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PROPERTIES OF SOME FAMILIES OF MEROMORPHIC MULTIVALENT FUNCTIONS INVOLVING CERTAIN LINEAR OPERATOR

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Abstract

Making use of a linear operator, which is defined here by means of the Hadamard product (or convolution), we introduce two novel subclasses $\Omega_{a,c}(p,A,B,\lambda)$ and $\Omega^{+}_{a,c}(p,A,B,\lambda)$ of meromorphically multivalent functions. The main object of this paper is to investigate the various important properties and characteristics of those subclasses of meromorphically multivalent functions. We extend the familiar concept of neighborhoods of analytic functions to these subclasses of meromorphically multivalent functions. We also derive many results for the Hadamard products of functions belonging to the class $\Omega^{+}_{a,c}(p,\alpha,\beta,\gamma,\lambda)$.

1 Introduction

Let Σ_p denote the class of functions of the form :

$$f(z) = z^{-p} + \sum_{k=1}^{\infty} a_{k-p} z^{k-p} \quad (p \in \mathbb{N} = \{1,2,...\}), \tag{1}$$

which are analytic and p-valent in the punctured unit disc $U^* = \{z : z \in \mathbb{C}$ and $0 < |z| < 1\} = U\backslash\{0\}$; where $U = \{z : z \in \mathbb{C}$ and $|z| < 1\}$. For functions $f(z) \in \Sigma_p$ given by (1) and $g(z) \in \Sigma_p$ given by

$$g(z) = z^{-p} + \sum_{k=1}^{\infty} b_{k-p} z^{k-p} \quad (p \in \mathbb{N}),$$

we define the Hadamard product (or convolution) of $f(z)$ and $g(z)$ by

$$(f * g)(z) = z^{-p} + \sum_{k=1}^{\infty} a_{k-p} b_{k-p} z^{k-p} = (g * f)(z).$$

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In terms of the Pochhammer symbol $(\theta)_n$ given by

$$(\theta)_n = \frac{\Gamma(\theta+n)}{\Gamma(\theta)} = \begin{cases} 1 & (n=0) \\ \theta(\theta+1)....(\theta+n-1) & (n \in \mathbb{N}), \end{cases}$$

we define the function $\varphi(a,c;z)$ by

$$\varphi(a,c;z) = z^{-p} + \sum_{k=1}^{\infty} \frac{(a)_k}{(c)_k} z^{k-p} \tag{2}$$

$$\left(z \in U^*; a \in \mathbb{R}; c \in \mathbb{R}\backslash\mathbb{Z}_0^-; \mathbb{Z}_0^- = \{0,-1,-2,...\}\right).$$

Corresponding to the function $\varphi_p(a,c;z)$, Liu [15] and Liu and Srivastava [16] have introduced a linear operator $\ell_p(a,c)$ which is defined by means of the following Hadamard product (or convolution) :

$$\ell_p(a,c)f(z) = \varphi_p(a,c;z) * f(z) \quad (f(z) \in \Sigma_p) . \tag{3}$$

Just as in [15] and [16], it is easily verified from the definitions (2) and (3) that

$$z(\ell_p(a,c)f(z))^{'} = a\ell_p(a+1,c)f(z) - (a+p)\ell_p(a,c)f(z) .$$

We also note, for any integer $n > -p$ and for $f(z) \in \Sigma_p$, that

$$\ell_p(n+p,1)f(z) = D^{n+p-1}f(z) = \frac{1}{z^p(1-z)^{n+p}} * f(z) ,$$

where $D^{n+p-1}f(z)$ is the differential operator studied by (among others) Uralegaddi and Somanatha [25] and Aouf [7].

Let

$$\begin{aligned} F_{p,a,c,\lambda}(z) &= (1-\lambda)L_p(a,c)f(z) + \frac{\lambda}{p} z(L_p(a,c)f(z))^{'} \\ (f &\in \Sigma_p; p \in \mathbb{N}; 0 \le \lambda < \frac{1}{2}) , \end{aligned} \tag{4}$$

so that, obviously,

$$F_{p,a,c,\lambda}(z) = \frac{1-2\lambda}{z^p} + \sum_{k=1}^{\infty} \left[1-\lambda+\lambda(\frac{k-p}{p})\right] \frac{(a)_k}{(c)_k} a_{k-p} z^{k-p} \tag{5}$$

$$(p \in \mathbb{N}; 0 \le \lambda < \frac{1}{2}) ,$$

since $f(z) \in \Sigma_p$ is given by (1). From (5), it is easily verified that

$$zF^{'}_{p,a,c,\lambda}(z) = aF_{p,a+1,c,\lambda}(z) - (a+p)F_{p,a,c,\lambda}(z) . \tag{6}$$

For fixed parameters A, B, p and λ with $-1 \leq B < A \leq 1, p \in \mathbb{N}$ and $0 \leq \lambda < \frac{1}{2}$, we say that a function $f(z) \in \Sigma_p$ is in the class $\Omega_{a,c}(p, A, B, \lambda)$ of meromorphically p-valent functions in U^* if the function $F_{p,a,c,\lambda}(z)$ defined by (4) satisfies the following inequality:

$$\left| \frac{z^{p+1}F'_{p,a,c,\lambda}(z) + p(1-2\lambda)}{Bz^{p+1}F'_{p,a,c,\lambda}(z) + Ap(1-2\lambda)} \right| < 1 \qquad (z \in U^*) \ . \tag{7}$$

Let Σ_p^* denote the class of functions of the form :

$$f(z) = z^{-p} + \sum_{k=p}^{\infty} |a_k| \, z^k \qquad (p \in \mathbb{N}) \tag{8}$$

which are analytic and p-valent in U^*. Furthermore, we say that a function $f(z) \in \Omega^+_{a,c}(p, A, B, \lambda)$ whenever $f(z)$ is of the form (8).

We note that the following interesting relationships with some of the special function classes which were investigated recently :

(i) $\Omega_{a,c}(p, \alpha A, \alpha B, 0) = S_{a,c}(A, B, \alpha)$ and $\Omega^+_{a,c}(p, \alpha A, \alpha B, 0) = S^*_{a,c}(A, B, \alpha)$ $(\alpha > 0; -1 \leq B < A \leq 1; -1 \leq B \leq 0$ and $|B\alpha| \leq 1)$ (Liu [15]);

(ii) $\Omega^+_{a,a}(A, B, 0) = H^*(p; A, B)(0 \leq B \leq 1; -B \leq A < B)$ (Mogra [18]);

(iii) $\Omega^+_{a,a}(1, (1-2\gamma\alpha)\beta, (1-2\gamma)\beta, 0) = \Sigma_d(\alpha, \beta, \gamma)$ $(0 \leq \alpha < 1; \frac{1}{2} \leq \gamma \leq 1, 0 < \beta \leq 1)$ (Cho et al. [10]);

(iv) $\Omega^+_{a,a}(1.A, B, 0) = \Sigma_d(A, B)$ $(-1 \leq B < A \leq 1, -1 \leq B < 0)$ (Cho [9]).

Also we note that :
$\Omega^+_{a,c}(p, (1-2\gamma\frac{\alpha}{p})\beta, (1-2\gamma)\beta, \lambda) = \Sigma^+_{a,c}(p, \alpha, \beta, \gamma, \lambda)$

$$= \left\{ f(z) \in \Sigma_p^* : \left| \frac{z^{p+1}F'_{p,a,c,\lambda}(z) + p(1-2\lambda)}{(2\gamma-1)z^{p+1}F'_{p,a,c,\lambda}(z) + (2\lambda\alpha - p)(1-2\lambda)} \right| < \beta \, , \right.$$

$$\left. (z \in U^*; 0 \leq \alpha < p; p \in \mathbb{N}; \frac{1}{2} \leq \gamma \leq 1; 0 < \beta \leq 1) \right\} \ .$$

Meromorphically multivalent functions have been extensively studied by (for example) Mogra ([17, 18]), Uralegaddi and Ganigi [24], Aouf ([4, 5, 6]) , Srivastava et al. [23], Owa et al. [19], Joshi and Aouf [13], Joshi and Srivastava [14], Aouf et al.[8], Raina and Srivastava [20] and Yang [26].

In this paper we investigate the various important propertries and characteristics of the classes $\Omega_{a,c}(p, A, B, \lambda)$ and $\Omega^+_{a,c}(p, A, B, \lambda)$. Following the recent investigations by Altintas et al. [3, p.1668], we extend the concept of neighborhoods of analytic functions, which was considered earlier by (for example) Goodman [11] and Ruscheweyh [21], to mermorphically multivalent functions belonging to the classes $\Omega_{a,c}(p, A, B, \lambda)$ and $\Omega^+_{a,c}(p, A, B, \lambda)$. We also derive many results for the Hadamard products of functions belonging to the class $\Omega^+_{a,c}(p, \alpha, \beta, \gamma, \lambda)$.

2 Inclusion properties of the class $\Omega_{a,c}(p, A, B, \lambda)$

We begin by recalling the following result (Jack's lemma), which we shall apply in proving our first inclusion theorem (Theorem 1 below).

Lemma 1. *([12]) Let the (nonconstant) function $w(z)$ be analytic in U with $w(0) = 0$. If $|w(z)|$ attains its maximum value on the circle $|z| = r < 1$ at a point $z_0 \in U$, then*

$$z_0 w^{'}(z_0) = \xi w(z_0)\,,$$

where ξ is a real number and $\xi \geq 1$.

Theorem 1. *Let $a > 0$, then*

$$\Omega_{a+1,c}(p, A, B, \lambda) \subset \Omega_{a,c}(p, A, B, \lambda)\ .$$

Proof. Let $f(z) \in \Omega_{a+1,c}(p, A, B, \lambda)$ and suppose that

$$z^{p+1} F^{'}_{p,a,c,\lambda}(z) = -p(1-2\lambda)\frac{1 + Aw(z)}{1 + Bw(z)}\ , \tag{9}$$

where the function $w(z)$ is either analytic or meromorphic in U, with $w(0) = 0$. Then, by using (6) and (9), we have

$$z^{p+1} F^{'}_{p,a+1,c,\lambda}(z) = -p(1-2\lambda)\frac{1 + Aw(z)}{1 + Bw(z)} - \frac{p(1-2\lambda)(A-B)}{a} \cdot \frac{zw^{'}(z)}{(1 + Bw(z))^2}\,. \tag{10}$$

We claim that $|w(z)| < 1$ for $z \in U$. Otherwise there exists a point $z_0 \in U$ such that $\max\limits_{|z| \leq |z_0|} |w(z)| = |w(z_0)| = 1$. Applying Jack's lemma, we have $z_0 w^{'}(z_0) = \xi w(z_0)(\xi \geq 1)$. Writing $w(z_0) = e^{i\theta}(0 \leq \theta \leq 2\pi)$ and putting $z = z_0$ in (10), we get

$$\begin{aligned} &\left|\frac{z_0^{p+1} F^{'}_{p,a+1,c,\lambda}(z_0) + p(1-2\lambda)}{Bz_0^{p+1} F^{'}_{p,a+1,c,\lambda}(z_0) + Ap(1-2\lambda)}\right|^2 - 1 \\ = \ &\frac{\left|a + \xi + aBe^{i\theta}\right|^2 - \left|a + B(a-\xi)e^{i\theta}\right|^2}{\left|a + B(a-\xi)e^{i\theta}\right|^2} \\ = \ &\frac{\xi^2(1 - B^2) + 2a\xi(1 + B^2 + 2B\cos\theta)}{\left|a + B(a-\xi)e^{i\theta}\right|^2} \geq 0\,, \end{aligned}$$

which obviously contradicts our hypothesis that $f(z) \in \Omega_{a+1,c}(p, A, B, \lambda)$. Thus we must have $|w(z)| < 1 (z \in U)$, and so from (9), we conclude that $f(z) \in \Omega_{a,c}(p, A, B, \lambda)$, which evidently completes the proof of Theorem 1. ∎

Theorem 2. *Let μ be a complex number such that* $\mathrm{Re}\mu > 0$. *If $f(z) \in \Omega_{a,c}(p, A, B, \lambda)$, then the function*

$$G_{p,a,c,\lambda}(z) = \frac{\mu}{z^{\mu+p}} \int_0^z t^{\mu+p-1} F_{p,a,c,\lambda}(t) dt \tag{11}$$

is also in the same class $\Omega_{a,c}(p, A, B, \lambda)$.

Proof. From (11), we have

$$zG'_{p,a,c,\lambda}(z) = \mu F_{p,a,c,\lambda}(z) - (\mu + p) G_{p,a,c,\lambda}(z) \ . \tag{12}$$

Put

$$z^{p+1} G'_{p,a,c,\lambda}(z) = -p(1-2\lambda) \frac{1 + Aw(z)}{1 + Bw(z)} \ , \tag{13}$$

where $w(z)$ is either analytic or meromorphic in U with $w(0) = 0$. Then, by using (12) and (13), we have

$$\begin{aligned} z^{p+1} F'_{p,a,c,\lambda}(z) \quad &= \quad -p(1-2\lambda) \frac{1 + Aw(z)}{1 + Bw(z)} \\ &\quad - \frac{p(1-2\lambda)(A-B)}{\mu} \cdot \frac{zw'(z)}{(1 + Bw(z))^2} \ . \end{aligned}$$

The remaining part of the proof is similar to that of Theorem 1 and so is omitted. ∎

Theorem 3. *$f(z) \in \Omega_{a,c}(p, A, B, \lambda)$ if and only if*

$$G_{p,a,c,\lambda}(z) = \frac{a}{z^{a+p}} \int_0^z t^{a+p-1} F_{p,a,c,\lambda}(t) dt \in \Omega_{a+1,c}(p, A, B, \lambda) \, .$$

Proof. In view of the definition of $G_{p,a,c,\lambda}(z)$, we have

$$aF_{p,a,c,\lambda}(z) = (a + p) G_{p,a,c,\lambda}(z) + zG'_{p,a,c,\lambda}(z) \ . \tag{14}$$

By using (6) and (14), we have

$$aF_{p,a,c,\lambda}(z) = aG_{p,a+1,c,\lambda}(z) \, .$$

The desired result follows immediately. ∎

3 Properties of the class $\Omega^{+}_{a,c}(p,A,B,\lambda)$

In this section we assume further that $a>0, c>0, -1\le B<A\le 1, -1\le B\le 0$ and $0\le\lambda<\frac{1}{2}$.

Theorem 4. *Let $f(z)\in\Sigma^{*}_{p}$ be given by (8). Then $f(z)\in\Omega^{+}_{a,c}(p,A,B,\lambda)$ if and only if*

$$\sum_{k=p}^{\infty} k\left[1+\lambda(\frac{k-p}{p})\right](1-B)\,\frac{(a)_{k+p}}{(c)_{k+p}}\,|a_k|\le (A-B)p(1-2\lambda)\,. \tag{15}$$

Proof. Let $f(z)\in\Omega^{+}_{a,c}(p,A,B,\lambda)$ is given by (8). Then, from (7) and (8), we have

$$\begin{aligned}&\left|\frac{z^{p+1}F'_{p,a,c,\lambda}(z)+p(1-2\lambda)}{Bz^{p+1}F'_{p,a,c,\lambda}(z)+Ap(1-2\lambda)}\right|\\ =&\left|\frac{\sum_{k=p}^{\infty} k\left[1+\lambda(\frac{k-p}{p})\right]\frac{(a)_{k+p}}{(c)_{k+p}}\,|a_k|\,z^{k+p}}{(A-B)p(1-2\lambda)+\sum_{k=p}^{\infty} Bk\left[1+\lambda(\frac{k-p}{p})\right]\frac{(a)_{k+p}}{(c)_{k+p}}\,|a_k|\,z^{k+p}}\right|<1(z\in U)\end{aligned}\tag{16}$$

Since $|\mathrm{Re} z|\le|z|$ $(z\in\mathbb{C})$, we have

$$\mathrm{Re}\left\{\frac{\sum_{k=p}^{\infty} k\left[1+\lambda(\frac{k-p}{p})\right]\frac{(a)_{k+p}}{(c)_{k+p}}\,|a_k|}{(A-B)p(1-2\lambda)+\sum_{k=p}^{\infty} Bk\left[1+\lambda(\frac{k-p}{p})\right]\frac{(a)_{k+p}}{(c)_{k+p}}\,|a_k|}\right\}<1\,.$$

Choose values of z on the real axis so that $z^{p+1}F'_{p,a,c,\lambda}(z)$ is real. Upon clearing the denominator in (16) and letting $z\to 1^{-}$ through real values we obtain (15).

In order to prove the converse, we assume that the inequality (15) holds true. Then, if we let $z\in\partial U$, we find from (8) and (15) that

$$\begin{aligned}&\left|\frac{z^{p+1}F'_{p,a,c,\lambda}(z)+p(1-2\lambda)}{Bz^{p+1}F'_{p,a,c,\lambda}(z)+Ap(1-2\lambda)}\right|\\ \le&\ \frac{\sum_{k=p}^{\infty} k\left[1+\lambda(\frac{k-p}{p})\right]\frac{(a)_{k+p}}{(c)_{k+p}}\,|a_k|}{(A-B)p(1-2\lambda)+\sum_{k=p}^{\infty} Bk\left[1+\lambda(\frac{k-p}{p})\right]\frac{(a)_{k+p}}{(c)_{k+p}}\,|a_k|}\\ <&\ 1\quad (z\in\partial U=\{z: z\in\mathbb{C}\text{ and }|z|=1\})\,.\end{aligned}$$

Hence, by the maximum modulus theorem, we have $f(z)\in\Omega^{+}_{a,c}(p,A,B,\lambda)$. This completes the proof of Theorem 4. ∎

Corollary 1. *If the function $f(z)$ defined by (8) is in the class $\Omega^{+}_{a,c}(p,A,B,\lambda)$, then*

$$|a_k| \leq \frac{(A-B)p(1-2\lambda)(c)_{k+p}}{k\left[1+\lambda(\frac{k-p}{p})\right](1-B)(a)_{k+p}} \qquad (k \geq p; p \in \mathbb{N}),$$

with equality for the function $f(z)$ given by

$$f(z) = z^{-p} + \frac{(A-B)p(1-2\lambda)(c)_{k+p}}{k\left[1+\lambda(\frac{k-p}{p})\right](1-B)(a)_{k+p}} z^k \quad (k \geq p; p \in \mathbb{N}). \tag{17}$$

Putting $A = (1-2\gamma\frac{\alpha}{p})\beta$ and $B = (1-2\gamma)\beta\,(0 \leq \alpha < p,\ 0 < \beta \leq 1,\ \frac{1}{2} \leq \gamma \leq 1$ and $p \in \mathbb{N})$ in Theorem 4, we obtain

Corollary 2. *A function $f(z)$ defined by (8) is in the class $\Omega^{+}_{a,c}(p,\alpha,\beta,\gamma,\lambda)$ if and only if*

$$\sum_{k=p}^{\infty} k\left[1+\lambda(\frac{k-p}{p})\right](1+2\beta\gamma-\beta)\frac{(a)_{k+p}}{(c)_{k+p}}|a_k| \leq 2\beta\gamma(p-\alpha)(1-2\lambda).$$

The following property is an easy consequence of Theorem 4.

Theorem 5. *Let each of the functions $f_j(z)$ defined by*

$$f_j(z) = z^{-p} + \sum_{k=p}^{\infty} |a_{k,j}|\, z^k \qquad (j = 1,2,...,m) \tag{18}$$

be in the class $\Omega^{+}_{a,c}(p,A,B,\lambda)$. Then the function $h(z)$ defined by

$$h(z) = \sum_{j=1}^{m} \zeta_j f_j(z) \quad (\zeta_j \geq 0 \text{ and } \sum_{j=1}^{m} \zeta_j = 1)$$

is also in the class $\Omega^{+}_{a,c}(p,A,B,\lambda)$.

Next we prove the following growth and distortion properties for the class $\Omega^{+}_{a,c}(p,A,B,\lambda)$.

Theorem 6. *If a function $f(z)$ defined by (8) is in the class $\Omega^{+}_{a,c}(p,A,B,\lambda)$, then*

$$\begin{aligned} &\left\{\frac{(p+m-1)!}{(p-1)!} - \frac{(c)_{2p}}{(a)_{2p}}\left[\frac{(A-B)(1-2\lambda)}{(1-B)}\right]\frac{p!}{(p-m)!}r^{2p}\right\} r^{-(p+m)} \\ \leq\ &\left|f^{(m)}(z)\right| \leq \\ &\left\{\frac{(p+m-1)!}{(p-1)!} - \frac{(c)_{2p}}{(a)_{2p}}\left[\frac{(A-B)(1-2\lambda)}{(1-B)}\right]\frac{p!}{(p-m)!}r^{2p}\right\} r^{-(p+m)} \end{aligned}$$

$$(0 < |z| = r < 1; a > c > 0; m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}; p \in \mathbb{N}; p > m).$$

The result is sharp for the function $f(z)$ given by

$$f(z)=z^{-p}+\frac{(A-B)(1-2\lambda)(c)_{2p}}{(1-B)(a)_{2p}}z^{p} \quad (p\in\mathbb{N}) .$$

Proof. In view of Theorem 4, we have

$$\frac{p(1-B)(a)_{2p}}{p!(c)_{2p}}\sum_{k=p}^{\infty}k!|a_k|\leq\sum_{k=p}^{\infty}k\left[1+\lambda(\frac{k-p}{p})\right](1-B)\frac{(a)_{k+p}}{(c)_{k+p}}|a_k|\leq(A-B)p(1-2\lambda)\,,$$

which yields

$$\sum_{k=p}^{\infty}k!|a_k|\leq\frac{(c)_{2p}p!(A-B)(1-2\lambda)}{(a)_{2p}(1-B)} \quad (p\in\mathbb{N})\,. \tag{19}$$

Now, by differentiating both sides of (8) m times with respect to z, we have

$$f^{(m)}(z)=(-1)^m\frac{(p+m-1)!}{(p-1)!}z^{-(p+m)}+\sum_{k=p}^{\infty}\frac{k!}{(k-m)!}|a_k|z^{k-m}, \tag{20}$$

$$(p\in\mathbb{N},m\in\mathbb{N}_0;p>m)$$

and Theorem 6 follows easily from(19) and (20). ∎

Next we determine the radii of meromorphically p-valent starlikeness of order $\delta(0\leq\delta<p)$ and meromorphically p-valent convexity of order $\delta(0\leq\delta<p)$ for functions in the class $\Omega_{a,c}^{+}(p,A,B,\lambda)$.

Theorem 7. *Let the function $f(z)$ defined by (8) be in the class $\Omega_{a,c}^{+}(p,A,B,\lambda)$, then we have :*

(i) $f(z)$ is meromorphically p-valent starlike of order $\delta(0\leq\delta<p)$ in the disc $|z|<r_1$, that is,

$$\mathrm{Re}\left\{-\frac{\mathrm{z}\mathrm{f}'(\mathrm{z})}{\mathrm{f}(\mathrm{z})}\right\}>\delta \quad (|\mathrm{z}|<\mathrm{r}_1;0\leq\delta<\mathrm{p};\mathrm{p}\in\mathbb{N})\,,$$

where

$$r_1=\inf_{k\geq p}\left\{\frac{(a)_{k+p}}{(c)_{c+p}}\cdot\frac{k\left[1+\lambda(\frac{k-p}{p})\right](1-B)(p-\delta)}{(A-B)p(1-2\lambda)(k+\delta)}\right\}^{\dfrac{1}{k+p}} . \tag{21}$$

(ii) $f(z)$ is meromorphically p-valent convex of order $\delta(0\leq\delta<p)$ in the disc $|z|<r_2$, that is,

$$\mathrm{Re}\left\{-(1+\frac{\mathrm{z}\mathrm{f}''(\mathrm{z})}{\mathrm{f}'(\mathrm{z})})\right\}>\delta \quad (|\mathrm{z}|<\mathrm{r}_2;0\leq\delta<\mathrm{p};\mathrm{p}\in\mathbb{N})\,,$$

where

$$r_2 = \inf_{k\geq p}\left\{\frac{(a)_{k+p}}{(c)_{c+p}}\cdot\frac{\left[1+\lambda(\frac{k-p}{p})\right](1-B)(p-\delta)}{(A-B)(1-2\lambda)(k+\delta)}\right\}^{\frac{1}{k+p}}. \tag{22}$$

Each of these results is sharp for the function $f(z)$ given by (17).

Proof. (i) From the definition (8), we easily get

$$\left|\frac{\frac{zf'(z)}{f(z)}+p}{\frac{zf'(z)}{f(z)}-p+2\delta}\right| \leq \frac{\sum\limits_{k=p}^{\infty}(k+p)|a_k||z|^{k+p}}{2(p-\delta)-\sum\limits_{k=p}^{\infty}(k-p+2\delta)|a_k||z|^{k+p}}.$$

Thus, we have the desired inequality

$$\left|\frac{\frac{zf'(z)}{f(z)}+p}{\frac{zf'(z)}{f(z)}-p+2\delta}\right| \leq 1 \quad (0\leq\delta<p; p\in\mathbb{N}),$$

if

$$\sum_{k=p}^{\infty}(\frac{k+\delta}{p-\delta})\,|a_k|\,|z|^{k+p}\leq 1. \tag{23}$$

Hence, by Theorem 4, (23) will be true if

$$(\frac{k+\delta}{p-\delta})|z|^{k+p}\leq\left\{\frac{(a)_{k+p}}{(c)_{k+p}}\cdot\frac{k\left[1+\lambda(\frac{k-p}{p})\right](1-B)}{(A-B)p(1-2\lambda)}\right\} \quad (k\geq p; p\in\mathbb{N}). \tag{24}$$

The last inequality (24) leads us immediately to the disc $|z|<r_1$, where r_1 is given by (21).

(ii) In order to prove the second assertion of Theorem 7, we find from the definition (8) that

$$\left|\frac{1+\frac{zf''(z)}{f'(z)}+p}{1+\frac{zf''(z)}{f'(z)}-p+2\delta}\right| \leq \frac{\sum\limits_{k=p}^{\infty}(k+p)|a_k||z|^{k+p}}{2(p-\delta)-\sum\limits_{k=p}^{\infty}(k-p+2\delta)|a_k||z|^{k+p}}.$$

Thus we have the desired inequality

$$\left|\frac{1+\frac{zf''(z)}{f'(z)}+p}{1+\frac{zf''(z)}{f'(z)}-p+2\delta}\right| \leq 1 \quad (0\leq\delta<p; p\in\mathbb{N}),$$

if
$$\sum_{k=p}^{\infty} \frac{k(k+\delta)}{p(p-\delta)}\,|a_k|\,|z|^{k+p} \leq 1\,. \tag{25}$$

Hence, by Theorem 4, (25) will be true if

$$\frac{k(k+\delta)}{p(p-\delta)}|z|^{k+p} \leq \left\{ \frac{(a)_{k+p}}{(c)_{k+p}} \cdot \frac{k\left[1+\lambda(\frac{k-p}{p})\right](1-B)}{(A-B)p(1-2\lambda)} \right\} \quad (k \geq p; p \in \mathbb{N})\,. \tag{26}$$

The last inequality (26) readily yields the disc $|z| < r_2$, where r_2 defined by (22), and the proof of Theorem 7 is completed by merely verifying that each assertion is sharp for the function $f(z)$ given by (17). ∎

4 Neighborhoods

Following the earlier works (based upon the familiar concept of neighborhoods of analytic functions) by Goodman [11] and Ruscheweyh [21], and (more recently) by Altintas et al. ([1], [2] and [3]), Liu [15], and Liu and Srivastava [16], we begin by introducing here the δ-neighborhood of a function $f(z) \in \Sigma_p$ of the form (1) by means of the definition given below :

$$N_\delta(f) = \Bigg\{ g \in \Sigma_p : g(z) = z^{-p} + \sum_{k=1}^{\infty} b_{k-p} z^{k-p} \quad \text{and}$$

$$\sum_{k=1}^{\infty} \frac{(1+|B|)(k+p)\left[1-\lambda+\lambda(\frac{k-p}{p})\right]}{(A-B)p(1-2\lambda)} \cdot \frac{(a)_k}{(c)_k}\,|b_{k-p} - a_{k-p}| \leq \delta\,,$$

$$(a>0; c>0; -1 \leq B < A \leq 1; 0 \leq \lambda < \frac{1}{2}; p \in \mathbb{N}; \delta > 0) \Bigg\}\,. \tag{27}$$

Making use of the definition (27), we now prove Theorem 8 below.

Theorem 8. *Let the function $f(z)$ defined by (1) be in the class $\Omega_{a,c}(p,A,B,\lambda)$. If $f(z)$ satisfies the following condition :*

$$\frac{f(z)+\epsilon z^{-p}}{1+\epsilon} \in \Omega_{a,c}(p,A,B,\lambda) \quad (\epsilon \in \mathbb{C}, |\epsilon| < \delta, \delta > 0)\,, \tag{28}$$

then
$$N_\delta(f) \subset \Omega_{a,c}(p,A,B,\lambda)\ . \tag{29}$$

Proof. It is easily seen from (7) that $g(z) \in \Omega_{a,c}(p, A, B, \lambda)$ if and only if for any complex number σ with $|\sigma| = 1$,

$$\frac{z^{p+1}F'_{p,a,c,\lambda}(z) + p(1-2\lambda)}{Bz^{p+1}F'_{p,a,c,\lambda}(z) + Ap(1-2\lambda)} \neq \sigma \quad (z \in U), \tag{30}$$

which is equivalent to

$$\frac{(g * h)(z)}{z^{-p}} \neq 0 \quad (z \in U)\,, \tag{31}$$

where, for convenience,

$$\begin{aligned} h(z) &= z^{-p} + \sum_{k=1}^{\infty} c_{k-p} z^{k-p} \\ &= z^{-p} + \sum_{k=1}^{\infty} \left\{ \frac{(1-\sigma B)(k-p)\left[1-\lambda+\lambda(\frac{k-p}{p})\right]}{(B-A)p(1-2\lambda)\sigma} \right\} \frac{(a)_k}{(c)_k} z^{k-p}\,. \end{aligned} \tag{32}$$

From (32), we have

$$\begin{aligned} |c_{k-p}| &= \left| \left\{ \frac{(1-\sigma B)(k-p)\left[1-\lambda+\lambda(\frac{k-p}{p})\right]}{(B-A)p(1-2\lambda)\sigma} \right\} \cdot \frac{(a)_k}{(c)_k} \right| \\ &\leq \left(\frac{(1+|B|)(k+p)\left[1-\lambda+\lambda(\frac{k-p}{p})\right]}{(A-B)p(1-2\lambda)} \right) \frac{(a)_k}{(c)_k} \quad (k, p \in \mathbb{N}; 0 \leq \lambda < \frac{1}{2})\,. \end{aligned}$$

Now, if $f(z) = z^{-p} + \sum\limits_{k=1}^{\infty} a_{k-p} z^{k-p} \in \Sigma_p$ satisfies the condition (28), then (31) yields

$$\left| \frac{(f * h)(z)}{z^{-p}} \right| \geq \delta \quad (z \in U; \delta > 0)\,.$$

By letting

$$g(z) = z^{-p} + \sum_{k=1}^{\infty} b_{k-p} z^{k-p} \in N_\delta(f)\,,$$

so that

$$\left| \frac{[g(z) - f(z)] * h(z)}{z^{-p}} \right| = \left| \sum_{k=1}^{\infty} (b_{k-p} - a_{k-p}) c_{k-p} z^k \right|$$

$$\begin{aligned} &\leq |z| \sum_{k=1}^{\infty} \frac{(1+|B|)(k+p)\left[1-\lambda+\lambda(\frac{k-p}{p})\right]}{(A-B)p(1-2\lambda)} \cdot \frac{(a)_k}{(c)_k} \, |b_{k-p} - a_{k-p}| \\ &< \delta \quad (z \in U; \delta > 0)\,. \end{aligned}$$

Thus we have (31), and hence also (30)for any $\sigma \in \mathbb{C}$ such that $|\sigma| = 1$, which implies that $g(z) \in \Omega_{a,c}(p, A, B, \lambda)$. This evidently proves the assertion (29) of Theorem 8. ∎

We now define the δ-neighborhood of a function $f(z) \in \Sigma_p^*$ of the form (8) as follows

$$N_\delta^+(f) = \Bigg\{ g \in \Sigma_p : g(z) = z^{-p} + \sum_{k=p}^{\infty} |b_k| z^k \quad \text{and}$$

$$\sum_{k=p}^{\infty} \frac{(1+|B|)k\left[1+\lambda(\frac{k-p}{p})\right]}{(A-B)p(1-2\lambda)} \cdot \frac{(a)_{k+p}}{(c)_{k+p}} \,||b_k| - |a_k|| \le \delta\,,$$

$$(a > 0; c > 0; -1 \le B < A \le 1; 0 \le \lambda < \frac{1}{2}; p \in \mathbb{N}; \delta > 0) \Bigg\}\,.$$

Theorem 9. *Let the function $f(z)$ defined by (27) be in the class $\Omega_{a+1,c}^+(p,$ $A, B, \lambda), -1 \le B < A \le 1,\ -1 \le B \le 0,\ 0 \le \lambda < \frac{1}{2}$ and $p \in \mathbb{N}$, then*

$$N_\delta^+(f) \subset \Omega_{a,c}^+(p, A, B, \lambda)\, (\delta = \frac{2p}{a+2p})\,.$$

The result is sharp.

Proof. Making use the same method as in the proof of Theorem 8, we can show that [cf. Eq. (32)]

$$\begin{aligned} h(z) &= z^{-p} + \sum_{k=p}^{\infty} c_k z^k \\ &= z^{-p} + \sum_{k=p}^{\infty} \frac{(1-\sigma B)k\left[1+\lambda(\frac{k-p}{p})\right]}{(B-A)p(1-2\lambda)\sigma} \cdot \frac{(a)_{k+p}}{(c)_{k+p}} z^k\,. \end{aligned}$$

Thus, under the hypothesis $-1 \le B < A \le 1, -1 \le B \le 0, 0 \le \lambda < \frac{1}{2}$ and $p \in \mathbb{N}$, if $f(z) \in \Omega_{a+1,c}^+(p, A, B, \lambda)$ is given by (8), we obtain

$$\begin{aligned} \left|\frac{(f*h)(z)}{z^{-p}}\right| &= \left|1 + \sum_{k=p}^{\infty} c_k |a_k| z^{k+p}\right| \\ &\ge 1 - \sum_{k=p}^{\infty} \frac{(1-B)k\left[1+\lambda(\frac{k-p}{p})\right]}{(A-B)p(1-2\lambda)} \cdot \frac{(a)_{k+p}}{(c)_{k+p}} |a_k| \\ &\ge 1 - \frac{a}{a+2p} \sum_{k=p}^{\infty} \frac{(1-B)k\left[1+\lambda(\frac{k-p}{p})\right]}{(A-B)p(1-2\lambda)} \cdot \frac{(a+1)_{k+p}}{(c)_{k+p}} |a_k|\,. \end{aligned}$$

Also, from Theorem 4, we obtain

$$\left|\frac{(f*h)(z)}{z^{-p}}\right| \geq 1 - \frac{a}{a+2p} = \frac{2p}{a+2p} = \delta\,.$$

The remaining part of the proof of Theorem 9 is similar to that of Theorem 8, and we skip the details involved.

To show the sharpness, we consider the functions $f(z)$ and $g(z)$ given by

$$f(z) = z^{-p} + \frac{(A-B)(1-2\lambda)}{(1-B)} \cdot \frac{(c)_{2p}}{(a+1)_{2p}}\, z^p \in \Omega^{+}_{a+1,c}(p, A, B, \lambda)$$

and

$$g(z) = z^{-p} + \left[\frac{(A-B)(1-2\lambda)}{(1-B)} \cdot \frac{(c)_{2p}}{(a+1)_{2p}} + \frac{(A-B)(1-2\lambda)\delta^{'}}{(1-B)} \cdot \frac{(c)_{2p}}{(a)_{2p}}\right] z^p\,,$$

where $\delta^{'} > \delta = \frac{2p}{a+2p}$. Clearly, the function $g(z)$ belongs to $N^{+}_{\delta'}(f)$. On the other hand, we find from Theorem 4 that $g(z)$ is not in the class $\Omega^{+}_{a,c}(p, A, B, \lambda)$. Thus the proof of Theorem 9 is completed. ∎

Next we prove the following result.

Theorem 10. *Let $f(z) \in \Sigma_p$ be given by (1) and define the partial sums $s_1(z)$ and $s_n(z)$ as follows :*

$$s_1(z) = z^{-p} \text{ and } s_n(z) = z^{-p} + \sum_{k=1}^{n-1} a_{k-p} z^{k-p} \qquad (n \in \mathbb{N}\backslash\{1\})\,.$$

Suppose also that

$$\sum_{k=1}^{\infty} d_k\, |a_{k-p}| \leq 1 \left(d_k = \frac{(1+|B|)(k+p)\left[1-\lambda+\lambda(\frac{k-p}{p})\right]}{(A-B)p(1-2\lambda)} \cdot \frac{(a)_k}{(c)_k} \right). \tag{33}$$

If $a > 0, c > 0, 0 \leq \lambda < \frac{1}{2}$ and $p \in \mathbb{N}$. Then we have

(i) $f(z) \in \Omega_{a,c}(p, A, B, \lambda)$

$$\text{(ii) } \mathrm{Re}\left\{\frac{\mathrm{f(z)}}{\mathrm{s_n(z)}}\right\} > 1 - \frac{1}{\mathrm{d_n}} \quad (\mathrm{z} \in \mathrm{U}; \mathrm{n} \in \mathbb{N})\,. \tag{34}$$

and

$$\text{(iii) } \mathrm{Re}\left\{\frac{\mathrm{s_n(z)}}{\mathrm{f(z)}}\right\} > \frac{\mathrm{d_n}}{1+\mathrm{d_n}} \quad (\mathrm{z} \in \mathrm{U}; \mathrm{n} \in \mathbb{N})\,. \tag{35}$$

The estimates in (34) and (35) are sharp for each $n \in \mathbb{N}$.

Proof. (i) It is not difficult to see that

$$z^{-p} \in \Omega_{a,c}(p, A, B, \lambda) \quad (p \in \mathbb{N}) .$$

Thus, from Theorem 8 and the hypothesis (33) of Theorem 10, we have

$$N_1(z^{-p}) \subset \Omega_{a,c}(p, A, B, \lambda) \quad (p \in \mathbb{N}) ,$$

which shows that $f(z) \in \Omega_{a,c}(p, A, B, \lambda)$ as asserted by Theorem 10.

(ii) For the coefficients d_k given by (33), it is not difficult to verify that

$$d_{k+1} > d_k > 1 \quad (a > c > 0; 0 \leq \lambda < \frac{1}{2}; k \in \mathbb{N}) .$$

Therefore, we have

$$\sum_{k=1}^{n-1} |a_{k-p}| + d_n \sum_{k=n}^{\infty} |a_{k-p}| \leq \sum_{k=1}^{\infty} d_k |a_{k-p}| \leq 1 , \tag{36}$$

where we have used the hypothesis (33) again.

By setting

$$h_1(z) = d_n \left[\frac{f(z)}{s_n(z)} - (1 - \frac{1}{d_n}) \right] = 1 + \frac{d_n \sum\limits_{k=n}^{\infty} a_{k-p} z^k}{1 + \sum\limits_{k=1}^{n-1} a_{k-p} z^k} ,$$

and applying (36), we find that

$$\left| \frac{h_1(z) - 1}{h_1(z) + 1} \right| \leq \frac{d_n \sum\limits_{k=n}^{\infty} |a_{k-p}|}{2 - 2 \sum\limits_{k=1}^{n-1} |a_{k-p}| - d_n \sum\limits_{k=n}^{\infty} |a_{k-p}|} \leq 1 \quad (z \in U) ,$$

which readily yields the assertion (34) of Theorem 10. If we take

$$f(z) = z^{-p} - \frac{z^{n-p}}{d_n} , \tag{37}$$

then

$$\frac{f(z)}{s_n(z)} = 1 - \frac{z^n}{d_n} \to 1 - \frac{1}{d_n} \ \text{ as } z \to 1^- ,$$

which shows that the bound in (34) is the best possible for each $n \in \mathbb{N}$.

(iii) Just as in Part (ii) above, if we put

$$\begin{aligned} h_2(z) &= (1 + d_n) \left(\frac{s_n(z)}{f(z)} - \frac{d_n}{1 + d_n} \right) \\ &= 1 - \frac{(1 + d_n) \sum\limits_{k=n}^{\infty} a_{k-p} z^n}{1 + \sum\limits_{k=1}^{\infty} a_{k-p} z^k} , \end{aligned}$$

and make use of (36), we can deduce that

$$\left|\frac{h_2(z)-1}{h_2(z)+1}\right| \leq \frac{(1+d_n)\sum\limits_{k=n}^{\infty}|a_{k-p}|}{2-2\sum\limits_{k=1}^{n-1}|a_{k-p}|+(1-d_n)\sum\limits_{k=n}^{\infty}|a_{k-p}|} \leq 1 \quad (z \in U)\,,$$

which leads us immediately to the assertion (35) of Theorem 10

The bound in (35) is sharp for each $n \in N$, with the extremal function $f(z)$ given by (37). The proof of Theorem 10 is thus completed. ∎

5 Convolution properties for the class $\Omega^{+}_{a,c}(p,\alpha,\beta,\gamma,\lambda)$

For the functions $f_j(z)(j = 1,2)$ defined by (18) we denote by $(f_1 \otimes f_2)(z)$ the Hadamard product (or convolution) of the functions $f_1(z)$ and $f_2(z)$, that is,

$$(f_1 \otimes f_2)(z) = z^{-p} + \sum_{k=p}^{\infty}|a_{k,1}||a_{k,2}|z^{k}\,.$$

Throughout this section, we assume further that $a > c > 0$.

Theorem 11. *Let the functions $f_j(z)(j = 1,2)$ defined by (18) be in the class $\Omega^{+}_{a,c}(p,\alpha,\beta,\gamma,\lambda)$, then $(f_1 \otimes f_2)(z) \in \Omega^{+}_{a,c}(p,\zeta,\beta,\gamma,\lambda)$, where*

$$\zeta = p - \frac{2\beta\gamma(p-\alpha)^2(1-2\lambda)}{p(1+2\beta\gamma-\beta)}\,\frac{(c)_{2p}}{(a)_p}\,.$$

The result is sharp for the functions $f_j(z)(j = 1,2)$ given by

$$f_j(z) = z^{-p} + \frac{2\beta\gamma(p-\alpha)(1-2\lambda)}{p(1+2\beta\gamma-\beta)}\cdot\frac{(c)_{2p}}{(a)_{2p}}z^{p} \quad (j = 1,2; p \in \mathbb{N})\,. \tag{38}$$

Proof. Employing the technique used earlier by Schild and Silverman [22], we need to find the largest ζ such that

$$\sum_{k=p}^{\infty}\frac{k\left[1+\lambda(\frac{k-p}{p})\right](1+2\beta\gamma-\beta)}{2\beta\gamma(p-\zeta)(1-2\lambda)}\cdot\frac{(a)_{k+p}}{(c)_{k+p}}|a_{k,1}||a_{k,2}| \leq 1$$

for $f_j(z) \in \Omega^{+}_{a,c}(p,\alpha,\beta,\gamma,\lambda)(j = 1,2)$. Since $f_j(z) \in \Omega^{+}_{a,c}(p,\alpha,\beta,\gamma,\lambda)(j = 1,2)$, we readily see that

$$\sum_{k=p}^{\infty}\frac{k\left[1+\lambda(\frac{k-p}{p})\right](1+2\beta\gamma-\beta)}{2\beta\gamma(p-\alpha)(1-2\lambda)}\cdot\frac{(a)_{k+p}}{(c)_{k+p}}|a_{k,j}| \leq 1 \ (j = 1,2)\,.$$

Therefore, by the Cauchy-Schwarz inequality, we obtain

$$\sum_{k=p}^{\infty} \frac{k\left[1+\lambda(\frac{k-p}{p})\right](1+2\beta\gamma-\beta)}{2\beta\gamma(p-\alpha)(1-2\lambda)} \cdot \frac{(a)_{k+p}}{(c)_{k+p}} \sqrt{|a_{k,1}||a_{k,2}|} \leq 1\,. \tag{39}$$

This implies that we only need to show that

$$\frac{1}{(p-\zeta)} \cdot |a_{k,1}||a_{k,2}| \leq \frac{1}{(p-\alpha)} \cdot \sqrt{|a_{k,1}||a_{k,2}|} \quad (k \geq p)$$

or, equivalently, that

$$\sqrt{|a_{k,1}||a_{k,2}|} \leq \frac{(p-\zeta)}{(p-\alpha)} \quad (k \geq p).$$

Hence, by the inequality (39), it is sufficient to prove that

$$\frac{2\beta\gamma(p-\alpha)(1-2\lambda)}{k\left[1+\lambda(\frac{k-p}{p})\right](1+2\beta\gamma-\beta)} \cdot \frac{(c)_{k+p}}{(a)_{k+p}} \leq \frac{(p-\zeta)}{(p-\alpha)} \quad (k \geq p)\,. \tag{40}$$

It follows from (40) that

$$\zeta \leq p - \frac{2\beta\gamma(p-\alpha)^2(1-2\lambda)}{k\left[1+\lambda(\frac{k-p}{p})\right](1+2\beta\gamma-\beta)} \cdot \frac{(c)_{k+p}}{(a)_{k+p}} \quad (k \geq p)\,.$$

Now, defining the function $\varphi(k)$ by

$$\varphi(k) = p - \frac{2\beta\gamma(p-\alpha)^2(1-2\lambda)}{k\left[1+\lambda(\frac{k-p}{p})\right](1+2\beta\gamma-\beta)} \cdot \frac{(c)_{k+p}}{(a)_{k+p}} \quad (k \geq p)\,.$$

But
$\varphi(k+1) - \varphi(k) =$

$$\frac{2\beta\gamma(p-\alpha)^2(1-2\lambda)}{(1+2\beta\gamma-\beta)} \cdot \frac{(c)_{k+p}}{(a)_{k+p}}\,.$$

$$\cdot \left\{ \frac{(k+1)(a+k+p)\left[1+\lambda(\frac{k+1-p}{p})\right] - k(c+k+p)\left[1+\lambda(\frac{k-p}{p})\right]}{k(k+1)(a+k+p)\left[1+\lambda(\frac{k-p}{p})\right]\left[1+\lambda(\frac{k+1-p}{p})\right]} \right\} > 0$$

for $a > c > 0, -1 \leq B < A \leq 1, -1 \leq B \leq 0, 0 \leq \lambda < \frac{1}{2}$ and $p \in \mathbb{N}$. Then we see that $\varphi(k)$ is an increasing function of $k (k \geq p)$. Therefore, we conclude that

$$\zeta \leq \varphi(p) = p - \frac{2\beta\gamma(p-\alpha)^2(1-2\lambda)}{p(1+2\beta\gamma-\beta)} \cdot \frac{(c)_{2p}}{(a)_{2p}}\,,$$

which evidently completes the proof of Theorem 11. ∎

Theorem 12. *Let the function $f_1(z)$ defined by (18) be in the class $\Omega^+_{a,c}(p,\alpha_1,\beta,\gamma,\lambda)$. Suppose also that the function $f_2(z)$ defined by (18) be in the class $\Omega^+_{a,c}(p,\alpha_2,\beta,\gamma,\lambda)$, then $(f_1 \otimes f_2)(z) \in \Omega^+_{a,c}(p,\tau,\beta,\gamma,\lambda)$ where*

$$\tau = p - \frac{2\beta\gamma(p-\alpha_1)(p-\alpha_2)(1-2\lambda)}{p(1+2\beta\gamma-\beta)} \cdot \frac{(c)_{2p}}{(a)_{2p}} .$$

The result is sharp for the functions $f_j(z)(j=1,2)$ given by

$$f_1(z) = z^{-p} + \frac{2\beta\gamma(p-\alpha_1)(1-2\lambda)}{p(1+2\beta\gamma-\beta)} \cdot \frac{(c)_{2p}}{(a)_{2p}} z^p \quad (p \in \mathbb{N})$$

and

$$f_2(z) = z^{-p} + \frac{2\beta\gamma(p-\alpha_2)(1-2\lambda)}{p(1+2\beta\gamma-\beta)} \cdot \frac{(c)_{2p}}{(a)_{2p}} z^p \quad (p \in \mathbb{N}) .$$

Theorem 13. *Let the functions $f_j(z)(j=1,2)$ defined by (18) be in the class $\Omega^+_{a,c}(p,\alpha,\beta,\gamma,\lambda)$. Then the function $h(z)$ defined by*

$$h(z) = z^{-p} + \sum_{k=p}^{\infty}(|a_{k,1}|^2 + |a_{k,2}|^2)z^k$$

belongs to the class $\Omega^+_{a,c}(p,\xi,\beta,\gamma,\lambda)$, where

$$\xi = p - \frac{4\beta\gamma(p-\alpha)^2(1-2\lambda)}{p(1+2\beta\gamma-\beta)} \cdot \frac{(c)_{2p}}{(a)_{2p}} .$$

This result is sharp for the functions $f_j(z)(j=1,2)$ defined by (38).

Proof. Noting that

$$\sum_{k=p}^{\infty}\left(\frac{k\left[1+\lambda(\frac{k-p}{p})\right](1+2\beta\gamma-\beta)}{2\beta\gamma(p-\alpha)(1-2\lambda)} \cdot \frac{(a)_{k+p}}{(c)_{k+p}}\right)^2 |a_{k,j}|^2$$
$$\leq \left(\sum_{k=p}^{\infty}\frac{k\left[1+\lambda(\frac{k-p}{p})\right](1+2\beta\gamma-\beta)}{2\beta\gamma(p-\alpha)(1-2\lambda)} \cdot \frac{(a)_{k+p}}{(c)_{k+p}}|a_{k,j}|\right)^2 \leq 1 \quad (j=1,2),$$

for $f_j(z) \in \Omega^+_{a,c}(p,\alpha,\beta,\gamma,\lambda)(j=1,2)$, we have

$$\sum_{k=p}^{\infty}\frac{1}{2}\left[\frac{k\left[1+\lambda(\frac{k-p}{p})\right](1+2\beta\gamma-\beta)}{2\beta\gamma(p-\alpha)(1-2\lambda)}\right]^2\left(\frac{(a)_{k+p}}{(c)_{k+p}}\right)^2(|a_{k,1}|^2+|a_{k,2}|^2) \leq 1.$$

Therefore, we have to find the largest φ such that

$$\frac{1}{(p-\xi)} \leq \frac{k\left[1+\lambda(\frac{k-p}{p})\right](1+2\beta\gamma-\beta)}{4\beta\gamma(p-\alpha)^2(1-2\lambda)} \cdot \frac{(a)_{k+p}}{(c)_{k+p}} \quad (k \geq p),$$

that is, that

$$\xi \leq p - \frac{4\beta\gamma(p-\alpha)^2(1-2\lambda)}{k\left[1+\lambda(\frac{k-p}{p})\right](1+2\beta\gamma-\beta)} \cdot \frac{(c)_{k+p}}{(a)_{k+p}} \quad (k \geq p).$$

Now, defining a function $\Psi(k)$ by

$$\Psi(k) = p - \frac{4\beta\gamma(p-\alpha)^2(1-2\lambda)}{k\left[1+\lambda(\frac{k-p}{p})\right](1+2\beta\gamma-\beta)} \cdot \frac{(c)_{k+p}}{(a)_{k+p}} \quad (k \geq p).$$

We observe that $\Psi(k)$ is an increasing function of $k(k \geq p)$. Thus, we conclude that

$$\xi \leq \Psi(p) = p - \frac{4\beta\gamma(p-\alpha)^2(1-2\lambda)}{p(1+2\beta\gamma-\beta)} \cdot \frac{(c)_{2p}}{(a)_{2p}},$$

which completes the proof of Theorem 13. ∎

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