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ON THE STABILITY AND BOUNDEDNESS OF SOLUTIONS OF NONLINEAR THIRD ORDER DIFFERENTIAL EQUATIONS WITH DELAY

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Abstract

By defining a Lyapunov functional, we investigate the stability and boundedness of solutions to nonlinear third order differential equation with constant delay, r :

$$\begin{aligned}&x'''(t)+g(x(t),x'(t))x''(t)+f(x(t-r),x'(t-r))+h(x(t-r))\\&=p(t,x(t),x'(t),x(t-r),x'(t-r),x''(t)),\end{aligned}$$

when $p(t,x(t),x'(t),x(t-r),x'(t-r),x''(t))=0$ and $\neq 0$, respectively. Our results achieve a stability result which exists in the relevant literature of ordinary nonlinear third order differential equations without delay to the above functional differential equation for the stability and boundedness of solutions. An example is introduced to illustrate the importance of the results obtained.

1 Introduction

By a recent paper, which has been published in 2007, Zhang and Si [6] investigated the asymptotic stability of solutions to the following nonlinear third order scalar differential equation without delay:

$$x'''(t)+g(x'(t))x''(t)+f(x(t),x'(t))+h(x(t))=0.$$

In this paper, instead of the above equation discussed in [6], we consider nonlinear third order differential equation with constant delay, r :

$$\begin{aligned}&x'''(t)+g(x(t),x'(t))x''(t)+f(x(t-r),x'(t-r))+h(x(t-r))\\&=p(t,x(t),x'(t),x(t-r),x'(t-r),x''(t)),\end{aligned}\tag{1}$$

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This equation, (1), can be expressed as the following system:

$$\begin{aligned} x'(t) &= y(t), \\ y'(t) &= z(t), \\ z'(t) &= -g(x(t),y(t))z(t) - f(x(t),y(t)) - h(x(t)) \\ &+ \int_{t-r}^{t} f_x(x(s),y(s))y(s)ds + \int_{t-r}^{t} f_y(x(s),y(s))z(s)ds \\ &+ \int_{t-r}^{t} h'(x(s))y(s)ds + p(t,x(t),y(t),x(t-r),y(t-r),z(t)), \end{aligned} \tag{2}$$

where r is a positive constant which will be determined later; the primes in equation (1) denote differentiation with respect to t, $t \in \Re^+$, $\Re^+ = [0,\infty)$; g, f, h and p are continuous functions in their respective arguments on $\Re^2$, $\Re^2$, $\Re$ and $\Re^+ \times \Re^5$, respectively, with $f(x,0) = h(0) = 0$. The continuity of the functions g, f, h and p guarantees the existence of the solution of equation (1) (see [2, pp.14]). In addition, it is assumed that the derivatives $g_x(x,y) \equiv \frac{\partial}{\partial x}g(x,y)$, $f_x(x,y) \equiv \frac{\partial}{\partial x}f(x,y)$, $f_y(x,y) \equiv \frac{\partial}{\partial y}f(x,y)$ and $h'(x) \equiv \frac{dh}{dx}$ exist and are continuous; the functions g, f, h and p satisfy a Lipschitz condition in x, y, z, $x(t-r)$ and $y(t-r)$. Then the solution is unique (see [2, pp.15]). Throughout the paper $x(t)$, $y(t)$ and $z(t)$ are also abbreviated as x, y and z, respectively.

The motivation for the present paper has come by the paper in [6]. Our purpose is to achieve the result established in [6] to nonlinear functional differential equation (1) for the asymptotic stability of the null solution and boundedness of all solutions of this equation, when $p \equiv 0$ and $p \neq 0$ in (1), respectively. We also give an example for the illustrations of the topic. It is worth mentioning that all papers registered in the references of this paper have been published without including an explanatory example on the subject (see Sinha [3], Zhang and Si [6] and Tunç ([4], [5]).

2 Preliminaries

We will give some basic information for the general non-autonomous delay differential system. Consider the general non-autonomous delay differential system:

$$\dot{x} = F(t,x_t), x_t = x(t+\theta), -r \le \theta \le 0, t \ge 0, \tag{3}$$

where $F : [0,\ \infty) \times C_H \to \Re^n$ is a continuous mapping, $F(t,0) = 0$, and we suppose that F takes closed bounded sets into bounded sets of $\Re^n$. Here $(C,\ \|.\|)$ is the Banach space of continuous function $\phi : [-r,0] \to \Re^n$ with supremum norm, $r > 0$; C_H is the open H -ball in C ; $C_H := \{\phi \in (C[-r,0],\ \ \Re^n) : \|\phi\| < H\}$.

Definition1. (See [1].) Let $F(t,0) = 0$. The null solution of (3) is:

(i) stable if for each $\varepsilon > 0$ and $t_1 \ge t_0 \ge 0$ there exists $\delta > 0$ such that $[\phi \in C(t_1), \|\phi\| < \delta,\ \ t \ge t_1]$ imply that $|x(t,t_1,\phi)| < \varepsilon$.

(ii) asymptotically stable if it is stable and if for each $t_1 \ge t_0 \ge 0$ there is an $\eta > 0$ such that $[\phi \in C(t_1),\ \ \|\phi\| < \eta]$ imply that $x(t,t_1,\phi) \to 0$ as $t \to \infty$.

Definition 2. (See [1].) Let $V(t,\phi)$ be a continuous functional defined for $t \geq 0$, $\phi \in C_H$. The derivative of V along solutions of (3) will be denoted by $\dot{V}$ and is defined by the following relation:

$$\dot{V}(t,\phi) = \limsup_{h\to 0} \frac{V(t+h, x_{t+h}(t_0,\phi)) - V(t, x_t(t_0,\phi))}{h},$$

where $x(t_0,\phi)$ is the solution of (3) with $x_{t_0}(t_0,\phi) = \phi$.

It should be noted that the function $x(t_0,\phi)$ here represents the solution of (3) with the initial condition $\phi \in C_H$ at $t = t_0$, $t_0 \geq 0$.

We also consider the general autonomous delay differential system

$$\dot{x} = G(x_t), \tag{4}$$

which is a special case of (3), and the following lemma is given:

Lemma1. (See[3].) Suppose $G(0) = 0$. Let V be a continuous functional defined on $C_H = C$ with $V(0) = 0$, and let $u(s)$ be a function, non-negative and continuous for $0 \leq s < \infty$, $u(s) \to \infty$ as $s \to \infty$ with $u(0) = 0$. If for all $\phi \in C$, $u(|\phi(0)|) \leq V(\phi)$, $\dot{V}(\phi) \leq 0$, then the solution $x = 0$ of (4) is stable.

If we define $Z = \{\phi \in C_H : \dot{V}(\phi) = 0\}$, then the solution $x = 0$ of (4) is asymptotically stable, provided that the largest invariant set in Z is $Q = \{0\}$.

3 Main results

First, for the case $p(t,x,y,x(t-r),y(t-r),z) \equiv 0$, the following result is introduced

Theorem 1. In addition to the basic assumptions imposed on the functions g , f and h that appearing in (1), we assume that the following conditions hold: There are positive constants a, b, μ, δ, λ_1, λ_2, K and L such that

$g(x,y) \geq a+\mu$, $yg_x(x,y) \leq 0$, $f(x,y)sgny \geq (b+\delta)|y|$,

$-K \leq f_x(x,y) \leq 0$, $|f_y(x,y)| \leq L$, $0 < h'(x) < ab$ and $sgnh(x) = sgnx$.

Then the null solution of equation (1) is asymptotically stable, provided that

$$r < \min\left\{\frac{2a\delta}{a^2b + aK + aL + 2\lambda_1},\ \frac{2\mu}{ab + K + L + 2\lambda_2}\right\}$$

with

$$\lambda_1 = \frac{a^2b}{2} + \frac{ab}{2} + \frac{aK}{2} + \frac{K}{2}$$

and

$$\lambda_2 = \frac{aL}{2} + \frac{L}{2}.$$

Proof. To prove this theorem, we define the following Lyapunov functional $V = V(x_t, y_t, z_t)$:

$$
\begin{aligned}
V &= a\int_0^x h(\xi)d\xi + h(x)y + \tfrac{1}{2}(ay+z)^2 + a\int_0^y [g(x,\eta)-a]\eta d\eta \\
&+ \int_0^y f(x,\eta)d\eta + \lambda_1 \int_{-r}^{0}\int_{t+s}^{t} y^2(\theta)d\theta ds + \lambda_2 \int_{-r}^{0}\int_{t+s}^{t} z^2(\theta)d\theta ds,
\end{aligned} \tag{5}
$$

where λ_1 and λ_2 are the positive constants defined above.

Now, we have $V(0,0,0) = 0$ and the functional V can be rearranged as the following:

$$
\begin{aligned}
V &= a\int_0^x h(\xi)d\xi - \tfrac{1}{2b}h^2(x) + \tfrac{b}{2}\left[y + \tfrac{h(x)}{b}\right]^2 + \tfrac{1}{2}(ay+z)^2 \\
&+ a\int_0^y [g(x,\eta)-a]\eta d\eta + \int_0^y f(x,\eta)d\eta - \tfrac{b}{2}y^2 \\
&+ \lambda_1 \int_{-r}^{0}\int_{t+s}^{t} y^2(\theta)d\theta ds + \lambda_2 \int_{-r}^{0}\int_{t+s}^{t} z^2(\theta)d\theta ds.
\end{aligned} \tag{6}
$$

By using the assumptions $0 < h'(x) < ab$, $sgnh(x) = sgnx$ and $g(x,y) \geq a + \mu$, we have

$$
\begin{aligned}
a\int_0^x h(\xi)d\xi - \tfrac{1}{2b}h^2(x) &= a\int_0^x h(\xi)d\xi - \tfrac{1}{b}\int_0^x h(\xi)h'(\xi)d\xi \\
&= \int_0^x [a - b^{-1}h'(\xi)]h(\xi)d\xi \\
&= b^{-1}\int_0^x [ab - h'(\xi)]h(\xi)d\xi > 0
\end{aligned}
$$

and

$$
a\int_0^y [g(x,\eta)-a]\eta d\eta \geq \frac{a\mu}{2}y^2.
$$

On the other hand, the assumption $f(x,y)sgny \geq (b+\delta)\,|y|$ yields that

$$
f(x,y) \geq (b+\delta)y \quad \text{when} \quad y > 0, \text{and hence} \quad f(x,y)y \geq (b+\delta)y^2
$$

and

$$
f(x,y) \leq (b+\delta)y \quad \text{when} \quad y < 0, \text{and hence} \quad f(x,y)y \geq (b+\delta)y^2.
$$

It also follows that

$$
\int_0^y f(x,\eta)d\eta - \frac{b}{2}y^2 = \int_0^y [f(x,\eta) - b\eta]d\eta \geq 0.
$$

Gathering aforementioned estimates into (6) we obtain

$$
\begin{aligned}
V &\geq b^{-1}\int_0^x [ab - h'(\xi)]h(\xi)d\xi + \tfrac{a\mu}{2}y^2 + \tfrac{1}{2}(ay+z)^2 \\
&+ \lambda_1 \int_{-r}^{0}\int_{t+s}^{t} y^2(\theta)d\theta ds + \lambda_2 \int_{-r}^{0}\int_{t+s}^{t} z^2(\theta)d\theta ds.
\end{aligned}
$$

It is evident, from the terms included in last inequality, that there exist sufficiently small positive constants D_i , $(i = 1,\ 2,\ 3)$, such that

$$\begin{aligned} V &\geq D_1 x^2 + D_2 y^2 + D_3 z^2 + \lambda_1 \int_{-r}^{0} \int_{t+s}^{t} y^2(\theta)d\theta ds + \lambda_2 \int_{-r}^{0} \int_{t+s}^{t} z^2(\theta)d\theta ds \\ &\geq D_4(x^2 + y^2 + z^2) + \lambda_1 \int_{-r}^{0} \int_{t+s}^{t} y^2(\theta)d\theta ds + \lambda_2 \int_{-r}^{0} \int_{t+s}^{t} z^2(\theta)d\theta ds \\ &\geq D_4(x^2 + y^2 + z^2), \end{aligned}$$

and hence

$$x^2 + y^2 + z^2 \leq D_4^{-1} V(x_t, y_t, z_t)$$

and

$$y^2 + z^2 \leq D_4^{-1} V(x_t, y_t, z_t)$$

since the integrals $\int_{-r}^{0} \int_{t+s}^{t} y^2(\theta)d\theta ds$ and $\int_{-r}^{0} \int_{t+s}^{t} z^2(\theta)d\theta ds$ are non-negative, where $D_4 = \min\{D_1,\ D_2,\ D_3\}$. Now, it can be shown that there exists a continuous function $u(s) \geq 0$ with $u(|\phi(0)|) \geq 0$ such that $u(|\phi(0)|) \leq V(\phi)$.

Now, the time derivative of functional V along the system (2) leads that

$$\begin{aligned} \frac{dV}{dt} &= -af(x,y)y + h'(x)y^2 - \{g(x,y) - a\}z^2 + ay \int_{0}^{y} g_x(x,\eta)\eta d\eta \\ &+ y \int_{0}^{y} f_x(x,\eta)d\eta + (ay + z) \int_{t-r}^{t} h'(x(s))y(s)ds \\ &+ (ay + z) \int_{t-r}^{t} f_x(x(s), y(s))y(s)ds + (ay + z) \int_{t-r}^{t} f_y(x(s), y(s))z(s)ds \\ &+ \lambda_1 y^2 r - \lambda_1 \int_{t-r}^{t} y^2(s)ds + \lambda_2 z^2 r - \lambda_2 \int_{t-r}^{t} z^2(s)ds. \end{aligned}$$

By help of the assumptions of the theorem and the inequality $2\,|st| \leq s^2 + u^2$, it follows the existence of the following:

$$\begin{aligned} -af(x,y)y + h'(x)y^2 &\leq -aby^2 - a\delta y^2 + h'(x)y^2 \\ &= -\{ab - h'(x)\}y^2 - a\delta y^2 \\ &\leq -a\delta y^2; \end{aligned}$$

$$-\{g(x,y) - a\}z^2 \leq -\mu z^2;$$

$$ay \int_{0}^{y} g_x(x,\eta)\eta d\eta \leq 0;$$

$$y \int_{0}^{y} f_x(x,\eta)d\eta \leq 0;$$

$$ay\int_{t-r}^{t} h'(x(s))y(s)ds \leq \frac{a^2br}{2}y^2 + \frac{a^2b}{2}\int_{t-r}^{t} y^2(s)ds;$$

$$z\int_{t-r}^{t} h'(x(s))y(s)ds \leq \frac{abr}{2}z^2 + \frac{ab}{2}\int_{t-r}^{t} y^2(s)ds;$$

$$ay\int_{t-r}^{t} f_x(x(s),y(s))y(s)ds \leq \frac{aKr}{2}y^2 + \frac{aK}{2}\int_{t-r}^{t} y^2(s)ds;$$

$$z\int_{t-r}^{t} f_x(x(s),y(s))y(s)ds \leq \frac{Kr}{2}z^2 + \frac{K}{2}\int_{t-r}^{t} y^2(s)ds;$$

$$ay\int_{t-r}^{t} f_y(x(s),y(s))z(s)ds \leq \frac{aLr}{2}y^2 + \frac{aL}{2}\int_{t-r}^{t} z^2(s)ds;$$

$$z\int_{t-r}^{t} f_y(x(s),y(s))z(s)ds \leq \frac{Lr}{2}z^2 + \frac{L}{2}\int_{t-r}^{t} z^2(s)ds.$$

The above estimates imply that

$$\begin{aligned}\tfrac{dV}{dt} \quad &\leq -\left\{a\delta - \left(\tfrac{a^2b}{2} + \tfrac{aK}{2} + \tfrac{aL}{2} + \lambda_1\right)r\right\}y^2 \\ &- \left\{\mu - \left(\tfrac{ab}{2} + \tfrac{K}{2} + \tfrac{L}{2} + \lambda_2\right)r\right\}z^2 \\ &+ \left\{\tfrac{a^2b}{2} + \tfrac{ab}{2} + \tfrac{aK}{2} + \tfrac{K}{2} - \lambda_1\right\}\int_{t-r}^{t} y^2(s)ds \\ &+ \left\{\tfrac{aL}{2} + \tfrac{L}{2} - \lambda_2\right\}\int_{t-r}^{t} z^2(s)ds.\end{aligned}$$

If we choose

$$\lambda_1 = \frac{a^2b}{2} + \frac{ab}{2} + \frac{aK}{2} + \frac{K}{2}$$

and

$$\lambda_2 = \frac{aL}{2} + \frac{L}{2},$$

then it follows that

$$\begin{aligned}\tfrac{d}{dt}V(x_t,y_t,z_t) \quad &\leq -\left\{a\delta - \left(\tfrac{a^2b}{2} + \tfrac{aK}{2} + \tfrac{aL}{2} + \lambda_1\right)r\right\}y^2 \\ &- \left\{\mu - \left(\tfrac{ab}{2} + \tfrac{K}{2} + \tfrac{L}{2} + \lambda_2\right)r\right\}z^2.\end{aligned}$$

Hence, we conclude that

$$\frac{d}{dt}V(x_t,y_t,z_t) \leq -D_5y^2 - D_6z^2 \leq 0$$

for some positive constants D_5 and D_6 provided that

$$r < \min\left\{\frac{2a\delta}{a^2b + aK + aL + 2\lambda_1},\ \frac{2\mu}{ab + K + L + 2\lambda_2}\right\}.$$

It can also be followed that the largest invariant set in Z is $Q = \{0\}$, where $Z = \{\phi \in C_H : \dot{V}(\phi) = 0\}$. That is, the only solution of equation (1) for which $\frac{d}{dt}V(x_t, y_t, z_t) = 0$ is the solution $x \equiv 0$. The above discussion guarantees that the null solution of equation (1) is asymptotically stable and completes the proof of Theorem 1.

In the case $p(t, x, y, x(t-r), y(t-r), z) \neq 0$, we prove the following result:

Theorem 2. In addition to the assumptions of Theorem 1, we assume the following condition holds for continuous p that appearing in (1):

$$|p(t, x, y, x(t-r), y(t-r), z)| \leq q(t)$$

where $q \in L^1(0, \infty)$, $L^1(0, \infty)$ is space of Lebesgue integrable functions.

Then, there exists a finite positive constant M such that the solution $x(t)$ of equation (1) defined by the initial function

$$x(t) = \phi(t), x'(t) = \phi'(t), x''(t) = \phi''(t)$$

satisfies the inequalities

$$|x(t)| \leq \sqrt{M}, |x'(t)| \leq \sqrt{M}, |x''(t)| \leq \sqrt{M}$$

for all $t \geq t_0 \geq 0$, where $\phi \in C^2([t_0 - r, t_0],\ \Re)$, provided that

$$r < \min\left\{\frac{2a\delta}{a^2b + aK + aL + 2\lambda_1},\ \frac{2\mu}{ab + K + L + 2\lambda_2}\right\}.$$

Remark. We use Lyapunov functional $V = V(x_t, y_t, z_t)$, which is given by (5), to prove Theorem 2.

Proof. For the case $p(t, x, y, x(t-r), y(t-r), z) \neq 0$, on differentiating (5) along the system (2), we have readily that

$$\frac{d}{dt}V(x_t, y_t, z_t) \leq -D_5y^2 - D_6z^2 + (ay + z)p(t, x, y, x(t-r), y(t-r), z).$$

Clearly, we observe that

$$\begin{aligned}\tfrac{d}{dt}V(x_t, y_t, z_t) \quad &\leq (a\,|y| + |z|)\,|p(t, x, y, x(t-r), y(t-r), z)| \\ &\leq D_7(|y| + |z|)q(t),\end{aligned}$$

where $D_7 = \max\{1, a\}$.

Now, the inequalities $|y| < 1 + y^2$ and $|z| < 1 + z^2$ lead

$$\frac{d}{dt}V(x_t, y_t, z_t) \leq D_7(2 + y^2 + z^2)\, q(t).$$

By

$$y^2 + z^2 \leq D_4^{-1}\, V(x_t, y_t, z_t)$$

it follows that

$$\begin{aligned}\frac{d}{dt}V(x_t,y_t,z_t) &\le D_7(2+D_4^{-1}V(x_t,y_t,z_t))q(t)\\ &= 2D_7q(t)+D_7D_4^{-1}V(x_t,y_t,z_t)q(t).\end{aligned} \tag{7}$$

Integrating (7)from 0 to t and using the assumption $q\in L^1(0,\infty)$ and Gronwall-Reid-Bellman inequality, we get

$$\begin{aligned}V(x_t,y_t,z_t) &\le V(x_0,y_0,z_0)+2D_7A+D_7D_4^{-1}\int_0^t(V(x_s,y_s,z_s))q(s)ds\\ &\le (V(x_0,y_0,z_0)+2D_7A)\exp\left(D_7D_4^{-1}\int_0^t q(s)ds\right)\\ &\le (V(x_0,y_0,z_0)+2D_7A)\exp(D_7D_4^{-1}A)=M_1<\infty,\end{aligned}$$

where $M_1>0$ is a constant, $M_1=(V(x_0,y_0,z_0)+2D_7A)\ \exp(D_7D_4^{-1}A)$ and $A=\int_0^\infty q(s)ds.$

Under the above discussion, we arrive at the following:

$$x^2(t)+y^2(t)+z^2(t)\le D_4^{-1}V(x_t,y_t,z_t)\le M,$$

where $M=M_1D_4^{-1}$. Therefore, one can conclude that

$$|x(t)|\le\sqrt{M},|y(t)|\le\sqrt{M},|z(t)|\le\sqrt{M}$$

for all $t\ge t_0\ge 0$. That is,

$$|x(t)|\le\sqrt{M},|x'(t)|\le\sqrt{M},|x''(t)|\le\sqrt{M}$$

for all $t\ge t_0\ge 0$.

The proof of Theorem 2 is now complete.

Example. Consider nonlinear third order delay differential equation

$$\begin{aligned}&x'''(t)+(9+(x'(t))^2)x''(t)+4x'(t-r)+2arctgx(t-r)\\ &=\frac{1}{1+t^2+x^2(t)+x^2(t-r)+x'^2(t)+x'^2(t-r)+x''^2(t)}.\end{aligned} \tag{8}$$

It can be seen that differential equation (8) has the form (1) and may be expressed as following:

$$\begin{aligned}z'(t) &= -(9+y^2(t))z(t)-7y(t)-2arctgx(t)\\ &+2\int_{t-r}^t y(s)ds+\int_{t-r}^t 6z(s)ds+2\int_{t-r}^t\frac{y(s)}{1+x^2(s)}ds\\ &+\frac{1}{1+t^2+x^2(t)+x^2(t-r)+y^2(t)+y^2(t-r)+z^2(t)},\end{aligned} \tag{9}$$

By comparing (9) with (1) and taking into account the assumptions of the theorems, it follows the following:

$$g(y)=9+y^2,$$

$$9 + y^2 \geq 9 = 8 + 1 = a + \mu,$$
$$a = 8, \mu = 1;$$
$$f(y) = 7y, f(0) = 0,$$
$$f(y)y \geq (6)y^2 = (5+1)y^2 = (b+\delta)y^2,$$
$$b = 5, \delta = 1;$$
$$h(x) = 2arctgx, h(0) = 0,$$
$$h'(x) = \frac{2}{1+x^2},$$
$$0 < h'(x) = \frac{2}{1+x^2} < 16 = ab,$$
$$sgn\ arctgx = sgn\ x;$$
$$f'(y) = 4 = L$$
$$\vdots$$
$$\begin{aligned} p(t,x,x(t-r),y,y(t-r),z) &= \tfrac{1}{1+t^2+x^2(t)+x^2(t-r)+y^2(t)+y^2(t-r)+z^2(t)} \\ &\leq \tfrac{1}{1+t^2} \end{aligned}$$

and
$$\int_0^\infty q(s)ds = \int_0^\infty \frac{1}{1+s^2}ds = \frac{\pi}{2} < \infty,$$
that is, $q \in L^1(0,\infty)$.

Thus, all the assumptions of Theorem 1 and Theorem 2 hold. That is, the null solution of equation (8) is asymptotic stability and all solutions of the same equation are bounded, when $p \equiv 0$ and $p \neq 0$ in (8), respectively.

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