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WEAK CONVERGENCE OF PRODUCT OF SUMS OF INDEPENDENT VARIABLES WITH MISSING VALUES

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Abstract

Let (X_n) be a sequence of independent and non-identically distributed random variables. We assume that only observations of (X_n) at certain points are available. We study limit properties in the sense of weak convergence in the space D[0, 1] of certain processes based on an incomplete sample from $\{X_1, X_2 \dots, X_n\}$. This is an extension of the results of Matula and Stepien [2009. Weak convergence of products of sums of independent and non-identically distributed random variables. J. Math. Anal. Appl. 353, 49-54].

1 Introduction and the main result

Let $(X_n)_{n\geq 1}$ be a sequence of independent, positive and square integrable random variables. More, we assume that only observations at certain points are available. Denote observed random variables among $\{X_1, \dots, X_n\}$ by $\widetilde{X}_1$, ..., $\widetilde{X}_{N_n}$. Here the random variable N_n represents the number of observed rv's among the first n terms of the sequence (X_n). If every term of X_n is observed with probability p, independently of other terms, N_n is binomial random variable. But we shall assume that observed random variables are determined by a general point process. This model was considered in Mladenovic and Piterbarg (2008).

Assumption A. $X_1, X_2, ...$ does not depend on N_n and

$$\frac{N_n}{n} \xrightarrow{P} c_0 > 0 \qquad \text{as} \quad n \to +\infty.$$

Let us denote:

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$\mu_k = E\widetilde{X}_k$, $\tau_k^2 = Var\widetilde{X}_k$, $S_{N_n} = \sum_{i=1}^{N_n} \widetilde{X}_i$ and $\sigma_{N_n}^2 = Var(S_{N_n})$.

Theorem 1. Let $\widetilde{X}_1$, ..., $\widetilde{X}_{N_n}$ represent the sequence of the observed random variables from the sequence of independent, positive and square integrable random variables. Let $ES_{N_n} \longrightarrow \infty$ as $n \to \infty$. Also assume that these variables satisfy next conditions:

$$\lim_{n\to\infty} \sigma_{N_n}^{-2} E\{\sum_{i=1}^{N_n} (\widetilde{X}_i - E\widetilde{X}_i)^2 I(|\widetilde{X}_i - E\widetilde{X}_i| \geq \varepsilon\sigma_{N_n})\} = 0, \qquad \text{for } \varepsilon > 0, \tag{1}$$

$$\sum_{j=1}^{\infty} p_j \sum_{i=1}^{j} \frac{\tau_{i+1}^2}{(ES_i)^2} < \infty. \tag{2}$$

and

$$\frac{\sum_{i=1}^{N_n} \tau_i^2}{\sigma_{N_n}^2} \xrightarrow{P} 1 \tag{3}$$

where $P\{N_n = j\} = p_j$.

Then

$$Z_{N_n}(t) := (\prod_{i=1}^{M_{N_n}(t)} (\frac{S_i}{ES_i})^{\frac{\tau_{i+1}^2 ES_i}{\sigma_i^2}})^{1/\sigma_{N_n}} \xrightarrow{d} exp(\int_0^t \frac{W(x)}{x} dx) \tag{4}$$

in D[0,1] as $n \to \infty$, where $M_{N_n}(t) = \max\{i \leq N_n | \sigma_i^2 \leq t\sigma_N^2\}$ and $Z_{N_n}(t) = 1$ for t such that $M_{N_n}(t) = 0$.

Remark. Condition (1.1) is called random Lindeberg condition and it is defined in Rychlik (1979).

Proof. Since $\log(1+x) = x + R(x)$, for $|x| \leq 1/2$, where $|R(x)| \leq 2x^2$ and by putting $C_i = \frac{S_i - ES_i}{ES_i}$ we get:

$$\log(\frac{S_i}{ES_i}) = C_i + R(C_i)I[|C_i| \leq 1/2] + (\log(1+C_i) - C_i)I[|C_i| > 1/2].$$

Now we have:

$$\log Z_{N_n}(t) = \frac{1}{\sigma_{N_n}} \sum_{i=1}^{M_{N_n}(t)} \frac{\tau_{i+1}^2 ES_i}{\sigma_i^2} \log(\frac{S_i}{ES_i})$$

$$= \frac{1}{\sigma_{N_n}} \sum_{i=1}^{M_{N_n}(t)} \frac{\tau_{i+1}^2 ES_i}{\sigma_i^2} C_i + \frac{1}{\sigma_{N_n}} \sum_{i=1}^{M_{N_n}(t)} \frac{\tau_{i+1}^2 ES_i}{\sigma_i^2} R(C_i) I[|C_i| \le 1/2]$$

$$+ \frac{1}{\sigma_{N_n}} \sum_{i=1}^{M_{N_n}(t)} \frac{\tau_{i+1}^2 ES_i}{\sigma_i^2} (\log(1+C_i) - C_i) I[|C_i| > 1/2]$$

Let us denote

$$A'_N(t) = \frac{1}{\sigma_{N_n}} \sum_{i=1}^{M_{N_n}(t)} \frac{\tau_{i+1}^2 ES_i}{\sigma_i^2} R(C_i) I(|C_i| \le 1/2)$$

and

$$A''_N(t) = \frac{1}{\sigma_{N_n}} \sum_{i=1}^{M_{N_n}(t)} \frac{\tau_{i+1}^2 ES_i}{\sigma_i^2} (\log(1+C_i) - C_i) I[|C_i| > 1/2].$$

We have that:

$$E \max_{0 \le t \le 1} |A'_N(t)| \le E \max_{0 \le t \le 1} \frac{1}{\sigma_{N_n}} \sum_{i=1}^{M_{N_n}(t)} \frac{\tau_{i+1}^2 ES_i}{\sigma_i^2} |R(C_i)| I(|C_i| \le 1/2)$$

$$\le E \frac{1}{\sigma_{N_n}} \sum_{i=1}^{N_n} \frac{\tau_{i+1}^2 ES_i}{\sigma_i^2} |R(C_i)| I(|C_i| \le 1/2)$$

$$= \frac{1}{\sigma_{N_n}} \sum_{j=1}^{\infty} E(\sum_{i=1}^{N_n} \frac{\tau_{i+1}^2 ES_i}{\sigma_i^2} |R(C_i)| I(|C_i| \le 1/2) | N_n = j) P\{N_n = j\}$$

$$= \frac{1}{\sigma_{N_n}} \sum_{j=1}^{\infty} E(\sum_{i=1}^{j} \frac{\tau_{i+1}^2 ES_i}{\sigma_i^2} |R(C_i)| I(|C_i| \le 1/2)) P\{N_n = j\}$$

$$\le \frac{2}{\sigma_{N_n}} \sum_{j=1}^{\infty} \sum_{i=1}^{j} \frac{\tau_{i+1}^2 ES_i}{\sigma_i^2} E(\frac{S_i - ES_i}{ES_i})^2 P\{N_n = j\}$$

$$= \frac{2}{\sigma_{N_n}} \sum_{j=1}^{\infty} p_j \sum_{i=1}^{j} \frac{\tau_{i+1}^2}{ES_i}.$$

In order to prove that $\lim_{n\to\infty} \frac{2}{\sigma_{N_n}} \sum_{j=1}^{n} p_j \sum_{i=1}^{j} \frac{\tau_{i+1}^2}{ES_i} = 0$ let $\varepsilon > 0$. According to (1.2) we can find n_0 such that $\sum_{j=n_0}^{\infty} p_j \sum_{i=1}^{j} \frac{\tau_{i+1}^2}{(ES_i)^2} < \varepsilon$.

Now, for every $n > n_0$ we have:

$$\frac{2}{\sigma_{N_n}} \sum_{j=1}^{n} p_j \sum_{i=1}^{j} \frac{\tau_{i+1}^2}{ES_i} = \frac{2}{\sigma_{N_n}} \sum_{j=1}^{n_0-1} p_j \sum_{i=1}^{j} \frac{\tau_{i+1}^2}{ES_i} + \frac{2}{\sigma_{N_n}} \sum_{j=n_0}^{n} p_j \sum_{i=1}^{j} \frac{\tau_{i+1}}{ES_i} \tau_{i+1}$$

$$\le \frac{2}{\sigma_{N_n}}\sum_{j=1}^{n_0-1}p_j\sum_{i=1}^{j}\frac{\tau_{i+1}^2}{ES_i}+\frac{2}{\sigma_{N_n}}\{\sum_{j=n_0}^{n}p_j(\sum_{i=1}^{j}\frac{\tau_{i+1}^2}{(ES_i)^2})^{1/2}(\sum_{i=1}^{j}\tau_{i+1}^2)^{1/2}\}$$

$$=\frac{2}{\sigma_{N_n}}\sum_{j=1}^{n_0-1}p_j\sum_{i=1}^{j}\frac{\tau_{i+1}^2}{ES_i}+\frac{2}{\sigma_{N_n}}\{\sum_{j=n_0}^{\infty}(p_j\sum_{i=1}^{j}\frac{\tau_{i+1}^2}{(ES_i)^2})^{1/2}(p_j\sum_{i=1}^{j}\tau_{i+1}^2)^{1/2}\}$$

$$\le \frac{2}{\sigma_{N_n}}\sum_{j=1}^{n_0-1}p_j\sum_{i=1}^{j}\frac{\tau_{i+1}^2}{ES_i}+\frac{2}{\sigma_{N_n}}(\sum_{j=n_0}^{\infty}p_j\sum_{i=1}^{j}\frac{\tau_{i+1}^2}{(ES_i)^2})^{1/2}(\sum_{j=n_0}^{\infty}p_j\sum_{i=1}^{j}\tau_{i+1}^2)^{1/2}$$

$$\le \frac{2}{\sigma_{N_n}}\sum_{j=1}^{n_0-1}p_j\sum_{i=1}^{j}\frac{\tau_{i+1}^2}{ES_i}+\frac{2}{\sigma_{N_n}}\sqrt{\varepsilon}\sigma_{N_n+1},$$

since we have that:

$$\sigma_{N_n+1}^2=Var(\sum_{i=1}^{N_n+1}\widetilde{X}_i)=E(\sum_{i=1}^{N_n+1}\widetilde{X}_i)^2-(E(\sum_{i=1}^{N_n+1}\widetilde{X}_i))^2$$

$$=\sum_{j=1}^{\infty}p_jE(\sum_{i=1}^{j+1}\widetilde{X}_i)^2-(\sum_{j=1}^{\infty}p_j\sum_{i=1}^{j+1}E\widetilde{X}_i)^2$$

$$=\sum_{j=1}^{\infty}p_j\sum_{i=1}^{j+1}\tau_i^2+\sum_{j=1}^{\infty}p_j(\sum_{i=1}^{j+1}E\widetilde{X}_i)^2-(\sum_{j=1}^{\infty}p_j\sum_{i=1}^{j+1}E\widetilde{X}_i)^2$$

$$\ge\sum_{j=1}^{\infty}p_j\sum_{i=1}^{j+1}\tau_i^2\ge\sum_{j=n_0}^{\infty}p_j\sum_{i=1}^{j}\tau_{i+1}^2.$$

Also, (1.1) and (1.3) allow us (see Rychlik, Theorem 2) to use the fact that

$$\lim_{n\to\infty}E(\max_{1\le i\le N_n}\frac{\tau_i^2}{\sigma_{N_n}^2})=0, \tag{5}$$

which implies

$$\frac{\sigma_{N_n+1}^2}{\sigma_{N_n}^2}\to 1 \tag{6}$$

as $n\to\infty$.

Therefore we have:

$$\limsup_{n\to\infty}\frac{2}{\sigma_{N_n}}\sum_{j=1}^{n}p_j\sum_{i=1}^{j}\frac{\tau_{i+1}^2}{ES_i}\le 2\varepsilon$$

for arbitrarily $\varepsilon>0$, proving that

$$E \max_{0\le t\le 1} |A'_N(t)| \to 0, \tag{7}$$

as $n \to \infty$.

Now, by the assumption (1.2) and the inequality $\frac{\tau_{i+1}^2}{(ES_{i+1})^2} < \frac{\tau_{i+1}^2}{(ES_i)^2}$ we have that $\sum_{j=1}^{\infty} p_j \sum_{i=1}^{j} \frac{\tau_i^2}{(ES_i)^2} < \infty$.

According to Theorem 6.7 of Petrov we have that $C_{N_n} = \frac{S_{N_n} - ES_{N_n}}{ES_{N_n}} \xrightarrow{a.s} 0$. Therefore $I(|C_i| > 1/2) = 0$ almost surely for sufficiently large i and

$$E \max_{0\le t\le 1} |A''_N(t)| \le \frac{1}{\sigma_{N_n}} E \sum_{i=1}^{N_n} \frac{\tau_{i+1}^2 ES_i}{\sigma_i^2} (\log(1+C_i) - C_i) I[|C_i| > 1/2]$$

$$= \frac{1}{\sigma_{N_n}} \sum_{j=1}^{n} p_j E \sum_{i=1}^{j} \frac{\tau_{i+1}^2 ES_i}{\sigma_i^2} (\log(1+C_i) - C_i) I[|C_i| > 1/2]$$

Thus

$$E \max_{0\le t\le 1} |A''_N(t)| \to 0, \tag{8}$$

as $n \to \infty$.

Now it is suffices to show that

$$Y_{N_n}(t) := \frac{1}{\sigma_{N_n}} \sum_{i=1}^{M_{N_n}(t)} \frac{\tau_{i+1}^2}{\sigma_i^2} (S_i - ES_i) \xrightarrow{d} \int_0^t \frac{W(x)}{x} \tag{9}$$

in D[0,1] as $n \to \infty$.

For fixed $\varepsilon > 0$ we define a process

$$Y_{N_n,\varepsilon}(t) = \begin{cases} \frac{1}{\sigma_{N_n}} \sum\limits_{i=M_{N_n}(\varepsilon)+1}^{M_{N_n}(t)} \frac{\tau_{i+1}^2}{\sigma_i^2} (S_i - ES_i), & \varepsilon < t \le 1, \\ 0, & 0 \le t \le \varepsilon. \end{cases}$$

We have:

$$E \max_{0\le t\le 1} |Y_{N_n}(t) - Y_{N_n,\varepsilon}(t)| \le E \frac{1}{\sigma_{N_n}} \sum_{i=1}^{M_{N_n}(\varepsilon)} \frac{\tau_{i+1}^2}{\sigma_i^2} |S_i - ES_i|$$

$$= \frac{1}{\sigma_{N_n}} \sum_{m_n(\varepsilon)=1}^{\infty} p_j E \sum_{i=1}^{m_n(\varepsilon)} \frac{\tau_{i+1}^2}{\sigma_i^2} |S_i - ES_i| \le \frac{1}{\sigma_{N_n}} \sum_{m_n(\varepsilon)=1}^{\infty} p_j \sum_{i=1}^{m_n(\varepsilon)} \frac{\tau_{i+1}^2}{\sigma_i}$$

$$= \frac{1}{\sigma_{N_n}} E \sum_{i=1}^{M_{N_n}(\varepsilon)} \frac{\tau_{i+1}^2}{\sigma_i} = \frac{\sigma_{M_{N_n}(\varepsilon)}}{\sigma_{N_n}} \frac{\sigma_{M_{N_n}(\varepsilon)+1}}{\sigma_{M_{N_n}(\varepsilon)}} \frac{1}{\sigma_{M_{N_n}(\varepsilon)+1}} E \sum_{i=1}^{M_{N_n}(\varepsilon)} \frac{\tau_{i+1}^2}{\sigma_i}. \tag{10}$$

According to (1.3) we have that

$$\frac{\tau_{N_n+1}^2}{\sigma_{N_n+1}^2 - \sigma_{N_n}^2} \xrightarrow{P} 1, \tag{11}$$

since

$$\frac{\tau_{N_n+1}^2}{\sigma_{N_n+1}^2 - \sigma_{N_n}^2} = \frac{\sum_{i=1}^{N_n+1} \tau_i^2 - \sum_{i=1}^{N_n} \tau_i^2}{\sigma_{N_n+1}^2 - \sigma_{N_n}^2}$$

$$= \frac{\frac{\sigma_{N_n+1}^2}{\sigma_{N_n+1}^2} \sum_{i=1}^{N_n+1} \tau_i^2 - \frac{\sigma_{N_n}^2}{\sigma_{N_n}^2} \sum_{i=1}^{N_n} \tau_i^2}{\sigma_{N_n+1}^2 - \sigma_{N_n}^2} \xrightarrow{P} 1.$$

Let us put $X_{N_n} = E \sum_{i=1}^{N_n} \frac{\tau_{i+1}^2}{\sigma_i}$ and $Y_{N_n} = \sigma_{N_n+1}$.
We have according to (1.11):

$$\frac{X_{N_n} - X_{N_n-1}}{Y_{N_n} - Y_{N_n-1}} = \frac{E(\frac{\tau_{N_n+1}^2}{\sigma_{N_n+1}})}{\sigma_{N_n+1} - \sigma_{N_n}} = \frac{E(\frac{\tau_{N_n+1}^2}{\sigma_{N_n+1}})(\sigma_{N_n+1} + \sigma_{N_n})}{\sigma_{N_n+1}^2 - \sigma_{N_n}^2} \longrightarrow 2.$$

Now applying Stolz theorem to sequences of real numbers X_{N_n} and Y_{N_n} we have that:

$$\frac{1}{\sigma_{M_{N_n+1}(\varepsilon)}} E \sum_{i=1}^{M_{N_n}(\varepsilon)} \frac{\tau_{i+1}^2}{\sigma_i} \longrightarrow 2. \tag{12}$$

as $n \to \infty$.

Also we have, by the definition of the function $M_{N_n(t)}$ that: $\frac{\sigma_{M_{N_n(\varepsilon)}}^2}{\sigma_{N_n}^2} \leq \varepsilon$.

Finally we have from (1.10), (1.12) and the above remark that:

$$\limsup_{n\to\infty} E \max_{0\leq t\leq 1} |Y_{N_n}(t) - Y_{N_n,\varepsilon}(t)| \leq 2\sqrt{\varepsilon}. \tag{13}$$

Now we shall prove that

$$\int_\varepsilon^t \frac{W_{N_n}(x)}{x} dx = \frac{1}{\sigma_{N_n}} \sum_{i=M_{N_n}(\varepsilon)+1}^{M_{N_n}(t)} \frac{\tau_{i+1}^2}{\sigma_i^2}(S_i - ES_i) + B_{N_n(t)} \tag{14}$$

where $B_{N_n(t)} \xrightarrow{P} 0$ in $D[0,1]$, as $n \to \infty$ and $W_{N_n}(t) = \sum_{k=1}^{M_{N_n}(t)} \frac{(X_k - \mu_k)}{\sigma_{N_n}}$.

We have:

$$\int_{\varepsilon}^{t} \frac{W_{N_n}(x)}{x} dx = \int_{\varepsilon}^{\frac{\sigma^2_{M_{N_n}(\varepsilon)+1}}{\sigma^2_{N_n}}} \frac{W_{N_n}(x)}{x} dx + \sum_{i=M_{N_n}(\varepsilon)+1}^{M_{N_n}(t)} \int_{\frac{\sigma_i^2}{\sigma^2_{N_n}}}^{\frac{\sigma^2_{i+1}}{\sigma^2_{N_n}}} \frac{W_{N_n}(x)}{x} dx$$

$$- \int_{t}^{\frac{\sigma^2_{M_{N_n}(t)+1}}{\sigma^2_{N_n}}} \frac{W_{N_n}(x)}{x} dx$$

$$= \frac{S_{M_{N_n}(\varepsilon)} - ES_{M_{N_n}(\varepsilon)}}{\sigma_{N_n}} \log \frac{\sigma^2_{M_{N_n}(\varepsilon)+1}}{\varepsilon \sigma^2_{N_n}} + \sum_{i=M_{N_n}(\varepsilon)+1}^{M_{N_n}(t)} \frac{S_i - ES_i}{\sigma_{N_n}} \log \frac{\sigma^2_{i+1}}{\sigma^2_i}$$

$$- \frac{S_{M_{N_n}(t)} - ES_{M_{N_n}(t)}}{\sigma_{N_n}} \log \frac{\sigma^2_{M_{N_n}(t)+1}}{t \sigma^2_{N_n}}$$

Let us denote:

$$B'_{N_n} = \frac{S_{M_{N_n}(\varepsilon)} - ES_{M_{N_n}(\varepsilon)}}{\sigma_{N_n}} \log \frac{\sigma^2_{M_{N_n}(\varepsilon)+1}}{\varepsilon \sigma^2_{N_n}},$$

$$Y'_{N_n,\varepsilon}(t) = \sum_{i=M_{N_n}(\varepsilon)+1}^{M_{N_n}(t)} \frac{S_i - ES_i}{\sigma_{N_n}} \log \frac{\sigma^2_{i+1}}{\sigma^2_i}$$

and

$$B''_{N_n} = \frac{S_{M_{N_n}(t)} - ES_{M_{N_n}(t)}}{\sigma_{N_n}} \log \frac{\sigma^2_{M_{N_n}(t)+1}}{t \sigma^2_{N_n}}.$$

For $t \in (0,1)$ we have

$$1 < \frac{\sigma^2_{M_{N_n}(t)+1}}{t \sigma^2_{N_n}} \leq 1 + \frac{E(\tau^2_{M_{N_n}(t)+1})}{t \sigma^2_{N_n}} \leq 1 + \frac{1}{t} \frac{E(\max_{1 \leq k \leq N_n+1} \tau_k^2)}{\sigma^2_{N_n}} \tag{15}$$

and according to (1.5) we have that

$$E|B'_{N_n}| \to 0. \tag{16}$$

as $n \to \infty$.

Also, for $t \in (\varepsilon, 1)$ and for fixed $\delta > 0$ we have:

$$P(\max_{\varepsilon \leq t \leq 1} |B''_{N_n(t)}| \geq \delta) \leq P(\frac{1}{\varepsilon} \frac{\max_{1 \leq k \leq N_n+1} \tau_k^2}{\sigma^2_{N_n}} \max_{\varepsilon \leq t \leq 1} |\frac{S_{M_{N_n}(t)} - ES_{M_{N_n}(t)}}{\sigma_{N_n}}| \geq \delta)$$

$$\leq Var(\frac{S_{N_n} - ES_{N_n}}{\sigma_{N_n}})\frac{1}{\varepsilon^2\delta^2}E(\frac{\max_{1\leq k\leq N_n+1}\tau_k^2}{\sigma_{N_n}^2} \geq \delta)^2 \to 0, \quad \text{as } n \to \infty, \tag{17}$$

by the Kolmogorov inequality and (1.5).

Since $\frac{\tau_{i+1}^2}{\sigma_i^2} \to 0$ we can find i_0 such that for $i \geq i_0$ we have $\frac{\tau_{i+1}^2}{\sigma_i^2} \leq 1/2$. So, if we take sufficiently large N_n (such that $P(M_{N_n+1} \geq i_0) = 1$) we obtain:

$$E\max_{\varepsilon\leq t\leq 1}|Y_{N_n,\varepsilon}(t) - Y'_{N_n,\varepsilon}(t)| \leq \frac{1}{\sigma_{N_n}}E\sum_{i=1}^{N_n}(S_i - ES_i)(\log(\frac{\sigma_i^2 + \tau_{i+1}^2}{\sigma_i^2}) - \frac{\tau_{i+1}^2}{\sigma_i^2})$$

$$\leq \frac{1}{\sigma_{N_n}}\sum_{j=1}^{\infty}p_j\sum_{i=1}^{j}E|S_i - ES_i||\log(\frac{\sigma_i^2 + \tau_{i+1}^2}{\sigma_i^2}) - \frac{\tau_{i+1}^2}{\sigma_i^2}|$$

$$\leq \frac{1}{\sigma_{N_n}}\sum_{j=1}^{\infty}p_j\sum_{i=1}^{j}E|S_i - ES_i||\log(\frac{\sigma_i^2 + \tau_{i+1}^2}{\sigma_i^2}) - \frac{\tau_{i+1}^2}{\sigma_i^2}| \leq \frac{2}{\sigma_{N_n}}E\sum_{j=1}^{N_n}\frac{\tau_{i+1}^4}{\sigma_i^3}. \tag{18}$$

Let us put : $X_{N_n} = E\sum_{i=1}^{N_n}\frac{\tau_{i+1}^4}{\sigma_i^3}$ and $Y_{N_n} = \sigma_{N_n+1}$. Similarly as in (1.11) we conclude that :

$$\frac{X_{N_n} - X_{N_n-1}}{Y_{N_n} - Y_{N_n-1}} = \frac{E(\frac{\tau_{N_n+1}^4}{\sigma_{N_n}^3})}{\sigma_{N_n+1} - \sigma_{N_n}}$$

$$= \frac{E(\frac{\tau_{N_n+1}^4}{\sigma_{N_n}^3})(\sigma_{N_n+1} + \sigma_{N_n})}{\sigma_{N_n+1}^2 - \sigma_{N_n}^2} \to 0.$$

Again applying Stolz theorem to sequences of real numbers X_{N_n} and Y_{N_n} we have that:

$$E\max_{\varepsilon\leq t\leq 1}|Y_{N_n,\varepsilon}(t) - Y'_{N_n,\varepsilon}(t)| \to 0, \quad \text{as} \quad n \to \infty. \tag{19}$$

Now (1.14) follows from (1.16), (1.17) and (1.19).

Finally, for a fixed $\varepsilon > 0$ define mapping:

$$H_\varepsilon(f)(t) = \begin{cases} \int_\varepsilon^t \frac{f(x)}{x}dx, & \varepsilon < t \leq 1, \\ 0, & 0 \leq t \leq \varepsilon. \end{cases}$$

By continuity of the mapping $H_\varepsilon(.)$ on D[0,1] we get:

$$Y_{N_n,\varepsilon}(t) \xrightarrow{d} H_\varepsilon(W)(t)$$

in D[0,1] as $n \to \infty$. Further proof can be lead similarly as in Li-Xin Zhang.

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