal{C})$ and $(Y, \mathcal{D})$ is said to be continuous if $f^{-1}(D) \in \mathcal{C}$ whenever $D \in \mathcal{D}$. $\boldsymbol{Cls}$ is the construct with closure spaces as objects and continuous maps as morphisms [19].

Another isomorphic description is obtained by means of a closure operator [13]. The closure operation $cl : P(X) \to P(X)$ associated with a closure space $(X, \mathcal{C})$ is

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defined in the usual way by $x \in clA \iff (\forall C \in \mathcal{C} : x \in C \Rightarrow C \cap A \neq \emptyset)$ where $A \subset X$ and $x \in X$. This closure need not be finitely additive, but it does satisfy the conditions $cl\emptyset = \emptyset$, $(A \subset B \Rightarrow clA \subset clB)$, $A \subset clA$, and $cl(clA) = clA$ whenever A and B are subsets of X. Continuity is then characterized in the usual way [19].

Finally, closure spaces can also be equivalently described by means of neighborhood collections of the points. These neighborhood collections satisfy the usual axioms, except for the fact that the collections need not be filters. So in a closure space $(X, \mathcal{C})$ the neighborhood collection of a point x is a non empty stack (in the sense that with every $V \in \mathcal{N}(x)$ also every W with $V \subset W$ belongs to $\mathcal{N}(x)$), where every $V \in \mathcal{N}(x)$ contains x and $\mathcal{N}(x)$ satisfies the open kernel condition [19].

The notions of "closedness" and "strong closedness" in set based topological categories are introduced by Baran [5], [6] and it is shown in [8] that these notions form an appropriate closure operator in the sense of Dikranjan and Giuli [20] in some well-known topological categories. Moreover, various generalizations of each of T_i, $i = 0, 1, 2, 3, 4$ separation properties for an arbitrary topological category over ***Set***, the category of sets are given and the relationship among various forms of each of these notions are investigated by Baran in [5], [7], [9] and [10].

Recently, complete objects in the category of T_0 closure spaces is characterized by D. Deses et al. [19] and a cartesian closed topological hull and quasitopos hull of the construct ***Cls*** of closure spaces is constructed by V. Claes et al.(see [16] and [17]).

The main goal of this paper is

1. to characterize the (strongly) closed and (strongly) open subspace of a closure space,
2. to give the characterization of each of the various notions of connected, and T_i, $i = 0, 1, 2$ closure spaces,
3. to examine how these generalizations are related.

2 Preliminaries

A closure space $(X, \mathcal{C})$ is a pair, where X is a set and $\mathcal{C}$ is a subset of the power set $P(X)$ satisfying the conditions that X and $\emptyset$ belong to $\mathcal{C}$ and that $\mathcal{C}$ is closed for arbitrary unions. A function $f : (X, \mathcal{C}) \to (Y, \mathcal{D})$ between closure spaces $(X, \mathcal{C})$ and $(Y, \mathcal{D})$ is said to be continuous if $f^{-1}(D) \in \mathcal{C}$ whenever $D \in \mathcal{D}$. ***Cls*** is the category with closure spaces as objects and continuous maps as morphisms [19].

Cls is a topological category [21] and ***Top***, the category of topological spaces, is embedded in ***Cls*** as a full bicoreflective subconstruct [16].

Note that a source $\{f_i : (X, \mathcal{C}) \to (Y_i, \mathcal{C}_i), i \in I\}$ is initial in ***Cls*** iff $\mathcal{C} = \{U \subset X : U = \bigcup_{i \in I} f_i^{-1}(U_i), U_i \in \mathcal{C}_i\}$ [25].

Similarly, an epi sink $f_i : (Y_i, \mathcal{C}_i) \to (X, \mathcal{C})$ is final in ***Cls*** iff $\mathcal{C} = \{U \subset X : f_i^{-1}(U) \in \mathcal{C}_i$, for all $i \in I\}$.

In particular:

1. The embeddings $f : X \to Y$ are the injective maps such that a subset of X is open iff it is inverse image by f of an open set of Y.

2. Let $\{(X_i, \mathcal{C}_i)\}$ be a collection of closure spaces and X be the product of the sets X_i, i.e., $X = \Pi_i X_i$. The product structure on X is the class $\mathcal{C} = \{U \subset X : U = \bigcup_{i \in I} \pi_i^{-1}(U_i), U_i \in \mathcal{C}_i\}$, where $\pi_i : X \to X_i$ are the projection maps.

3. $(X, \mathcal{C})$ is a discrete space iff $\mathcal{C} = P(X)$ and it is an indiscrete space iff $\mathcal{C} = \{X, \emptyset\}$.

3 Connected Closure Spaces

In this section, the (strongly) closed and (strongly) open subobjects of an object are characterized in the category of closure spaces. Furthermore, the characterization of each of the various notions of the connected objects in this category are given.

Let B be a set and $p \in B$. Let $B \vee_p B$ be the wedge at p ([5] p.334), i.e., two disjoint copies of B identified at p, i.e., the pushout of $p : 1 \to B$ along itself (where 1 is the terminal object in ***Set***. More precisely, if i_1 and $i_2 : B \to B \vee_p B$ denote the inclusion of B as the first and second factor, respectively, then $i_1 p = i_2 p$ is the pushout diagram. A point x in $B \vee_p B$ will be denoted by $x_1 (x_2)$ if x is in the first (resp. second) component of $B \vee_p B$. Note that $p_1 = p_2$.

The principle $p-$axis map, $A_p : B \vee_p B \to B^2$ is defined by $A_p(x_1) = (x, p)$ and $A_p(x_2) = (p, x)$. The skewed $p-$axis map, $S_p : B \vee_p B \to B^2$ is defined by $S_p(x_1) = (x, x)$ and $S_p(x_2) = (p, x)$.

The fold map at p, $\bigtriangledown_p : B \vee_p B \to B$ is given by $\bigtriangledown_p(x_i) = x$ for $i = 1, 2$ [5], [6].

Note that the maps S_p and $\bigtriangledown_p$ are the unique maps arising from the above pushout diagram for which $S_p i_1 = (id, id) : B \to B^2$, $S_p i_2 = (p, id) : B \to B^2$, and $\bigtriangledown_p i_j = id, j = 1, 2$, respectively, where, $id : B \to B$ is the identity map and $p : B \to B$ is the constant map at p.

The infinite wedge product $\vee_p^\infty B$ is formed by taking countably many disjoint copies of B and identifying them at the point p. Let $B^\infty = B \times B \times ...$ be the countable cartesian product of B. Define $A_p^\infty : \vee_p^\infty B \to B^\infty$ by $A_p^\infty(x_i) = (p, p, ..., p, x, p, ...)$, where x_i is in the i-th component of the infinite wedge and x is in the i-th place in $(p, p, ..., p, x, p, ...)$ and $\bigtriangledown_p^\infty : \vee_p^\infty B \longrightarrow B$ by $\bigtriangledown_p^\infty(x_i) = x$ for all $i \in I$ [5], [6].

Note, also, that the map A_p^∞ is the unique map arising from the multiple pushout of $p : 1 \to B$ for which $A_p^\infty i_j = (p, p, ..., p, id, p, ...) : B \to B^\infty$, where the identity map, id, is in the j-th place [8].

Definition 1. *(cf. [5] or [6]) Let $\mathcal{U} : \mathcal{E} \longrightarrow$ **Set**(=the category of sets) be a topological functor, X an object in $\mathcal{E}$ with $\mathcal{U}(X) = B$. Let F be a nonempty subset of B. We denote by X/F the final lift of the epi $\mathcal{U}$–sink $q : \mathcal{U}(X) = B \to B/F = (B \backslash F) \cup \{*\}$, where q is the epi map that is the identity on $B \backslash F$ and identifying F with a point $*$ [5]. Let p be a point in B.*

1. *X is T_1 at p iff the initial lift of the $U-$source $\{S_p : B\vee_p B \longrightarrow U(X^2) = B^2$ and $\bigtriangledown_p : B\vee_p B \longrightarrow UD(B) = B\}$ is discrete, where D is the discrete functor which is a left adjoint to U.*
2. *p is closed iff the initial lift of the $U-$source $\{A_p^\infty : \vee_p^\infty B \longrightarrow U(X^\infty) = B^\infty$ and $\bigtriangledown : \vee_p^\infty B \longrightarrow UD(B) = B\}$ is discrete.*
3. *$F \subset X$ is closed iff $\{*\}$, the image of F, is closed in X/F or $F = \emptyset$.*
4. *$F \subset X$ is strongly closed iff X/F is T_1 at $\{*\}$ or $F = \emptyset$.*
5. *If $B = F = \emptyset$, then we define F to be both closed and strongly closed.*

Remark 1. *1. In **Top**, the notion of closedness coincide with the usual one [5] and M is strongly closed iff M is closed and for each $x \notin M$ there exists a neighbourhood of M missing x [5]. If a topological space is T_1, then the notions of closedness and strong closedness coincide [5].*
2. In general, for an arbitrary topological category, the notions of closedness and strong closedness are independent of each other (see [6] and [11]). Even if $X \in \mathcal{E}$ is T_1, where $\mathcal{E}$ is a topological category, then these notions are still independent of each other (see [6] and [11]).

Theorem 1. *Let $(X,\mathcal{C})$ be a closure space and $p \in X$. $(X,\mathcal{C})$ is T_1 at p iff $X = \{p\}$.*

Proof. Suppose $(X,\mathcal{C})$ is T_1 at p and $X \neq \{p\}$. Then there exists $x \in X$ with $x \neq p$. Since $(X,\mathcal{C})$ is T_1 at p, $\{x_1\} = \nabla_p^{-1}(\{x\}) \cup S_p^{-1}(W)$, where $W \in \mathcal{C}^2$ and $\mathcal{C}^2$ is the product structure on $X \times X$. Note that $\{x_1\} = \nabla_p^{-1}(\{x\}) \cup S_p^{-1}(W) \supset \nabla_p^{-1}(\{x\}) = \{x_1, x_2\}$ which is a contradiction. Hence, $X = \{p\}$.
Conversely, let $X = \{p\}$. Note that the only closure structure on X is given by $C = \{\emptyset, \{p\}\} = P(X)$. It follows from Definition 1 that $(X,\mathcal{C})$ is T_1 at p. □

Theorem 2. *Let $(X,\mathcal{C})$ be a closure space and $p \in X$. $\{p\} \subset X$ is closed iff $X = \{p\}$.*

Proof. Suppose $\{p\} \subset X$ is closed but X is not a singleton, i.e., there exists $x \in X$ with $x \neq p$. Note that $x_i \in \vee_p^\infty X$, $A_p^\infty(x_i) = (p,p,...,p,x,p,...) \in X^\infty$ and $\bigtriangledown_p^\infty(x_i) = x$ for all $i \in I$. Since $\{p\}$ is closed in X, by Definition 1 (2), the initial lift of A_p^∞ and $\bigtriangledown_p^\infty$*is* discrete and consequently, $\{x_i\} \in P(\vee_p^\infty X)$, the discrete structure on $\vee_p^\infty X$. We have $\{x_i\} = (\nabla_p^\infty)^{-1}(\{x\}) \cup (A_p^\infty)^{-1}(W)$, where $\{x\} \in P(X)$ and $W \in \mathcal{C}^\infty$, the product structure on X^∞. Since $(\nabla_p^\infty)^{-1}(\{x\}) = \{x_1, x_2, ...\}$, it follows that $\{x_i\} = (\nabla_p^\infty)^{-1}(\{x\}) \cup (A_p^\infty)^{-1}(W) \supset \{x_1, x_2, ...\}$ which is a contradiction. Hence, we must have $X = \{p\}$.
Conversely, if $X = \{p\}$, then $X^\infty = \{(p,p,...)\} = \vee_p^\infty X$. Consequently, by Definition 1, $X = \{p\}$ is closed. □

Theorem 3. *Let $(X,\mathcal{C})$ be a closure space. $\emptyset \neq F \subset X$ is closed iff $F = X$.*

Proof. Let $(X,\mathcal{C})$ be a closure space and F be a nonempty subset of X. By Definition 1, F is closed iff $\{*\}$ is closed in X/F iff, by Theorem 2, we have $X/F = \{*\}$ iff $F = X$. □

Theorem 4. *Let $(X, \mathcal{C})$ be a closure space. $\emptyset \neq F \subset X$ is strongly closed iff $F = X$.*

Proof. Let $(X, \mathcal{C})$ be a closure space and F be a nonempty subset of X. By Definition 1, F is strongly closed iff X/F is T_1 at $\{*\}$ iff, by Theorem 1, $X/F = \{*\}$ iff $F = X$. □

Note that in ***Top*** the notion of openness coincide with the usual one [11]. If a topological space is T_1, then the notions of openness and strong openness coincide [11].

We now give the characterization the various notions of connectedness in the category of closure spaces.

Definition 2. *Let $\mathcal{E}$ be a topological category over **Set** and X be an object in $\mathcal{E}$. X is D-connected iff any morphism from X to discrete object is constant (cf. [11], [14], [29], [35]).*

Note that for the category ***Top*** of topological spaces, the notion of D-connectedness coincides with the usual notion of connectedness.

Theorem 5. *Let $(X, \mathcal{C})$ be a closure space. $(X, \mathcal{C})$ is D-connected iff for any $U, V \in \mathcal{C}$ with $U \neq \emptyset \neq V$, $U \cap V = \emptyset$, then $U \cup V \neq X$.*

Proof. Suppose that $(X, \mathcal{C})$ is D-connected and the condition does not hold, i.e., there exists $U, V \in \mathcal{C}$ with $U \neq \emptyset \neq V$, $U \cap V = \emptyset$, then $U \cup V = X$. We define a function $f : (X, \mathcal{C}) \to (Y, P(Y))$ by $f(x) = a$, if $x \in U$ and $f(x) = b$, if $x \in V$. Let $W \in P(Y)$, the discrete structure on Y. If $a \in W$, then $f^{-1}(W) = U \in \mathcal{C}$. If $b \in W$, then $f^{-1}(W) = V \in \mathcal{C}$. If $a, b \in W$, then $f^{-1}(W) = X \in \mathcal{C}$ since $(X, \mathcal{C})$ is a closure space. Hence, f is continuous but it is not constant, a contradiction.
Conversely, assume that the condition holds. If $Y = \{a\}$, one point set, then every continuous function $f : (X, \mathcal{C}) \to (Y, P(Y))$ is constant. Suppose $|Y| \geq 2$ and $(X, \mathcal{C})$ is not D-connected. Then there exists a continuous function $f : (X, \mathcal{C}) \to (Y, P(Y))$ which is not constant. It follows that there exist $x, y \in X$ with $x \neq y$ and $f(x) \neq f(y)$. Let $U = f^{-1}(\{f(x)\})$ and $V = f^{-1}(\{f(x)\}^c)$, where $\{f(x)\}^c$ is the complement of $\{f(x)\}$. Note that $x \in U$, $y \in V$, $U \cap V = \emptyset$, $U \cup V = X$ and $U, V \in \mathcal{C}$ since f is continuous and $\{f(x)\}, \{f(x)\}^c$ are in $P(Y)$. This is a contradiction. Hence, $(X, \mathcal{C})$ is D-connected. □

Definition 3. *Let $\mathcal{E}$ be a complete category and C be a closure operator in the sense of Dikranjan and Giuli [20] of $\mathcal{E}$. An object X of $\mathcal{E}$ is called C-connected if the diagonal morphism $\delta_X =< 1_X, 1_X >: X \to X \times X$ is C-dense. $\nabla(C)$ denotes the full subcategory of C-connected objects [18].*

Note that if $\mathcal{E} = \textbf{\textit{Top}}$ and $C = K$, the usual Kuratowski closure operator, then $\nabla(K)$ is the category of irreducible spaces (i.e., of spaces X for which $X = F \cup G$ with closed sets F and G are possible only for $F = X$ or $G = X$) [18].

Let $(X, \mathcal{C})$ be a closure space and $M \subset X$. The (strong) closure of M is the intersection of all (strongly) closed subsets of X containing M and it is denoted by *cl*(M) (resp., *scl*(M)). Note that for any closure space $(X, \mathcal{C})$ and $M \subset X$, if

$M \neq \emptyset$, then $cl(M) = X = scl(M)$ and if $M = \emptyset$, then $cl(\emptyset) = \emptyset = scl(\emptyset)$ are closure operators of $\boldsymbol{Cls}$ in the sense of [20].

We have;

Corollary 1. *Every closure space (X, C) is cl(scl)-connected.*

Remark 2. *Let $(X, \mathcal{C})$ be a closure space. If $(X, \mathcal{C})$ is D-connected, then $(X, \mathcal{C})$ is cl(scl)-connected, but the converse of this implication is not true, in general. For example, suppose $X = \{a, b\}$ and $\mathcal{C} = P(X)$. Then, by Theorem 5 and Corollary 1, $(X, \mathcal{C})$ is cl(scl)$-$connected but it is not D-connected.*

4 Hausdorff Closure Spaces

Let B be a nonempty set, $B^2 = B \times B$ be cartesian product of B with itself and $B^2 \vee_\Delta B^2$ be two distinct copies of B^2 identified along the diagonal. A point (x, y) in $B^2 \vee_\Delta B^2$ will be denoted by $(x, y)_1((x, y)_2)$ if (x, y) is in the first (resp. second) component of $B^2 \vee_\Delta B^2$. Clearly $(x, y)_1 = (x, y)_2$ iff $x = y$ [5].

The principal axis map $A : B^2 \vee_\Delta B^2 \to B^3$ is given by $A(x, y)_1 = (x, y, x)$ and $A(x, y)_2 = (x, x, y)$. The skewed axis map $S : B^2 \vee_\Delta B^2 \to B^3$ is given by $S(x, y)_1 = (x, y, y)$ and $S(x, y)_2 = (x, x, y)$ and the fold map, $\nabla : B^2 \vee_\Delta B^2 \to B^2$ is given by $\nabla(x, y)_i = (x, y)$ for $i = 1, 2$ [5].

Definition 4. *(cf. [5], [9] and [38]) Let $\mathcal{U} : \mathcal{E} \to \boldsymbol{Set}$ be a topological functor, X an object in $\mathcal{E}$ with $\mathcal{U}(X) = B$.*

1. X is $\overline{T_0}$ iff the initial lift of the $\mathcal{U}$-source $\{A : B^2 \vee_\Delta B^2 \to \mathcal{U}(X^3) = B^3$ and $\nabla : B^2 \vee_\Delta B^2 \to \mathcal{U}\mathcal{D}(B^2) = B^2\}$ is discrete, where $\mathcal{D}$ is the discrete functor which is a left adjoint to $\mathcal{U}$.
2. X is T_0 iff X does not contain an indiscrete subspace with (at least) two points.
3. X is $Pre\overline{T_2}$ iff the initial lifts of the $\mathcal{U}$ -sources $\{A : B^2 \vee_\Delta B^2 \to \mathcal{U}(X^3) = B^3\}$ and $\{S : B^2 \vee_\Delta B^2 \to \mathcal{U}(X^3) = B^3\}$ coincide.
4. X is $PreT_2'$ iff the initial lift of the $\mathcal{U}$-source $\{S : B^2 \vee_\Delta B^2 \to \mathcal{U}(X^3) = B^3\}$ and the final lift of the $\mathcal{U}-$sink $\{i_1, i_2 : \mathcal{U}(X^2) = B^2 \to B^2 \vee_\Delta B^2\}$ coincide, where i_1 and i_2 are the canonical injections.
5. X is $\overline{T_2}$ iff X is $\overline{T_0}$ and $Pre\overline{T_2}$.
6. X is NT_2 iff X is T_0 and $Pre\overline{T_2}$.
7. X is MT_2 iff X is T_0 and $PreT_2'$.
8. X is ST_2 iff Δ, the diagonal, is strongly closed in X^2.
9. X is ΔT_2 iff Δ, the diagonal, is closed in X^2.

Remark 3. *1. Note that for the category* ***Top*** *of topological spaces,* $\overline{T_0}$, T_0, $Pre\overline{T_2}$, $PreT_2'$, *and* $\overline{T_2}$ *reduce to the usual* T_0, T_1, $PreT_2$ *(where a topological space is called* $PreT_2$ *if for any two distinct points, if there is a neighbourhood of one missing the other, then the two points have disjoint neighbourhoods), and* T_2 *separation axioms, respectively [5].*
2. Let $\mathcal{U} : \mathcal{E} \to \mathcal{B}$ *be a topological functor. In [8], it is shown that every indiscrete object of* $\mathcal{E}$ *is* $Pre\overline{T_2}$. *Furthermore, if* X *is* $PreT_2'$, *then it is* $Pre\overline{T_2}$ *object in* $\mathcal{E}$ *[10]. In an arbitrary topological category,* $Pre\overline{T_2}$ *and* $PreT_2'$ *objects are used to define various forms of each of* T_2 *objects and regular objects, and consequently normal objects, see [5] and [7].*
3. Mielke in [32] showed that pre-Hausdorff objects are used to characterize the decidable objects, [27] or [30] in a topos, where an object X *of* $\mathcal{E}$, *a topos, is said to be decidable if the diagonal* $\Delta \subset X^2$ *is a complemented subobject. Moreover, it is proved in [31] that the image of a topos in a topological category by a geometric morphism [27] or [30], i.e., an exact functor which has a right adjoint, is a* $PreT_2'$ *object. In particular,* $PreT_2'$ *objects play a role in the general theory of geometric realizations, their associated interval and corresponding homotopy structures [31].*
4. In [6] for an arbitrary topological category, it is proven that there is no relationship between $\overline{T_0}$ *and* T_0, *in general. Moreover, it is shown, in [10], that the notions of* $\overline{T_2}$, NT_2, MT_2, $\triangle T_2$, *and* ST_2 *are independent of each other, in general.*

Theorem 6. *Let* $(X, \mathcal{C})$ *be a closure space.* $(X, \mathcal{C})$ *is* $\overline{T_0}$ *iff* X *is a singleton.*

Proof. Suppose (X, C) is $\overline{T_0}$ and $X \neq \{x\}$. Then there exists $y \in X$ such that $x \neq y$. Since $A(x, y)_1 = (x, y, x) \in W$ and $W \in C^3$, where C^3 is the product structure on X^3, there exist $M, N \in C$ such that $W = \pi 1^{-1}(N) \cup \pi 2^{-1}(M) \cup \pi 3^{-1}(N) = (N \times X^2) \cup (X \times M \times X) \cup (X^2 \times N)$. Since $x \neq y$ we have $\nabla - 1((x, y)) = \{(x, y)_1, (x, y)_2\}$. However, $\{(x, y)_1\} = \nabla - 1(\{x, y\}) \cup A^{-1}(W) \supset \{(x, y)_1, (x, y)_2\}$. This is a contradiction. Hence, $X = \{x\}$. Conversely, if $X = \{x\}$, i.e., a singleton, then clearly, by Definition 4, (X, C) is $\overline{T_0}$. □

Theorem 7. *Let* $(X, \mathcal{C})$ *be a closure space.* $(X, \mathcal{C})$ *is* T_0 *iff for all different pair of points* $x, y \in X$, *there exists a set* $U_x \in \mathcal{C}$ *with* $x \in U_x$ *such that* $y \notin U_x$ *or there exists a set* $U_y \in \mathcal{C}$ *with* $y \in U_y$ *such that* $x \notin U_y$.

Proof. Suppose $(X, \mathcal{C})$ is T_0, i.e., X does not contain an indiscrete subspace with two points and $x, y \in X$ with $x \neq y$. Let $A = \{x, y\}$ and $\mathcal{C}_A = \{U \cap A : U \in \mathcal{C}\}$, the subspace structure on A. Since $\mathcal{C}_A$ is not indiscrete structure, then there exists a set $U_x \in \mathcal{C}_A$ with $x \in U_x$ such that $y \notin U_x$ or there exists a set $U_y \in \mathcal{C}_A$ with $y \in U_y$ such that $x \notin U_y$. Suppose $x \in U_x \in \mathcal{C}_A$. Then, there exists a set $U \in \mathcal{C}$ such that $x \in U$ and $U_x = U \cap A$. If $y \in U$, then $y \in U_x$. This is a contradiction. Hence, $y \notin U$. Similarly, if $y \in U_y \in \mathcal{C}_A$, then there exists a set $V \in \mathcal{C}$ such that $x \in V$ with $U_y = V \cap A$. If $x \in V$, then $x \in U_y$. This is a contradiction. Therefore, $x \notin V$.
Conversely, suppose the above condition holds. We want to show that X contains an indiscrete subspace with two points, i.e., $A = \{x, y\}$ and $\mathcal{C}_A = \{\emptyset, A, U_x \cap A\}$

with $x \neq y$. Since $A \in \mathcal{C}_A$, there exists $V_x \in \mathcal{C}$, $x \in V_x$ such that $V_x \cap A = A$. Thus, $A \subset V_x$, i.e., $y \in V_x$. This is a contradiction. Similarly, we can show for $x \in U_y$. Therefore, X does not contain an indiscrete subspace with two points. Hence, $(X, \mathcal{C})$ is T_0. □

Theorem 8. *Let $(X, \mathcal{C})$ be a closure space. $(X, \mathcal{C})$ is $Pre\overline{T_2}$ iff it is an indiscrete closure space.*

Proof. If $(X, \mathcal{C})$ is indiscrete closure space, then, by Remark 3.2, it is $Pre\overline{T_2}$.
Suppose that $(X, \mathcal{C})$ is $Pre\overline{T_2}$. If $X = \emptyset$ or $X = \{x\}$, then it is obvious that $(X, \mathcal{C})$ is indiscrete closure space and by Remark 3.2, $(X, \mathcal{C})$ is $Pre\overline{T_2}$. Suppose that X contains at least two points, i.e., there exist $x, y \in X$ with $x \neq y$. Assume that $A = \{x, y\} \subset X$. The induced structure on A is given by $\mathcal{C}_A = \{A \cap U : U \in \mathcal{C}\}$. We want to show that $(A, \mathcal{C}_A)$ is $Pre\overline{T_2}$ only if $\mathcal{C}_A$ is indiscrete structure on A. We have four cases.
Case 1. If $\mathcal{C}_A = \{\emptyset, A\}$, then, by Remark 3.2, $(A, \mathcal{C}_A)$ is $Pre\overline{T_2}$.
Case 2. If $\mathcal{C}_A = \{\emptyset, A, \{x\}\}$, then $\mathcal{C}^3_A = \{\emptyset, A^3, \{x\} \times A^2, A \times \{x\} \times A, A^2 \times \{x\}, (\{x\} \times A^2) \cup (A \times \{x\} \times A), (\{x\} \times A^2) \cup (A^2 \times \{x\}), (A \times \{x\} \times A) \cup (A^2 \times \{x\}), (\{x\} \times A^2) \cup (A \times \{x\} \times A) \cup (\{x\} \times A^2)\}$, where $\mathcal{C}^3_A$ is the product structure on A^3. We claim that $(A, \mathcal{C}_A)$ is not $Pre\overline{T_2}$. Let $W = A^2 \times \{x\} \in \mathcal{C}^3_A$ and $U = S^{-1}(W) = S^{-1}(A^2 \times \{x\}) = \{(x,x)_1 = (x,x)_2, (y,x)_1, (y,x)_2\}$. In fact, if $(x,y)_1 \in U = S^{-1}(W)$, then $S(x,y)_1 = (x,y,y) \notin W = A^2 \times \{x\}$, since $x \neq y$. Similarly, $(y,x)_1 \in U = S^{-1}(W)$, then $S(y,x)_1 = (y,x,x) \in W = A^2 \times \{x\}$. Hence $(y,x)_1 \in S^{-1}(W)$. If $(x,y)_2 \in U = S^{-1}(W)$, then $S(x,y)_2 = (x,x,y) \notin W = A^2 \times \{x\}$, since $x \neq y$. Similarly, if $(y,x)_2 \in U = S^{-1}(W)$, then $S(y,x)_2 = (y,y,x) \in W = A^2 \times \{x\}$. Therefore, $(y,x)_2 \in U = S^{-1}(W)$. Hence, $S^{-1}(W) = \{(x,x)_1 = (x,x)_2, (y,x)_1, (y,x)_2\}$. However, there is no $W' \in \mathcal{C}^3_A$ such that $A^{-1}(W') = S^{-1}(W)$. In fact, if $W = A^2 \times \{x\}$, then $A^{-1}(W) = \{(x,x)_1 = (x,x)_2, (x,y)_1, (y,x)_2\}$. In fact, if $(x,y)_1 \in A^{-1}(W)$, then $A(x,y)_1 = (x,y,x) \in W = A^2 \times \{x\}$. Thus, $(x,y)_1 \in A^{-1}(W)$. If $(y,x)_1 \in A^{-1}(W)$, then $A(y,x)_1 = (y,x,y) \notin W = A^2 \times \{x\}$, since $x \neq y$. Similarly, if $(x,y)_2 \in A^{-1}(W)$, then $A(x,y)_2 = (x,x,y) \notin W = A^2 \times \{x\}$, since $x \neq y$. If $(y,x)_2 \in A^{-1}(W)$, then $A(y,x)_2 = (y,y,x) \in W = A^2 \times \{x\}$. Hence, $(y,x)_2 \in A^{-1}(W)$. Similarly, we have $A^{-1}(\mathcal{C}^3_A) = \{\emptyset, A^2 \vee_\Delta A^2, \{(x,x)_1 = (x,x)_2, (x,y)_2, (x,y)_1\}, \{(x,x)_1 = (x,x)_2, (x,y)_2, (y,x)_1\}, \{(x,x)_1 = (x,x)_2, (x,y)_1, (y,x)_2\}, \{(x,x)_1 = (x,x)_2, (x,y)_2, (x,y)_1, (y,x)_1\}, \{(x,x)_1 = (x,x)_2, (x,y)_2, (x,y)_1, (y,x)_2\}, \{(x,x)_1 = (x,x)_2, (x,y)_2, (y,x)_1, (x,y)_1, (y,x)_2\}\}$. Therefore, $S^{-1}(W) \in S^{-1}(\mathcal{C}^3_A)$, but $S^{-1}(W) \notin A^{-1}(\mathcal{C}^3_A)$. Thus $A^{-1}(\mathcal{C}^3_A) \neq S^{-1}(\mathcal{C}^3_A)$.
Case 3. If $\mathcal{C}_A = \{\emptyset, A, \{y\}\}$, then the proof is similar to Case 2.
Case 4. If $\mathcal{C}_A = \{\emptyset, A, \{x\}, \{y\}\}$ then $\mathcal{C}^3_A = \{\emptyset, A^3, \{x\} \times A^2, A \times \{x\} \times A, A^2 \times \{x\}, \{y\} \times A^2, A \times \{y\} \times A, A^2 \times \{y\}, (\{x\} \times A^2) \cup (A \times \{x\} \times A), (\{x\} \times A^2) \cup (A^2 \times \{x\}), (A \times \{x\} \times A) \cup (A^2 \times \{x\}), (\{x\} \times A^2) \cup (A \times \{x\} \times A) \cup (\{x\} \times A^2), (\{y\} \times A^2) \cup (A \times \{y\} \times A), (\{y\} \times A^2) \cup (A^2 \times \{y\}), (A \times \{y\} \times A) \cup (A^2 \times \{y\}), (\{y\} \times A^2) \cup (A \times \{y\} \times A) \cup (\{y\} \times A^2)\}$. Let $W = A^2 \times \{x\}$, by the above argument $U = S^{-1}(W) = \{(x,x)_1 = (x,x)_2, (y,x)_1, (y,x)_2\}$ and $U = S^{-1}(W) \notin A^{-1}(\mathcal{C}^3_A)$. Therefore, $S^{-1}(\mathcal{C}^3_A) \neq A^{-1}(\mathcal{C}^3_A)$.

In general, suppose X contains at least two points. If $(X,\mathcal{C})$ is indiscrete closure space, then, by Remark 3.2, $(X,\mathcal{C})$ is $Pre\overline{T_2}$. Suppose $(X,\mathcal{C})$ is not an indiscrete closure space. Then, there exists $U\subset X$ such that $U\in\mathcal{C}$ with $\emptyset\neq U\neq X$.
1. Let $x\in U$, $y\notin U$ and $W'=X^2\times U\in\mathcal{C}^3$. Then, $A^{-1}(W')\notin A^{-1}(\mathcal{C}^3)$. In fact, as it is shown in Case 2, $W=A^2\times\{x\}\in\mathcal{C}^3_A$ and $A^{-1}(W)\notin A^{-1}(\mathcal{C}^3_A)$. Since $W=A^2\times\{x\}\in\mathcal{C}^3_A$, there exists $X^2\times U\in\mathcal{C}^3$ such that $W=A^2\times\{x\}=A^3\cap(X^2\times U)$. Since $A^{-1}(W)\notin A^{-1}(\mathcal{C}^3_A)$, we have $A^{-1}(X^2\times U)=A^{-1}(W')\notin A^{-1}(\mathcal{C}^3)$. On the other hand, if $A^{-1}(W')\in A^{-1}(\mathcal{C}^3)$, then $A^{-1}(W)\in A^{-1}(\mathcal{C}^3_A)$. This is a contradiction. Similarly, since $S^{-1}(W)\in S^{-1}(\mathcal{C}^3_A)$, then $S^{-1}(W')\in S^{-1}(\mathcal{C}^3)$. If it was $S^{-1}(W')\notin S^{-1}(\mathcal{C}^3)$, then, by the definition of $\mathcal{C}^3_A$, $S^{-1}(W)\notin S^{-1}(\mathcal{C}^3_A)$. This is a contradiction, as it shown in Case 2.
2. Let $y\in U$, $x\notin U$ and $W'=X^2\times U\in\mathcal{C}^3$. Then, as it is shown above $A^{-1}(W')\notin A^{-1}(\mathcal{C}^3)$ and $S^{-1}(W')\in S^{-1}(\mathcal{C}^3)$ by using Case 3.
3. Let $x,y\in U$ and $W'=X^2\times U\in\mathcal{C}^3$. Then, as it is shown above $A^{-1}(W')\notin A^{-1}(\mathcal{C}^3)$ and $S^{-1}(W')\in S^{-1}(\mathcal{C}^3)$ by using Case 4.
Consequently, a closure space $(X,\mathcal{C})$ is not $Pre\overline{T_2}$ unless it is an indiscrete space. □

Theorem 9. *Let $(X,\mathcal{C})$ be a closure space. $(X,\mathcal{C})$ is $PreT_2'$ iff it is an indiscrete closure space.*

Proof. Suppose that $(X,\mathcal{C})$ is $PreT_2'$. If $X=\emptyset$ or $X=\{x\}$ then it is obvious that $(X,\mathcal{C})$ is indiscrete closure space. Assume that X contains at least two points. Then, there exist $x,y\in X$ with $x\neq y$. Let $A=\{x,y\}\subset X$. Then $\mathcal{C}_A=\{A\cap U: U\in\mathcal{C}\}$ is the subspace structure on A. Let $\mathcal{C}^2_A$ be the product structure on A^2 and $\mathcal{C}^*_A$ be the final structure on $A^2\vee_\Delta A^2$ induced by canonical injections i_1 and $i_2: A^2\to A^2\vee_\Delta A^2$, i.e., $\mathcal{C}^*_A=\{U\subset A^2\vee_\Delta A^2: i_1^{-1}(U)\in\mathcal{C}^2_A \text{ and } i_2^{-1}(U)\in\mathcal{C}^2_A\}$. We show that by Definition 4 $(A,\mathcal{C}_A)$ is $PreT_2'$, i.e., $S^{-1}(\mathcal{C}^3_A)=\mathcal{C}^*_A$, where $\mathcal{C}^3_A$ is the product structure on A^3, iff $(A,\mathcal{C}_A)$ is indiscrete. We have four cases.
Case 1. If $\mathcal{C}_A=\{\emptyset,A\}$, the indiscrete structure, then it is easy to see that $S^{-1}(\mathcal{C}^3_A)=\{A^2\vee_\Delta A^2,\emptyset\}=\mathcal{C}^*_A$. So $(A,\mathcal{C}_A)$ is $PreT_2'$.
Case 2. If $\mathcal{C}_A=\{\emptyset,A,\{x\}\}$, then $\mathcal{C}^2_A=\{\emptyset,A^2,\{x\}\times A,A\times\{x\},(\{x\}\times A)\cup(A\times\{x\})\}$, where $\mathcal{C}^2_A$ is the product structure on A^2. We want to show that $(A,\mathcal{C}_A)$ is not $PreT_2'$. Let $W=A^2\times\{x\}\in\mathcal{C}^3_A$ and $S^{-1}(W)=S^{-1}(A^2\times\{x\})=\{(x,x)_1=(x,x)_2,(y,x)_1,(y,x)_2\}\in S^{-1}(\mathcal{C}^3_A)$, as it is shown in the proof of above theorem. However, there is no element in $\mathcal{C}^*_A$ equivalent to $S^{-1}(W)$. Let $U=(\{x\}\times A)\vee_\Delta A^2$. Note that $i_1^{-1}(U)=\{x\}\times A\in\mathcal{C}^2_A$ and $i_2^{-1}(U)=A^2\in\mathcal{C}^2_A$. Hence, $U\in\mathcal{C}^*_A$. Similarly, we get $C^*_A=\{\emptyset, A^2\vee_\Delta A^2, \{x\}\times A\vee_\Delta A^2, A^2\vee_\Delta\{x\}\times A, A\times\{x\}\vee_\Delta A^2, A^2\vee_\Delta A\times\{x\}, [\{x\}\times A\cup A\times\{x\}]\vee_\Delta A^2, A^2\vee_\Delta[\{x\}\times A\cup A\times\{x\}]\}$. Thus, $S^{-1}(\mathcal{C}^3_A)\neq\mathcal{C}^*_A$.
Case 3. If $\mathcal{C}_A=\{\emptyset,A,\{y\}\}$ then, similar to the above case, we can show that $S^{-1}(\mathcal{C}^3_A)\neq\mathcal{C}^*_A$.
Case 4. If $\mathcal{C}_A=\{\emptyset,A,\{x\},\{y\}\}$ then $\mathcal{C}^2_A=\{\emptyset,A^2,\{x\}\times A,A\times\{x\},(\{x\}\times A)\cup(A\times\{x\}),\{y\}\times A,A\times\{y\},(\{y\}\times A)\cup(A\times\{y\}),(\{x\}\times A)\cup(\{y\}\times A),(\{x\}\times A)\cup(A\times\{y\}),(A\times\{x\})\cup(A\times\{y\})\}$. Again, similar to Case 2, when we find $\mathcal{C}^*_A=\{\emptyset, A^2\vee_\Delta A^2, \{x\}\times A\vee_\Delta A^2, A^2\vee_\Delta\{x\}\times A, A\times\{x\}\vee_\Delta A^2, A^2\vee_\Delta A\times\{x\}, [\{x\}\times A\cup A\times\{x\}]\vee_\Delta A^2, A^2\vee_\Delta[\{x\}\times A\cup A\times\{x\}], \{y\}\times A\vee_\Delta A^2, A^2\vee_\Delta\{y\}\times A, A\times\{y\}\vee_\Delta A^2, A^2\vee_\Delta A\times\{y\}, [\{y\}\times A\cup A\times\{y\}]\vee_\Delta A^2, A^2\vee_\Delta[\{y\}\times A\cup A\times\{y\}],$

$[\{x\} \times A \cup A \times \{y\}] \vee_\Delta A^2$, $A^2 \vee_\Delta [\{x\} \times A \cup A \times \{y\}]$, $[A \times \{x\} \cup \{y\} \times A] \vee_\Delta A^2$, $A^2 \vee_\Delta [A \times \{x\} \cup \{y\} \times A]\}$, we can see that there is no element in $\mathcal{C}_A^*$ which is equivalent to $S^{-1}(W) = \{(x,x)_1 = (x,x)_2, (y,x)_1, (y,x)_2\}$, where $W = A^2 \times \{x\}$. Hence, $S^{-1}(\mathcal{C}_A^3) \neq \mathcal{C}_A^*$.
Therefore, closure space $(A, \mathcal{C}_A)$ is $PreT_2'$ if and only if it is an indiscrete space.
In general, suppose X contains at least two points. Let $\mathcal{C}^2$ be the product structure on X^2 and $\mathcal{C}^*$ be the final structure on $X^2 \vee_\Delta X^2$ induced by canonical injections i_1 and $i_2 : X^2 \to X^2 \vee_\Delta X^2$, i.e., $\mathcal{C}^* = \{U \subset X^2 \vee_\Delta X^2 : i_1^{-1}(U) \in \mathcal{C}^2$ and $i_2^{-1}(U) \in \mathcal{C}^2\}$. We show that $(X, \mathcal{C})$ is $PreT_2'$, i.e., $S^{-1}(\mathcal{C}^3) = \mathcal{C}^*$, where $\mathcal{C}^3$ is the product structure on X^3, iff $(X, \mathcal{C})$ is indiscrete. If $(X, \mathcal{C})$ is indiscrete closure space, then $S^{-1}(\mathcal{C}^3) = \{\emptyset, X^2 \vee_\Delta X^2\} = \mathcal{C}^*$. Thus $(X, \mathcal{C})$ is $PreT_2'$.
Conversely, suppose that $(X, \mathcal{C})$ is $PreT_2'$ but it is not an indiscrete closure space. Then, there exists $U \subset X$ such that $U \in \mathcal{C}$ with $\emptyset \neq U \neq X$.
1. Let $x \in U$, $y \notin U$, $A = \{x, y\}$ and $W' = X^2 \times U \in \mathcal{C}^3$. As it is shown in Case 2, $W = A^2 \times \{x\} \in \mathcal{C}_A^3$ and $S^{-1}(W) \notin \mathcal{C}_A^*$. Since, $W = A^2 \times \{x\} \in \mathcal{C}_A^3$, there exists $X^2 \times U \in \mathcal{C}^3$. Note that $W = A^2 \times \{x\} = A^3 \cap (X^2 \times U)$. Since $S^{-1}(W) \notin \mathcal{C}_A^*$, we have $S^{-1}(X^2 \times U) = S^{-1}(W') \notin \mathcal{C}^*$. On the other hand, if $S^{-1}(W') \in \mathcal{C}^*$, then $S^{-1}(W) \in \mathcal{C}_A^*$. This is a contradiction. Similarly, since $S^{-1}(W) \in S^{-1}(\mathcal{C}_A^3)$, then $S^{-1}(W') \in S^{-1}(\mathcal{C}^3)$. If it is $S^{-1}(W') \notin S^{-1}(\mathcal{C}^3)$, then, by the definition of $\mathcal{C}_A^3$, $S^{-1}(W) \notin S^{-1}(\mathcal{C}_A^3)$. This is a contradiction, as it is shown in Case 2.
2. Let $y \in U$, $x \notin U$ and $W' = X^2 \times U \in \mathcal{C}^3$. Then, as it is shown above $S^{-1}(W') \notin \mathcal{C}^*$ and $S^{-1}(W') \in S^{-1}(\mathcal{C}^3)$, by using Case 3.
3. Let $x, y \in U$ and $W' = X^2 \times U \in \mathcal{C}^3$. Then, as it is shown above $S^{-1}(W') \notin \mathcal{C}^*$ and $S^{-1}(W') \in S^{-1}(\mathcal{C}^3)$, by using Case 4.
Therefore, a closure space $(X, \mathcal{C})$ is not $PreT_2'$ unless it is an indiscrete space. □

Theorem 10. *Let $(X, \mathcal{C})$ be a closure space. X is $\overline{T_2}$ (resp. NT_2 or MT_2) iff X is a point or the empty set.*

Proof. It follows from Definition 4 and Theorems 6, 7, 8, and 9. □

Theorem 11. *Let $(X, \mathcal{C})$ be a closure space. X is $\triangle T_2$ (resp. ST_2) iff X is a point or the empty set.*

Proof. Assume $(X, \mathcal{C})$ is $\triangle T_2$ (resp. ST_2), i.e., $\triangle$ is (strongly) closed in X^2 by Definition 4. By Theorem 3 (Theorem 4), $\triangle = X^2$. It follows that X is a point or the empty set.

Converse is obvious. □

Remark 4. *1. A closure space $(X, \mathcal{C})$ is $Pre\overline{T_2}$ iff $(X, \mathcal{C})$ is $PreT_2'$.*
2. If a closure space $(X, \mathcal{C})$ is $\overline{T_0}$, then it is T_0 but the converse is not true generally. For example, let $X = \{a, b\}$ and $\mathcal{C} = \{\emptyset, X, \{a\}\}$. Then $(X, \mathcal{C})$ is T_0 but it is not $\overline{T_0}$.
3. All of $\overline{T_2}$, NT_2, MT_2, $\triangle T_2$, and ST_2 closure spaces are equivalent.

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Mehmet Baran
Department of Mathematics, Faculty of Arts and Sciences, Erciyes University, Kayseri, Turkey
E-mail: baran@erciyes.edu.tr

Deniz Tokat
Department of Mathematics, Faculty of Arts and Sciences, Erciyes University, Kayseri, Turkey
E-mail: detokat@hotmail.com

Muammer Kula
Department of Mathematics, Faculty of Arts and Sciences, Erciyes University, Kayseri, Turkey
E-mail: kulam@erciyes.edu.tr