(STRONG) WEAK EXHAUSTIVENESS AND (STRONG UNIFORM) CONTINUITY

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Abstract

In this paper we continue, in the realm of metric spaces, the study of exhaustiveness and weak exhaustiveness at a point of a net of functions initiated by Gregoriades and Papanastassiou in 2008. We prove that exhaustiveness at every point of a net of pointwise convergent functions is equivalent to uniform convergence on compacta. We extend exhaustiveness-type properties to subsets. First, we introduce the notion of strong exhaustiveness at a subset B for sequences of functions and prove its equivalence with strong exhaustiveness at $\mathcal{P}_0(B)$ of the sequence of the direct image maps, where the hypersets are equipped with the Hausdorff metric. Furthermore, we show that the notion of strong-weak exhaustiveness at a subset is the proper tool to investigate when the limit of a pointwise convergent sequence of functions fulfills the strong uniform continuity property, a new pregnant form of uniform continuity discovered by Beer and Levi in 2009.

1 Introduction

In 2008 Gregoriades and Papanastassiou introduced the notion of exhaustiveness at a point of a metric space both for sequences and nets of functions (see [9]). This new notion is closely related to equicontinuity and enables to consider the convergence of a net of functions in terms of properties of the whole net and not as properties of functions as single members. Exhaustiveness is a powerful tool to state Ascoli-type theorems and to describe the relation between pointwise convergence for functions and continuous convergence. So, in the third section, we continue the investigation of exhaustiveness at a point. We prove that exhaustiveness for a net of functions at every point of the domain is the property that must be added to pointwise convergence to have uniform convergence on compacta.

But, we are interested to the weaker notion of weak exhaustiveness at a point which gives a necessary and sufficient condition under which the pointwise limit of a sequence of arbitrary function (not necessarily continuous) is continuous (see [9]). This is a novel machinery to study the fundamental conundrum of Analysis: what precisely must be added to pointwise convergence of a sequence of continuous functions to preserve continuity. In 1883 Arzelá solved the problem ([2], [3]) and paved the way to several outstanding papers (for a comprehensive approach the interested reader may consult [7]). A more appropriate question to ask in this setting is the following: is there any topology on Y^X finer than pointwise convergence that has as intrinsic property to preserve continuity? The answer to this question was given by Bouleau's work in [10] and it falls out from a general theory. He introduced the sticky topology on $C(X,Y)$ as the coarsest topology preserving continuity. Its convergence is described by a criterion of convergence which as the Cauchy criterion does not involve the limit. We point out that also exhaustiveness and weakly exhaustiveness of a sequence of functions are defined in terms of properties of the whole sequence and not of properties of the functions as single members.

In the third section, among other results, we offer a direct proof of the equivalence for nets of arbitrary pointwise convergent functions between sticky convergence and weakly exhaustiveness at every point. The proof reveals the internal gear of these two formally far away conditions.

In the fourth section we introduce two notions of strong exhaustiveness at a subset B of a metric space (X,d). We investigate the relation among: the strong exhaustiveness of a sequence of functions $f_n : X \to Y$ at B, the properties at $\mathcal{P}_0(B)$ of the associated image maps $\hat{f}_n : \mathcal{P}_0(X) \to \mathcal{P}_0(Y)$, where the hypersets are equipped with the corresponding Hausdorff pseudometric, and the limit of $D_d(f_n(A), f_n(C))$, where A and C are near subsets of X. Finally, we study the notion of strong uniform continuity on a subset B of X, a pregnant notion introduced by Beer and Levi in 2009 [5], a strengthen of the notion of uniform continuity on B. This novel notion, which models the behavior of a continuous function on the enlargements of a compact subset of X, has several applications and plays an important role in General Topology and enables the notion of strong uniform convergence on bornologies ([5]). Recall that a bornology $\mathcal{B}$ on a metric space (X,d) is a family of subsets of X that is closed under taking finite unions, is hereditary and forms a cover of X (see [5], [6], [8], [12]). Note that since each bornology $\mathcal{B}$ contains singletons, we automatically get pointwise convergence whatever $\mathcal{B}$ may be. We prove that the pointwise limit of a sequence of functions is strongly uniformly continuous at B if and only if the sequence is strongly-weakly exhaustive at B.

As a result we solve the following question: what exactly must be added to pointwise convergence of a sequence of arbitrary functions to have the uniform continuity of the limit.

2 Preliminaries

All metric spaces (X, d) are assumed to contain at least two points. We denote the power set of a set A by $\mathcal{P}(A)$, and the nonempty subsets of A by $\mathcal{P}_0(A)$. If (Y, ρ) is a second metric space, we again denote by Y^X the set of all functions from X to Y which is (except when all points of X are isolated or Y is a singleton) properly larger than $C(X, Y)$, the set of all continuous functions from X to Y. Given any function $f \in Y^X$, the associated direct image map $\hat{f} : \mathcal{P}_0(X) \to \mathcal{P}_0(Y)$ is defined by $\hat{f}(A) = \{f(a) : a \in A\}$.

If $x_0 \in X$ and $\epsilon > 0$, we write $S(x_0, \epsilon)$ for the open ϵ-ball with center x_0. If A is a nonempty subset of X, we write $d(x_0, A)$ for the distance from x_0 to A, and if $A = \emptyset$ we agree that $d(x_0, A) = \infty$. We denote the ϵ-enlargement of A by

$$A^\epsilon = \{x : d(x, A) < \epsilon\} = \bigcup_{x \in A} S(x, \epsilon).$$

We define the Hausdorff distance between two nonempty subsets A and B in terms of enlargements:

$$H_d(A, B) = \inf\{\epsilon > 0 : A \subset B^\epsilon \ and \ B \subset A^\epsilon\}.$$

The Hausdorff distance so defined is an extended real valued pseudometric on $\mathcal{P}_0(X)$. Furthermore, the map $x \to \{x\}$ is an isometry of X into $\mathcal{P}_0(X)$.

If A and B are nonempty subsets of X, we define the gap between A and B by the formula:

$$D_d(A, B) = \inf\{d(a, b) : a \in A, b \in B\}$$

The set A and B are called near if $D_d(A, B) = 0$ (the relation of nearness so defined is called the metric proximity determined by d). A result that we will employ with respect to nearness is the celebrated Efremovich Lemma ([13]):

Lemma 2.1. *Let (X, d) be a metric space and $(x_n), (w_n)$ be sequences such that for every $n \in \mathbb{N}$ $d(x_n, w_n) > \epsilon$. Then there is an infinite subset N_1 of $\mathbb{N}$ such that for every $\{n, k\} \subset N_1$ we have $d(x_n, w_k) \geq \epsilon/4$. In particular $\{x_n : n \in N_1\}$ and $\{w_n : n \in N_1\}$ are not near.*

In 2009 Beer and Levi introduced in the realm of metric spaces the novel and powerful notion of strong uniform continuity on a subset B (see [5]).

Definition 2.2. Let (X, d) and (Y, ρ) be a metric space and let B be a subset of X. A function $f : X \to Y$ is *strongly uniformly continuous on B* if for every $\epsilon > 0$ there is $\delta > 0$ such that if $d(x, w) < \delta$ and $\{x, w\} \cap B \neq \emptyset$, then $\rho(f(x), f(w)) < \epsilon$.

Observe that strong uniform continuity od f on $\{x_0\}$ is equivalent to continuity at x_0, while strong uniform continuity on X is equivalent to global uniform continuity. Rephrasing, a bornology $\mathcal{B}$ on X is a family of non empty subsets of X that contains singletons, that is stable with respect to finite unions and whenever $B \in \mathcal{B}$ and $B_0 \subset B$, then $B_0 \in \mathcal{B}$. The smallest bornology on X is the family of

finite subsets of X, $\mathcal{F}_0(X)$, and the largest is the family of all nonempty subsets of X, $\mathcal{P}_0(X)$. Other important bornologies are: the family of nonempty d-bounded subsets, the family of nonempty d-totally bounded subsets and the family $\mathcal{K}$ of nonempty subsets of X with compact closure. By a base $\mathcal{B}_0$ for a bornology $\mathcal{B}$ we mean a subfamily of $\mathcal{B}$ that is cofinal with respect to inclusion, i.e. for each $B \in \mathcal{B}$ there is $B_0 \in \mathcal{B}_0$ such that $B \subset B_0$. If $\mathcal{B}$ is a family of nonempty subsets of X and (Y, ρ) a metric space, a function $f \in Y^X$ is called uniformly continuous (resp. strongly uniformly continuous) on $\mathcal{B}$ if for each $B \in \mathcal{B}$, $f \upharpoonright B$ is uniformly continuous (resp. strongly uniformly continuous) on B. Of course if $f \in C(X, Y)$ and K is a compact subset of X, then f is strongly uniformly continuous on K. Since $f \in Y^X$ is continuous at x if and only if f is strongly uniformly continuous at $\{x\}$, just looking at the bornology $\mathcal{F}_0(X)$ of finite sets, we see that strong uniform continuity on $\mathcal{F}_0(X)$, i.e. global continuity, is a stronger requirement than uniform continuity on $\mathcal{F}_0(X)$ which amounts to no requirement at all.

In our analysis a key role is played by the bornology of UC-sets introduced by Beer and Levi in [5]. We recall that a subset A of X is an UC-subset if and only if each function f continuous on X is strongly uniformly continuous on A.

3   Exhaustiveness at points

Let $\mathcal{B}$ be a bornology with closed base on X. The classical uniformity for the topology $\tau_{\mathcal{B}}$ of uniform convergence on $\mathcal{B}$ for $C(X, Y)$ has as a base for its entourages all sets of the form

$$[B, \epsilon] := \{(f, g) : \text{ for every } x \in B \ \rho(f(x), g(x)) < \epsilon\}, \quad (B \in \mathcal{B}, \epsilon > 0).$$

When $\mathcal{B} = \mathcal{F}$, we get the standard uniformity for the topology of pointwise convergence, $\tau_{\mathcal{F}} = \tau_p$; when $\mathcal{B} = \mathcal{K}$, we get the standard uniformity for the topology of uniform convergence on compacta, $\tau_{\mathcal{K}} = \tau_k$; when $\mathcal{B} = \mathcal{P}_0(X)$, we get the standard uniformity for the topology of uniform convergence on X, $\tau_{\mathcal{P}_0(X)} = \tau_u$. These uniformities make sense on Y^X as well.

Given a bornology $\mathcal{B}$ with closed base on X, Beer and Levi present a new uniformizable topology on the set of all functions Y^X from X to Y that preserves strong uniform continuity on $\mathcal{B}$.

Definition 3.1. [5] Let (X, d) and (Y, ρ) be metric spaces and let $\mathcal{B}$ be a bornology with closed base on X. Then the *topology of strong uniform convergence* $\tau_{\mathcal{B}}^s$ is determined by a uniformity on Y^X having as a base all sets of the form

$$[B, \epsilon]^s := \{(f, g) : \exists \delta > 0 \text{ for every } x \in B^\delta \ \rho(f(x), g(x)) < \epsilon\}, \ (B \in \mathcal{B}, \epsilon > 0).$$

When the bornology $\mathcal{B}$ is equal to the family of all finite subsets of X, $\mathcal{F}_0(X)$, the strong uniform convergence on $\mathcal{F}_0(X)$ is equal to the sticking topology τ_S defined by Bouleau in [10] and [11]. We point out that these topologies are the coarsest topologies preserving continuity of nets of continuous functions.

In the space $X = \mathbb{R}^{\mathbb{R}_+}$ of functions from $[0, \infty)$ into $\mathbb{R}$, Bouleau proved that there exists for the sticking topology a criterion of convergence which, as the Cauchy criterion, does not involve the limit. This criterion can be extended to nets in the setting of metric spaces.

Proposition 3.2. *Let (X, d) and (Y, ρ) be metric spaces and $(f_\alpha)_{\alpha \in \Lambda}$ be a net of functions from X to Y. The net $(f_\alpha)_{\alpha \in \Lambda}$ converges sticky if and only if for every x and for every $\epsilon > 0$ there is α_0 such that for all $\alpha > \alpha_0$ there exists V_x, a neighbourhood of x, with the property that for all $y \in V_x$ there exists β_y such that for all $\beta \geq \beta_y$ we have $\rho(f_\beta(y), f_\alpha(y)) < \epsilon$.*

We recall the notion of exhaustiveness which applies for both families and nets of functions [9].

Definition 3.3. Let (X, d) and (Y, ρ) be metric spaces and $(f_\alpha)_{\alpha \in \Lambda}$ be a net of functions from X to Y. $(f_\alpha)_{\alpha \in \Lambda}$ is *exhaustive at x* if for every $\epsilon > 0$ there is $\delta > 0$ and α_0 such that for all $y \in S(x, \delta)$ and for all $\alpha \geq \alpha_0$ we have $\rho(f_\alpha(y), f_\alpha(x)) < \epsilon$. The net $(f_\alpha)_{\alpha \in \Lambda}$ is *exhaustive* if it is exhaustive at every $x \in X$.

In [9] Gregoriades and Papanastassiou proved the following proposition:

Proposition 3.4. *Let (X, d) and (Y, ρ) be metric spaces and $(f_n)_{n \in \omega}$, f be functions from X to Y. If the sequence $(f_n)_{n \in \omega}$ pointwise converges to f and $(f_n)_{n \in \omega}$ is exhaustive at x, then f is continuous at x.*

We have the following proposition:

Proposition 3.5. *Let (X, d) and (Y, ρ) be metric spaces and $(f_n)_{n \in \omega}$, f be functions from X to Y. The following are equivalent:*

(i) *$(f_n)_{n \in \omega}$ pointwise converges to f and $(f_n)_{n \in \omega}$ is exhaustive,*

(ii) *f is continuous and $(f_n)_{n \in \omega}$ converges to f uniformly on compact sets,*

(iii) *f is continuous and $(f_n)_{n \in \omega}$ $\tau_{\mathcal{B}}^s$-converges to f, where $\mathcal{B}$ is the bornology of relatively compact subsets of X.*

PROOF. (i) $\Rightarrow$ (ii) The function f is continuous by Prop. 2.5 in [9]. Let $\epsilon > 0$ be fixed and K a compact subset of X. Let $\delta > 0$ be such that $\rho(f(z), f(u)) < \epsilon/4$ for every z, u with $d(z, u) < \delta$ and $\{z, u\} \cap K \neq \emptyset$. By assumption, for every $x \in X$, $(f_n)_{n \in \omega}$ is exhaustive at x. Since K is compact, there exist $x_1, ..., x_t \in K$ and respectively, $\delta_{x_i} < \delta, n_{x_i}$ for $i \leq t$ such that $K = \bigcup_{i \leq t} S(x_i, \delta_{x_i})$ and for every $y \in S(x_i, \delta_{x_i})$ we have $\rho(f_n(y), f_n(x_i)) < \epsilon/4$ for all $n \geq n_{x_i}$. By assumption $(f_n)_{n \in \omega}$ converges pointwise to f at x_i for $i = 1, ..., t$, hence there exists n_i such that for all $n \geq n_i$, we have that $\rho(f_n(x_i), f(x_i)) < \epsilon/4$. Let $n^* = \max\{n_{x_1}, ..., n_{x_t}, n_1, ..., n_t\}$. Hence for every $y \in K$, $y \in S(x_i, \delta_{x_i})$ for some $i \leq t$. Therefore for every $n \geq n^*$

$$\rho(f_n(y), f(y)) \leq \rho(f_n(y), f_n(x_i)) + \rho(f_n(x_i), f(x_i)) + \rho(f(x_i), f(y)) < \epsilon.$$

(ii) $\Rightarrow$ (i) It is sufficient to prove that $(f_n)_{n \in \omega}$ is exhaustive at every $x \in X$. Let $\epsilon > 0$ be fixed as well as $x \in X$. Since $(f_n)_{n \in \omega}$ converges pointwise to f at x, there exists n_0 such that for all $n \geq n_0$ we have that $\rho(f_n(x), f(x)) < \epsilon/4$. We claim that there is a $\delta > 0$ such that for every $y \in S(x, \delta)$ we have $\rho(f_n(x), f_n(y)) < \epsilon$ for every $n \geq n_0$. If not, there is a sequence (x_n) converging to x and an increasing sequence (k_n) of positive integers such that $k_n \geq n_0$ for every $n \in \omega$ and $\rho(f_{k_n}(x_n), f_{k_n}(x)) \geq \epsilon$ for every n. $K = \{x_n\} \cup \{x\}$ is a compact subset of X. By assumption, there is an n_1 such that for all $z \in K$ and for all $n \geq n_1$ we have that $\rho(f_n(z), f(z)) < \epsilon/4$. Since f is continuous at x, there is $\delta > 0$ such that for every $y \in S(x, \delta)$ we have $\rho(f(x), f(y)) < \epsilon/4$. Since $(x_n) \to x$ there is $x_n \in S(x, \delta)$ and for $k_n \geq \max\{n_0, n_1\}$ we have:

$$\rho(f_{k_n}(x_n), f_{k_n}(x)) \leq \rho(f_{k_n}(x_n), f(x_n)) + \rho(f(x_n), f(x)) + \rho(f(x), f_{k_n}(x)) < \epsilon$$

a contradiction.

The equivalence (ii) $\Leftrightarrow$ (iii) follows from $\tau_{\mathcal{B}}^s = \tau_k$, where τ_k denotes the compact open topology. ∎

The previous proposition does not hold for nets of functions, indeed we have the following proposition:

Proposition 3.6. *Let (X, d) be a metric space. The following are equivalent:*

(i) X is locally compact,

(ii) for every metric space (Y, ρ), every continuous function $f : X \to Y$ and every net $(f_\alpha)_{\alpha \in \Lambda}$ which converges uniformly on compacta to f, we have that $(f_\alpha)_{\alpha \in \Lambda}$ is exhaustive.

PROOF. (i) $\Rightarrow$ (ii) is clear. To prove (ii) $\Rightarrow$ (i), suppose that X is not locally compact. Let $x \in X$ be a point of X which has no compact neighborhood. Let $\mathcal{K}(x) = \{K \subset X : x \in K, K \text{ compact}\}$. Let $\mathcal{U}(x)$ denote the family of all open neighborhoods of x. On $\mathcal{U}(x) \times \mathcal{K}(x)$ define the following direction: $(U_1, K_1) \geq (U_2, K_2)$ if and only if $U_1 \subset U_2$ and $K_2 \subset K_1$. For every $(U, K) \in \mathcal{U}(x) \times \mathcal{K}(x)$ let $x_{U,K} \in U \setminus K$, and let $f_{U,K} : X \to [0, 1]$ be a continuous function such that $f_{U,K}(x_{U,K}) = 1$ and $f_{U,K}(z) = 0$ for every $z \in K$.

It is easy to verify that the net $(f_{U,K})$ converges uniformly on compact sets to the function identically equal to 0 and it is not exhaustive at x. ∎

Now, we show that exhaustiveness for a net of arbitrary pointwise convergence functions forces weaker classical convergences.

Proposition 3.7. *Let (X, d) and (Y, ρ) be metric spaces and $(f_\alpha)_{\alpha \in \Lambda}$, f be functions from X to Y. If the net $(f_\alpha)_{\alpha \in \Lambda}$ pointwise converges to f and $(f_\alpha)_{\alpha \in \Lambda}$ is exhaustive, then $(f_\alpha)_{\alpha \in \Lambda}$ converges sticky.*

PROOF. Let $\epsilon > 0$ be fixed as well as $x \in X$. By assumption, $(f_\alpha)_{\alpha \in \Lambda}$ is exhaustive at x, hence there exist a δ_0 and α_0 such that for all $y \in S(x, \delta_0)$ we

have $\rho(f_\alpha(y), f_\alpha(x)) < \epsilon/8$ for all $\alpha \geq \alpha_0$. Let $y \in S(x, \delta_0)$, since $(f_\alpha)_{\alpha \in \Lambda}$ converges pointwise to f, there exists α_1 such that for all $\alpha \geq \alpha_1$, we have that $\rho(f_\alpha(x), f(x)) < \epsilon/8$ and $\rho(f_\alpha(y), f(y)) < \epsilon/8$. Let $\alpha_2 \geq \alpha_0, \alpha_1$, hence for every $\alpha \geq \alpha_2$ there is $\delta = \delta_0$ and $\beta_0 = \alpha_2$ such that for all $\beta \geq \beta_0$

$$\rho(f_\beta(y), f_\alpha(y)) \leq \rho(f_\beta(y), f_\beta(x)) + \rho(f_\beta(x), f_\alpha(x)) + \rho(f_\alpha(x), f_\alpha(y)) < \epsilon.$$

∎

The coming example shows that the reverse implication in the previous proposition is not true.

Example 3.8. Let $X = Y = [0, 1]$ and f be a zero function. For every $n \in \omega$ let $f_n(1/n) = 1$ and $f_n(X \setminus (1/n - 1/2^n, 1/n + 1/2^n)) = \{0\}$. First we prove that $(f_n)_{n \in \omega}$ converges sticky. For every $x > 0$ and every $\epsilon > 0$ select n' such that $1/n' + 1/2^{n'} < x$. Let $n_0 = n' + 1$. For every $n \geq n_0$, $f_n(y) = 0$ for every $y \in (1/n' + 1/2^{n'}, 1]$, and the claim follows. Let $x = 0$, $\epsilon > 0$ and $n_0 = 1$. For every $n \geq n_0$, set $V_0 = [0, 1]$. If $y \in [0, 1]$, select n' such that $1/n' + 1/2^{n'} < y$ and set $n_y = n' + 1$. Again for every $p, q \geq n_y$ we have that $f_p(y) = f_q(y) = 0$. Finally we show that $(f_n)_{n \in \omega}$ is not exhaustive at 0. For $\epsilon > 1/2$ and for every $\delta > 0$ and n_0 there is $y \in S(0, \delta)$ and $n > n_0$ such that $|f_n(y) - f_n(0)| > 1/2$. In fact given n_0 it is suffices to select n' such that $n' > n_0$ and $1/n' < \delta$. Then $f_{n'}(1/n') = 1$ and $f_{n'}(0) = 0$.

Proposition 3.9. *Let (X, d) and (Y, ρ) be metric spaces and $(f_\alpha)_{\alpha \in \Lambda}$, f be functions from X to Y. If the net $(f_\alpha)_{\alpha \in \Lambda}$ pointwise converges to f and $(f_\alpha)_{\alpha \in \Lambda}$ is exhaustive, then $(f_\alpha)_{\alpha \in \Lambda}$ converges to f topologically.*

PROOF. The proof is trivial.

Remark 3.10. We observe that Example 3.8 shows also that sticky convergence does not imply neither topological convergence nor uniform convergence on compact sets. In fact, topological convergence fails since the sequence $\{1/n : n \in \omega\}$ converges to 0, but $f_n(1/n) = 1$ for every n and $f(0) = 0$. To show that $(f_n)_{n \in \omega}$ does not converges uniformly on compact sets, take $K = \{1/n : n \in \omega\} \cup \{0\}$.

Thus we need to investigate a weaker property than exhaustiveness defined for sequences by Gregoriades and Papanastassiou in [9].

Definition 3.11. Let (X, d) and (Y, ρ) be metric spaces and $(f_\alpha)_{\alpha \in \Lambda}$, be a net of functions from X to Y. The net $(f_\alpha)_{\alpha \in \Lambda}$ is *weakly exhaustive at x* if for every $\epsilon > 0$ there is $\delta > 0$ such that for all $y \in S(x, \delta)$ there exists $\alpha_y \in \Lambda$ such that for all $\alpha \geq \alpha_y$ we have $\rho(f_\alpha(y), f_\alpha(x)) < \epsilon$.
The net $(f_\alpha)_{\alpha \in \Lambda}$, is *weakly exhaustive* if it is weakly exhaustive at every x.

Using this new tool, Gregoriades and Papanastassiou proved in [9] the following theorem:

Theorem 3.12. *Let (X, d) and (Y, ρ) be metric spaces and $(f_n)_{n \in \omega}$, f be functions from X to Y such that the sequence $(f_n)_{n \in \omega}$ pointwise converges to f. Then $(f_n)_{n \in \omega}$ is weakly exhaustive if and only if f is continuous.*

This property can be easily extended to nets of functions. Now, we offer a direct proof of the equivalence between sticky convergence and weakly exhaustiveness at any point of X for nets of continuous functions.

Proposition 3.13. *Let (X, d) and (Y, ρ) be metric spaces and $(f_\alpha)_{\alpha \in \Lambda}$ be a net of function of $C(X, Y)$ which pointwise converges to a function f from X to Y. Then the following are equivalent:*

(i) $(f_\alpha)_{\alpha \in \Lambda}$ is weakly exhaustive,

(ii) $(f_\alpha)_{\alpha \in \Lambda}$ converges sticky.

PROOF. (i) $\Rightarrow$ (ii) Let $\epsilon > 0$ be fixed as well as $x \in X$. By assumption, $(f_\alpha)_{\alpha \in \Lambda}$ is weakly exhaustive at x, hence there exists a δ_0 such that such for all $y \in S(x, \delta_0)$ there is α_y such that for all $\alpha \geq \alpha_y$ we have $\rho(f_\alpha(y), f_\alpha(x)) < \epsilon/4$. Since $(f_\alpha)_{\alpha \in \Lambda}$ converges pointwise to f at x, there exists α_0 such that for all $\alpha \geq \alpha_0$, we have that $\rho(f_\alpha(x), f(x)) < \epsilon/8$. Thus for every $\alpha, \beta \geq \alpha_0$, $\rho(f_\beta(x), f_\alpha(x)) < \epsilon/4$. Since such f_α is continuous at x, there exists a δ_x^α such that for every $y \in S(x, \delta_x^\alpha)$, we have that $\rho(f_\alpha(x), f_\alpha(y)) < \epsilon/4$. Let $\delta = \min\{\delta_0, \delta_x^\alpha\}$. Hence for all $\alpha \geq \alpha_0$ there exists δ, with the property that for all $y \in S(x, \delta)$ there exists α_y such that for all $\beta \geq \alpha_y$ we have

$$\rho(f_\beta(y), f_\beta(y)) \leq \rho(f_\beta(y), f_\beta(x)) + \rho(f_\beta(x), f_\alpha(x)) + \rho(f_\alpha(x), f_\alpha(y)) \leq 3/4\epsilon.$$

(ii) $\Rightarrow$ (i) Let $\epsilon > 0$ be fixed as well as $x \in X$. By assumption, $(f_\alpha)_{\alpha \in \Lambda}$ converges sticky, there is α_0 such that for all $\alpha \geq \alpha_0$ there is a $\delta_0 > 0$ and for every $y \in S(x, \delta_0)$ there is α_y such that for every $\beta \geq \alpha_y$, $\rho(f_\beta(y), f_\alpha(y)) < \epsilon/8$. Since $(f_\alpha)_{\alpha \in \Lambda}$ converges pointwise to f at x, there exists α_1 such that for all $\alpha \geq \alpha_1$, we have that $\rho(f_\alpha(x), f(x)) < \epsilon/8$. Thus for every $\alpha, \beta \geq \alpha_1$, $\rho(f_\beta(x), f_\alpha(x)) < \epsilon/4$. Since f_α is continuous at x there exists a δ_x^α such that for every $y \in S(x, \delta_x^\alpha)$, we have that $\rho(f_\alpha(x), f_\alpha(y)) < \epsilon/4$. Let $\delta = \min\{\delta_0, \delta_x^\alpha\}$ and choose α_y^* greater than $\alpha_0, \alpha_1, \alpha_y$. Thus we have

$$\rho(f_\beta(y), f_\beta(x)) \leq \rho(f_\beta(y), f_\alpha(y)) + \rho(f_\alpha(y), f_\alpha(x)) + \rho(f_\alpha(x), f_\beta(x)) \leq \epsilon/2 < \epsilon.$$

∎

The notion of quasi-uniform convergence was introduced for the first time by Arzelà [2] in 1883 to give a necessary and sufficient condition for the continuity of a series of real valued continuous functions defined on compact intervals of $\mathbb{R}$. In 1948 P.S. Alexandroff in [1] studied the question for a sequence of continuous functions from a topological space X (not necessarily compact) to a metric space Y. We quote also the seminal paper of Bartle [4], where Arzelà's theorem is extended to nets of real valued continuous functions on a topological space.

The Alexandroff convergence can be restated for nets in the following way (see [7]).

Definition 3.14. Let $(f_\alpha)_{\alpha \in \Lambda}$ be a net of continuous functions from a topological space X to a metric space (Y, ρ) and let $f : X \to Y$. Then $(f_\alpha)_{\alpha \in \Lambda}$ is called

Alexandroff convergent to f provided it is pointwise convergent to f and for every $\epsilon > 0$, for every $\alpha_0 \in \Lambda$ there exist a cofinal subset Λ_0 of $\{\alpha : \alpha \geq \alpha_0\}$ and an open cover $\{\Gamma_\alpha : \alpha \in \Lambda_0\}$ of X such that for every $\alpha \in \Lambda_0$, for every $x \in \Gamma_\alpha$, we have $\rho(f_\alpha(x), f(x)) < \epsilon$.

Definition 3.15. [4] Let $(f_\alpha)_{\alpha \in \Lambda}$ be a net of real valued functions on an arbitrary set X and let $f : X \to \mathbb{R}$. Then $(f_\alpha)_{\alpha \in \Lambda}$ is said to *converge to f quasi-uniformly on X*, provided it pointwise converges to f, and for every $\epsilon > 0$ and α_0 there exists a finite number of indices $\alpha_1, \alpha_2, ..., \alpha_n \geq \alpha_0$ such that for each $x \in X$ at least one of the following inequalities holds:

$$|f_{\alpha_i}(x) - f(x)| < \epsilon \quad i = 1, ..., n.$$

Using Proposition 3.13 and Theorem 2.10 in [7] we state the following theorem:

Theorem 3.16. *Let (X, d) and (Y, ρ) be metric and $(f_\alpha)_{\alpha \in \Lambda}$ be a net of functions from X to Y. Suppose that $(f_\alpha)_{\alpha \in \Lambda}$ is a net in $C(X, Y)$ that is pointwise convergent to f. The following are equivalent:*

(i) $(f_\alpha)_{\alpha \in \Lambda}$ *is weakly exhaustive,*

(ii) $(f_\alpha)_{\alpha \in \Lambda}$ *is sticky convergent (or equivalently $\tau_{\mathcal{F}}^s$ -convergent to f),*

(iii) $(f_\alpha)_{\alpha \in \Lambda}$ *Alexandroff converges to f,*

(iv) $(f_\alpha)_{\alpha \in \Lambda}$ *is quasi-uniformly convergent to f on compacta.*

4   Strong exhaustiveness at families

Beer and Levi initiated the study of strong uniform continuity on a set B, so we introduce two appropriate forms of exhaustiveness at a set B.

Definition 4.1. Let (X, d) and (Y, ρ) be metric spaces, $(f_n)_{n \in \omega}$ be functions from X to Y and B a subset of X. The sequence $(f_n)_{n \in \omega}$ is *strongly exhaustive at B* if for every $\epsilon > 0$ there is $\delta > 0$ and n_0 such that if $d(x, y) < \delta$ and $B \cap \{x, y\} \neq \emptyset$ then $\rho(f_n(y), f_n(x)) < \epsilon$ for every $n \geq n_0$. Let $\mathcal{B}$ be a subset of $\mathcal{P}_0(X)$, we say that $(f_n)_{n \in \omega}$ is *strongly exhaustive at $\mathcal{B}$* if it is strongly exhaustive at any $B \in \mathcal{B}$.

Now we characterize the notion of strongly exhaustiveness at B for a sequence of functions in terms of the behaviour of the induced direct image maps, and a property of the gap functional with respect to images of near subsets.

Theorem 4.2. *Let (X, d) and (Y, ρ) be metric spaces, $(f_n)_{n \in \omega}$ be functions from X to Y and $\mathcal{B}$ a family of nonempty subsets of X. The following are equivalent:*

(i) $(f_n)_{n \in \omega}$ *is strongly exhaustive at $\mathcal{B}$,*

(ii) *for every $B \in \mathcal{B}$, $(\hat{f}_n)_{n \in \omega}$ is strongly exhaustive at $\mathcal{P}_0(B)$ with respect to $(\mathcal{P}_0(X), H_d)$ and $(\mathcal{P}_0(Y), H_\rho)$,*

(iii) for every $B \in \mathcal{B}$, whenever $C \in \mathcal{P}_0(B)$ and $D_d(C, A) = 0$, then
    $lim_n \, D_\rho(f_n(C), f_n(A)) = 0$.

PROOF. (i) $\Rightarrow$ (ii) Let $\epsilon > 0$ and $B \in \mathcal{B}$ be fixed. Let $B_0 \subset B$. By assumption, $(f_\alpha)_{\alpha \in \Lambda}$ is strongly exhaustive at B, hence there exist a δ_0 and n_0 such that if $d(x, y) < \delta$ and $B \cap \{x, y\} \neq \emptyset$ then $\rho(f_n(y), f_n(x)) < \epsilon$ for every $n \geq n_0$.

We claim that if $H_d(C, A) < \delta$ and $C \subset B$, then $H_\rho(\hat{f}_n(C), \hat{f}_n(A)) < \epsilon$ for every $n \geq n_0$. Let $C \in \mathcal{P}_0(B)$ such that $H_d(C, A) < \delta$ ad $n \geq n_0$. Then $\hat{f}_n(C) \subset (\hat{f}_n(A))^\epsilon$ and $\hat{f}_n(A) \subset (\hat{f}_n(C))^\epsilon$. Thus $H_\rho(\hat{f}_n(C), \hat{f}_n(A)) \leq \epsilon$ for every $n \geq n_0$.

(ii) $\Rightarrow$ (iii) Let $B \in \mathcal{B}$ and $C \in \mathcal{P}_0(B)$ such that $D_d(C, A) = 0$. For $n \in \omega$ let $a_n \in A$ be such that $d(a_n, C) < 1/n$. Let $C_n = C \cup \{a_n\}$ for every $n \in \omega$. Let $\epsilon > 0$. There is $\delta > 0$ and n_0 such that if $C \subset B$ and $H_d(C, H) < \delta$, then $H_\rho(\hat{f}_n(C), \hat{f}_n(H)) \leq \epsilon$ for every $n \geq n_0$. Let $k \in \omega$ be such that $1/k < \delta$. Then $H_d(C, C_l) < \delta$ for every $l \geq k$. Thus $H_\rho(\hat{f}_n(C), \hat{f}_n(C_l)) \leq \epsilon$ for every $n \geq n_0$ and $l \geq k$. Let $n \geq n_0$ and $l \geq k$ be fixed. Thus there is $c_l \in C$ such that $\rho(f_n(c_l), f_n(a_l)) < \epsilon$, i.e. $D_\rho(f_n(C), f_n(A)) < \epsilon$. (iii) $\Rightarrow$ (i) Assume that $(f_n)_{n \in \omega}$ fails to be strongly exhaustive at some $B \in \mathcal{B}$. For some positive ϵ we can pick for each n, $m_n \geq n$, $a_n \in B$ and $x_n \in X$ such that $d(a_n, x_n) < 1/n$, (m_n) is an increasing sequence of positive integers and $\rho(f_{m_n}(a_n), f_{m_n}(x_n)) > \epsilon$ for every n. By Efromovich Lemma there exists $\mathbb{N}_1 \subset \mathbb{N}$ infinite such that $D\rho(A', B') > 0$ where $A' = \{f_{m_n}(a_n) : n \in \mathbb{N}_1\}$ and $B' = \{f_{m_n}(x_n) : n \in \mathbb{N}_1\}$. The condition (iii) fails with $A = \{a_n : n \in \mathbb{N}_1\}$ and $B = \{x_n : n \in \mathbb{N}_1\}$. ∎

Now, we introduce the appropriate form of exhaustiveness for sequences of pointwise convergent functions to study strong uniform continuity of the limit.

Definition 4.3. Let (X, d) and (Y, ρ) be metric spaces, $(f_n)_{n \in \omega}$ be functions from X to Y and B a subset of X. The sequence $(f_n)_{n \in \omega}$ is *strongly-weakly exhaustive at B* if for every $\epsilon > 0$ there is a $\delta > 0$ such that $y \in B^\delta$ and for every $x \in B$ such that $d(x, y) < \delta$ there is $n_{(x,y)}$ and $\rho(f_n(y), f_n(x)) < \epsilon$ for every $n \geq n_{(x,y)}$. Let $\mathcal{B}$ be a subset of $\mathcal{P}_0(X)$, we say that $(f_n)_{n \in \omega}$ is *strongly-weakly exhaustive at $\mathcal{B}$* if it is strongly-weakly exhaustive at any $B \in \mathcal{B}$.

First of all we notice that being strongly-weakly exhaustive at a subset is a stronger requirement than weakly exhaustive at points.

Example 4.4. Example 3.8 shows that f is weakly exhaustive at any point but it is not strongly-weakly exhaustive at $(0, 1]$.

We prove the following theorem.

Theorem 4.5. *Let (X, d) and (Y, ρ) be metric spaces, $(f_n)_{n \in \omega}$, f be functions from X to Y such that $(f_n)_{n \in \omega}$ pointwise converges to f. Let $\mathcal{B}$ be a family of non empty subsets of X. The following are equivalent:*

(i) $(f_n)_{n \in \omega}$ is strongly weakly exhaustive at $\mathcal{B}$,

(ii) f is strongly uniformly continuous on $\mathcal{B}$.

PROOF. (i) $\Rightarrow$ (ii) Let $\epsilon > 0$ and $B \in \mathcal{B}$ be fixed. By assumption, $(f_n)_{n \in \omega}$ is strongly-weakly exhaustive at B, hence there exists $\delta > 0$ such that for all $y \in B^\delta$ and for every $x \in B$ with $d(x,y) < \delta$ there is $n_{(x,y)}$ and for every $n \geq n_{(x,y)}$ we have $\rho(f_n(y), f_n(x)) < \epsilon/4$. Take $y \in B^\delta$ and $x \in B$ with $d(x,y) < \delta$. Since $(f_n)_{n \in \omega}$ pointwise converges to f, there is n_0 and for every $n \geq n_0$, $\rho(f_n(x), f(x)) < \epsilon/4$ and $\rho(f_n(y), f(y)) < \epsilon/4$. Let $\bar{n} \geq \max\{n_0, n_{(x,y)}\}$, hence for every $n \geq \bar{n}$ we have that $\rho(f(x), f(y)) < \epsilon$.

(ii) $\Rightarrow$ (i) Let $\epsilon > 0$ and $B \in \mathcal{B}$ be fixed and f strongly uniformly continuous at B. Hence there exists $\delta > 0$ such that for all $y \in B^\delta$ and for every $x \in B$ with $d(x,y) < \delta$ we have $\rho(f(y), f(x)) < \epsilon/4$. Take $y \in B^\delta$ and $x \in B$ with $d(x,y) < \delta$. Since $(f_n)_{n \in \omega}$ pointwise converges to f, there is n_0 and for every $n \geq n_0$, $\rho(f_n(x), f(x)) < \epsilon/4$ and $\rho(f_n(y), f(y)) < \epsilon/4$. For every $n \geq n_0$ we have that $\rho(f_n(x), f_n(y)) < \epsilon$. ∎

In view of Theorem 6.7 in [5], we have the following corollary:

Corollary 4.6. *Let (X, d) and (Y, ρ) be metric spaces and $\mathcal{B}$ be a bornology with closed base. Let $(f_n)_{n \in \omega}$ be a net of strongly uniformly continuous functions from X to Y that is $\tau_{\mathcal{B}}$-convergent to $f \in Y^X$. The following are equivalent:*

(i) $(f_n)_{n \in \omega}$ is strongly weakly exhaustive at $\mathcal{B}$,

(ii) f is strongly uniformly continuous on $\mathcal{B}$,

(iii) $(f_n)_{n \in \omega}$ is $\tau_{\mathcal{B}}^s$-convergent to f.

Note that for $\mathcal{B} = \{X\}$ we get what must be exactly added to pointwise convergence to have uniform continuity of the limit.

Corollary 4.7. *Let (X, d) and (Y, ρ) be metric spaces, $(f_n)_{n \in \omega}$, f be functions from X to Y such that $(f_n)_{n \in \omega}$ pointwise converges to f. The following are equivalent:*

(i) $(f_n)_{n \in \omega}$ is strongly weakly exhaustive,

(ii) f is uniformly continuous.

Using Proposition 4.11 and Theorem 5.2 in [9] we have the following corollaries:

Corollary 4.8. *Let (X, d) and (Y, ρ) be metric spaces, $(f_n)_{n \in \omega}$, f be functions from X to Y such that $(f_n)_{n \in \omega}$ pointwise converges to f. Let $\mathcal{B}$ the bornology of the d-totally bounded subsets of X. The following are equivalent:*

(i) $(f_n)_{n \in \omega}$ is strongly-weakly exhaustive on $\mathcal{B}$,

(ii) f is strongly uniformly continuous on $\mathcal{B}$,

(iii) f is uniformly continuous on $\mathcal{B}$.

Corollary 4.9. *Let (X, d) and (Y, ρ) be metric spaces, $(f_n)_{n \in \omega}$, f be functions from X to Y such that $(f_n)_{n \in \omega}$ pointwise converges to f. Let $\mathcal{B}^{uc}$ the bornology of the family of UC-sets. If $A \in \mathcal{B}^{uc}$ and $(f_n)_{n \in \omega}$ is weakly exhaustive on X, then f is strongly uniformly continuous on A.*

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