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Hypercyclic and chaotic integrated C-cosine functions

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Dedicated to the Memory of Professor Sen-Yen Shaw

Abstract

The main purpose of the paper is to display the main structural properties of hypercyclic and chaotic integrated C-cosine functions. The notions of hypercyclicity, mixing and chaoticity of an α-times integrated C-cosine function ($\alpha \geq 0$) are defined by using distributional techniques. We provide several examples which justify our abstract theoretical approach.

1 Introduction and preliminaries

Let E be a complex Banach space. A linear operator T on E is said to be *hypercyclic* if there exists an element $x \in D_\infty(T) \equiv \bigcap_{n\in\mathbb{N}} D(T^n)$ whose orbit $\{T^n x : n \in \mathbb{N}_0\}$ is dense in E; T is said to be *topologically transitive,* resp. *topologically mixing*, if for every pair of open non-empty subsets U, V of E, there exists $n_0 \in \mathbb{N}$ such that $T^{n_0}(U) \cap V \neq \emptyset$, resp. if for every pair of open non-empty subsets U, V of E, there exists $n_0 \in \mathbb{N}$ such that, for every $n \in \mathbb{N}$ with $n \geq n_0$, $T^n(U) \cap V \neq \emptyset$. A *periodic point* for T is an element $x \in D_\infty(T)$ satisfying that there exists $n \in \mathbb{N}$ with $T^n x = x$. Finally, T is said to be *chaotic* if T is hypercyclic and the set of periodic points of T is dense in E.

The organization of this paper, which continues the researches of A. Bonilla, P. J. Miana [11] and T. Kalmes [31], is given as follows. In the second section, we introduce and systematically analyze the class of C-distribution cosine functions and slightly improve the results obtained in collaboration with P. J. Miana [49], [34]-[35]. In Definition 21 and Definition 22, we introduce the notions of various types of hypercyclicity of C-distribution cosine functions and integrated C-cosine functions. Motivated by the study of R. deLaubenfels, H. Emamirad and K.-G.

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Grosse-Erdmann [25], we clarify in Theorem 23 the equivalent conditions for hypercyclicity, mixing and chaoticity of C-distribution cosine functions. In Theorem 25 and Theorem 27, we significantly improve results given in [11] and [50]. The main objective in Theorem 33 and Theorem 35 is to provide sufficient conditions for mixing and chaoticity of certain classes of C-distribution cosine functions. The last two sections of the paper are devoted to the study of hypercyclic and chaotic cosine functions generated by squares of gradient operators ([28]-[31]) and disjoint hypercyclicity of cosine functions on weighted function spaces. The notion of subspace chaoticity introduced by J. Banasiak and M. Moszyński [5] plays an important role throughout the paper.

It is worth noting that several results established in this paper are obvious modifications of corresponding results from the theory of hypercyclic single valued operators. We are not primarily concerned with studying new concepts in the theory of hypercyclicity and our main intention is, in fact, to analyze the basic properties of a new important class of abstract second order (ill-posed) PDEs (cf. [3], [23], [26], [32], [34]-[35], [39], [42]-[44], [49] and [56]-[57] for further information in this direction).

Henceforth $L(E)$ stands for the space of all continuous linear mappings from E into E and $L(E) \ni C$ is an injective operator which satisfies $CA \subseteq AC$. Recall that the C-resolvent set of A, denoted by $\rho_C(A)$, is defined by

$$\rho_C(A) := \Big\{\lambda \in \mathbb{C} : \lambda - A \text{ is injective and } (\lambda - A)^{-1}C \in L(E)\Big\}.$$

For a closed linear operator A, $\mathrm{Kern}(A)$, $\mathrm{R}(A)$, $\rho(A)$, $\sigma(A)$ and $\sigma_p(A)$ denote its kernel space, range, resolvent set, spectrum and point spectrum, respectively. By $[D(A)]$ we denote the Banach space $D(A)$ equipped with the graph norm. Suppose F is a closed subspace of E. Then the part of A in F, denoted by A_F, is a linear operator defined by $D(A_F) := \{x \in D(A) \cap F : Ax \in F\}$ and $A_F x := Ax,\ x \in D(A_F)$.

Definition 1. Suppose A is a closed operator, $\alpha \geq 0$ and $0 < \tau \leq \infty$. If there exists a strongly continuous operator family $(C_\alpha(t))_{t\in[0,\tau)}$ such that:

(i) $C_\alpha(t)A \subseteq AC_\alpha(t)$, $t \in [0, \tau)$,

(ii) $C_\alpha(t)C = CC_\alpha(t),\ t \in [0, \tau)$, and

(iii) for every $x \in E$ and $t \in [0, \tau)$: $\int_0^t (t-s)C_\alpha(s)x\,ds \in D(A)$ and

$$A\int_0^t (t-s)C_\alpha(s)x\,ds = C_\alpha(t)x - \frac{t^\alpha}{\Gamma(\alpha+1)}Cx,$$

then it is said that A is a *subgenerator* of a (local) α-times integrated C-cosine function $(C_\alpha(t))_{t\in[0,\tau)}$. If $\tau = \infty$, then it is said that $(C_\alpha(t))_{t\geq 0}$ is an *exponentially bounded*, α-times integrated C-cosine function with a subgenerator A if, in addition, there are constants $M > 0$ and $\omega \in \mathbb{R}$ such that $||C_\alpha(t)|| \leq Me^{\omega t},\ t \geq 0$.

The set which consists of all subgenerators of $(C_\alpha(t))_{t\in[0,\tau)}$ need not be finite and a local α-times integrated C-cosine function need not be extendible beyond the interval $[0,\tau)$. In this paper, we primarily consider hypercyclicity and chaoticity of global integrated C-cosine functions. The (integral) generator $\hat{A}$ of $(C_\alpha(t))_{t\in[0,\tau)}$ defined by

$$\hat{A} := \Big\{(x,y) \in E \oplus E : C_\alpha(t)x - \frac{t^\alpha}{\Gamma(\alpha+1)}Cx = \int\limits_0^t (t-s)C_\alpha(s)y\,ds,\ t \in [0,\tau)\Big\}$$

is the maximal subgenerator of $(C_\alpha(t))_{t\in[0,\tau)}$ with respect to the set inclusion. We refer the reader to [39] for further information concerning integrated C-cosine functions and semigroups.

The Schwartz spaces of test functions $\mathcal{D} = C_0^\infty(\mathbb{R})$ and $\mathcal{E} = C^\infty(\mathbb{R})$ carry the usual inductive limit topologies. The topology of the space of rapidly decreasing functions $\mathcal{S}$ is induced by the following system of seminorms: $p_{m,n}(\psi) =: \sup_{x\in\mathbb{R}} |x^m\psi^{(n)}(x)|$, $\psi \in \mathcal{S}$, $m,n \in \mathbb{N}_0$. By $\mathcal{D}_0$ we denote the subspace of $\mathcal{D}$ which consists of the elements supported by $[0,\infty)$. Further on, $\mathcal{D}'(E) := L(\mathcal{D} : E)$, $\mathcal{E}'(E) := L(\mathcal{E} : E)$ and $\mathcal{S}'(E) := L(\mathcal{S} : E)$ are the spaces of continuous linear functions $\mathcal{D} \to E$, $\mathcal{E} \to E$ and $\mathcal{S} \to E$, respectively; $\mathcal{D}_0'(E)$, $\mathcal{E}_0'(E)$ and $\mathcal{S}_0'(E)$ are the subspaces of $\mathcal{D}'(E)$, $\mathcal{E}'(E)$ and $\mathcal{S}'(E)$, respectively, containing the elements supported by $[0,\infty)$. Denote by $\mathcal{B}$ the family of all bounded subsets of $\mathcal{D}$. Put $p_B(f) := \sup_{\varphi\in B} ||f(\varphi)||$, $f \in \mathcal{D}'(E)$, $B \in \mathcal{B}$. Then p_B, $B \in \mathcal{B}$ is a seminorm on $\mathcal{D}'(E)$ and the system $(p_B)_{B\in\mathcal{B}}$ defines the topology on $\mathcal{D}'(E)$. The topology on $\mathcal{E}'(E)$, resp., $\mathcal{S}'(E)$, is defined similarly. We employ the convolution product $*$ and the finite convolution product $*_0$ of measurable functions $\varphi,\psi : \mathbb{R} \to \mathbb{C}$:

$$\varphi * \psi(t) =: \int\limits_{-\infty}^{\infty} \varphi(t-s)\psi(s)\,ds,\ \varphi *_0 \psi(t) := \int\limits_0^t \varphi(t-s)\psi(s)\,ds,\ t \in \mathbb{R}$$

and refer the reader to [39, Section 1.3] for the basic properties of the convolution products $*$ and $*_0$ in the subspaces of scalar-valued distributions. The convolution of vector-valued distributions is taken in the sense of [41, Proposition 1.1]:

Proposition 2. *Suppose X, Y and Z are Banach spaces and $b : X \times Y \to Z$ is bilinear and continuous. Then there is a unique bilinear, separately continuous mapping $*_b : \mathcal{D}_0'(X) \times \mathcal{D}_0'(Y) \to \mathcal{D}_0'(Z)$ such that*

$$(S \otimes x) *_b (T \otimes y) = S * T \otimes b(x,y),$$

for all $S,T \in \mathcal{D}_0'$ and $x \in X$, $y \in Y$. Moreover, this mapping is continuous.

Definition 3. ([33]) Let $G \in \mathcal{D}_0'(L(E))$ satisfy $CG(\varphi) = G(\varphi)C$, $\varphi \in \mathcal{D}$. If $G(\varphi *_0 \psi)C = G(\varphi)G(\psi)$, $\varphi,\psi \in \mathcal{D}$, then G is called a *pre*$-(C-DS)$. If, additionally, $\mathcal{N}(G) = \bigcap_{\varphi\in\mathcal{D}_0} \mathrm{Kern}(G(\varphi)) = \{0\}$, then G is called a *C-distribution semigroup*, $(C-DS)$ in short. It is said that a pre$-(C-DS)$ is *dense* if the set $\mathcal{R}(G) =$

$\bigcup_{\varphi\in\mathcal{D}_0} \mathrm{R}(G(\varphi))$ is dense in E. A pre$-(C-DS)$ G is said to be *exponential* if there exists $\omega\in\mathbb{R}$ such that $e^{-\omega t}G\in\mathcal{S}_0'(L(E))$; the shorthand $(E-CDS)$ is used to denote an exponential $(C-DS)$.

Let G be a $(C-DS)$ and let $T\in\mathcal{E}_0'(\mathbb{C})$, i.e., T is a scalar-valued distribution with compact support in $[0,\infty)$. Then we define $G_1(T)$ on a subset of E by

$$y=G_1(T)x \text{ iff } G(T*\varphi)x=G(\varphi)y \text{ for all } \varphi\in\mathcal{D}_0.$$

Then $G_1(T)$ is a closed linear operator, $G_1(\delta)=I$ and the (infinitesimal) generator of a $(C-DS)$ G is defined by $A:=G_1(-\delta')$. In the case $C=I$, there is no risk for confusion and we do not distinguish G and G_1.

Put $\mathcal{D}_+:=\{f\in C^\infty([0,\infty)) : f \text{ is compactly supported}\}$ and define $\mathcal{K}:\mathcal{D}\to\mathcal{D}_+$ by $\mathcal{K}(\varphi)(t):=\varphi(t),\ t\geq 0,\ \varphi\in\mathcal{D}$. As is known, $\mathcal{D}_+$ is an (LF) space and there exists a linear continuous operator $\Lambda:\mathcal{D}_+\to\mathcal{D}$ which satisfies $\mathcal{K}\Lambda=I_{\mathcal{D}_+}$ ([52]).

2 C-distribution cosine functions, almost C-distribution cosine functions and integrated C-cosine functions

We begin this section by recalling the following notion ([34]). Let $\zeta\in\mathcal{D}_{[-2,-1]}$ be a fixed test function satisfying $\int_{-\infty}^{\infty}\zeta(t)dt=1$. Then, with ζ chosen in this way, we define $I(\varphi)$ $(\varphi\in\mathcal{D})$ as follows

$$I(\varphi)(\cdot):=\int_{-\infty}^{\cdot}\Big[\varphi(t)-\zeta(t)\int_{-\infty}^{\infty}\varphi(u)du\Big]dt.$$

Then $I(\varphi)\in\mathcal{D}$, $I(\varphi')=\varphi$, $\frac{d}{dt}I(\varphi)(t)=\varphi(t)-\zeta(t)\int_{-\infty}^{\infty}\varphi(u)du$, $t\in\mathbb{R}$ and, for every $G\in\mathcal{D}'(L(E))$, the primitive G^{-1} of G is defined by setting $G^{-1}(\varphi):=-G(I(\varphi))$, $\varphi\in\mathcal{D}$. It is clear that $G^{-1}\in\mathcal{D}'(L(E))$, $(G^{-1})'=G$, i.e., $-G^{-1}(\varphi')=G(I(\varphi'))=G(\varphi)$, $\varphi\in\mathcal{D}$ and that $\operatorname{supp}G\subseteq[0,\infty)$ implies $\operatorname{supp}G^{-1}\subseteq[0,\infty)$.

Definition 4. An element $\mathbf{G}\in\mathcal{D}_0'(L(E))$ is called a *pre*$-(C-DCF)$ iff $\mathbf{G}(\varphi)C=C\mathbf{G}(\varphi)$, $\varphi\in\mathcal{D}$ and

$$(C-DCF_1):\mathbf{G}^{-1}(\varphi*_0\psi)C=\mathbf{G}^{-1}(\varphi)\mathbf{G}(\psi)+\mathbf{G}(\varphi)\mathbf{G}^{-1}(\psi),\ \varphi,\psi\in\mathcal{D};$$

if, additionally,

$$(C-DCF_2):\qquad x=y=0 \text{ iff } \mathbf{G}(\varphi)x+\mathbf{G}^{-1}(\varphi)y=0,\ \varphi\in\mathcal{D}_0,$$

then $\mathbf{G}$ is called a C-*distribution cosine function*, in short $(C-DCF)$. A pre$-(C-DCF)$ $\mathbf{G}$ is called *dense* if the set $\mathcal{R}(\mathbf{G}):=\bigcup_{\varphi\in\mathcal{D}_0}\mathrm{R}(\mathbf{G}(\varphi))$ is dense in E.

Notice that (DCF_2) implies $\bigcap\limits_{\varphi\in\mathcal{D}_0}\mathrm{Kern}(\mathbf{G}(\varphi))=\{0\}$ and $\bigcap\limits_{\varphi\in\mathcal{D}_0}\mathrm{Kern}(\mathbf{G}^{-1}(\varphi))=\{0\}$, and that the assumption $\mathbf{G}\in\mathcal{D}_0'(L(E))$ implies $\mathbf{G}(\varphi)=0$, $\varphi\in\mathcal{D}_{(-\infty,0]}$.

Proposition 5. ([39])

(i) *Let* $\mathbf{G}\in\mathcal{D}_0'(L(E))$ *and* $\mathbf{G}(\varphi)C=C\mathbf{G}(\varphi)$, $\varphi\in\mathcal{D}$. *Then* $\mathbf{G}$ *is a pre-(C-DCF) in* E *iff* $\mathcal{G}\equiv\begin{pmatrix}\mathbf{G} & \mathbf{G}^{-1}\\ \mathbf{G}'-\delta\otimes C & \mathbf{G}\end{pmatrix}$ *is a pre-($\mathcal{C}$-DS) in* $E\oplus E$, *where* $\mathcal{C}\equiv\begin{pmatrix}C & 0\\ 0 & C\end{pmatrix}$. *Moreover,* $\mathcal{G}$ *is a ($\mathcal{C}$-DS) iff* $\mathbf{G}$ *is a pre-(C-DCF) which satisfies* $(C-DCF_2)$.

(ii) *Let* $\mathbf{G}\in\mathcal{D}_0'(L(E))$ *and* $\mathbf{G}(\varphi)C=C\mathbf{G}(\varphi)$, $\varphi\in\mathcal{D}$. *Then* $\mathbf{G}$ *is a (C-DCF) iff* (DCF_2) *holds and*

$$\mathbf{G}^{-1}(\varphi*\psi_+)C=\mathbf{G}^{-1}(\varphi)\mathbf{G}(\psi)+\mathbf{G}(\varphi)\mathbf{G}^{-1}(\psi),\ \varphi\in\mathcal{D}_0,\ \psi\in\mathcal{D}.$$

Assume $\mathbf{G}$ is a $(C-DCF)$ and $T\in\mathcal{E}_0'(\mathbb{C})$. Then the (infinitesimal) generator A of $\mathbf{G}$ is defined by

$$A:=G(\delta''):=\Big\{(x,y)\in E\oplus E:\mathbf{G}^{-1}(\varphi'')x=\mathbf{G}^{-1}(\varphi)y\text{ for all }\varphi\in\mathcal{D}_0\Big\}.$$

Then A is a closed linear operator and, by the proof of [39, Lemma 3.1.6], we have $C^{-1}AC=A$.

Theorem 6. ([39])

(i) *Let* A *be the generator of a (C-DCF)* $\mathbf{G}$. *Then* $\mathcal{A}\subseteq\mathcal{B}$, *where* $\mathcal{A}\equiv\begin{pmatrix}0 & I\\ A & 0\end{pmatrix}$ *and* $\mathcal{B}$ *is the generator of* $\mathcal{G}$. *Furthermore,* $(x,y)\in A\Leftrightarrow\Big(\binom{x}{0},\binom{0}{y}\Big)\in\mathcal{B}$.

(ii) *Let* $\mathbf{G}$ *be a (C-DCF) generated by* A. *Then the following holds:*

  (a) $(\mathbf{G}(\psi)x,\ \mathbf{G}(\psi'')x+\psi'(0)Cx)\in A$, $\psi\in\mathcal{D}$, $x\in E$.

  (b) $(\mathbf{G}^{-1}(\psi)x,-\mathbf{G}(\psi')x-\psi(0)Cx)\in A$, $\psi\in\mathcal{D}$, $x\in E$.

  (c) $\mathbf{G}(\psi)A\subseteq A\mathbf{G}(\psi)$, $\psi\in\mathcal{D}$.

  (d) $\mathbf{G}^{-1}(\psi)A\subseteq A\mathbf{G}^{-1}(\psi)$, $\psi\in\mathcal{D}$.

(iii) *A closed linear operator* A *is the generator of a (C-DCF)* $\mathbf{G}$ *iff for every* $\tau>0$ *there exist an integer* $n_\tau\in\mathbb{N}$ *and a local* n_τ*-times integrated* C*-cosine function* $(C_n(t))_{t\in[0,\tau)}$ *with the integral generator* A. *If this is the case, then the following equality holds:*

$$\mathbf{G}(\varphi)x=(-1)^n\int\limits_0^\tau\varphi^{(n)}(t)C_n(t)x\,dt,\ x\in E,\ \varphi\in\mathcal{D}_{(-\infty,\tau)}.$$

Recall that the exponential region $E(a,b)$ $(a,b>0)$ is defined in [2] by $E(a,b) := \{\lambda\in\mathbb{C} : \Re\lambda\geq b,\ |\Im\lambda|\leq e^{a\Re\lambda}\}$; set $E^2(a,b) := \{\lambda^2 : \lambda\in E(a,b)\}$.

Theorem 7. *Suppose $a>0$, $b>0$, $\alpha>0$, $M>0$, $E^2(a,b)\subseteq\rho_C(A)$, the mapping $\lambda\mapsto(\lambda^2-A)^{-1}C$, $\lambda\in E(a,b)$ is continuous and $||(\lambda^2-A)^{-1}C||\leq M(1+|\lambda|)^\alpha$, $\lambda\in E(a,b)$. Put $\tilde{\varphi}(\lambda) := \int\limits_{-\infty}^{\infty} e^{\lambda t}\varphi(t)dt$, $\varphi\in\mathcal{D}$ and*

$$\mathbf{G}(\varphi)x := \frac{1}{2\pi i}\int\limits_{\Gamma}\lambda\tilde{\varphi}(\lambda)(\lambda^2-A)^{-1}Cxd\lambda,\ x\in E,\ \varphi\in\mathcal{D},$$

where Γ is the upwards oriented boundary of $E(a,b)$. Then $\mathbf{G}$ is a (C-DCF) generated by $C^{-1}AC$.

Proof. The prescribed assumptions combined with [39, Proposition 2.1.24] imply that there exist $\beta\geq 0$ and $M_1>0$ such that $E(a,b)\subseteq\rho_{\mathcal{C}}(\mathcal{A})$, $||(\lambda-\mathcal{A})^{-1}\mathcal{C}||\leq M_1(1+|\lambda|)^\beta$, $\lambda\in E(a,b)$ and that the mapping $\lambda\mapsto(\lambda-\mathcal{A})^{-1}\mathcal{C}$, $\lambda\in E(a,b)$ is continuous. Put $\mathcal{G}(\varphi)\binom{x}{y} := \frac{1}{2\pi i}\int\limits_{\Gamma}\tilde{\varphi}(\lambda)(\lambda-\mathcal{A})^{-1}\mathcal{C}\binom{x}{y}d\lambda$, $x,y\in E$, $\varphi\in\mathcal{D}$. By [36, Theorem 2.1], $\mathcal{G}$ is a $(\mathcal{C}-DS)$ generated by $\mathcal{C}^{-1}\mathcal{A}\mathcal{C}$. Using [39, Proposition 2.1.24] again, one gets that, for every $x,y\in E$ and $\varphi\in\mathcal{D}$: $\mathcal{G}(\varphi) = \begin{pmatrix}\mathbf{G}_1(\varphi) & \mathbf{G}_2(\varphi)\\ \mathbf{G}_3(\varphi) & \mathbf{G}_1(\varphi)\end{pmatrix}$, where $\mathbf{G}_1(\varphi)x = \frac{1}{2\pi i}\int\limits_{\Gamma}\lambda\tilde{\varphi}(\lambda)(\lambda^2-A)^{-1}Cxd\lambda$, $\mathbf{G}_2(\varphi)x = \frac{1}{2\pi i}\int\limits_{\Gamma}\tilde{\varphi}(\lambda)(\lambda^2-A)^{-1}Cxd\lambda$ and $\mathbf{G}_3(\varphi)x = \frac{1}{2\pi i}\int\limits_{\Gamma}\tilde{\varphi}(\lambda)[\lambda^2(\lambda^2-A)^{-1}C-C]xd\lambda$, $x\in E$, $\varphi\in\mathcal{D}$. The proof of [36, Theorem 2.1] implies $\operatorname{supp}\mathbf{G}\subseteq[0,\infty)$, $\frac{1}{2\pi i}\int\limits_{\Gamma}\tilde{\varphi}(\lambda)d\lambda = \varphi(0)$, $\varphi\in\mathcal{D}$ and $\lambda\widetilde{I(\varphi)}(\lambda) = -\widetilde{I(\varphi)'}(\lambda) = \widetilde{\varphi - \int\limits_{-\infty}^{\infty}\varphi(t)dt\zeta}(\lambda) = \tilde{\varphi}(\lambda)$, $\lambda\in\mathbb{C}$. Therefore, $\mathbf{G}_2 = \mathbf{G}_1^{-1}$ and $\mathbf{G}_3 = \mathbf{G}_1' - \delta\otimes C$. By Theorem 6(i), we get that $\mathbf{G}_1$ is a $(C-DCF)$. Denote by B the generator of G. Then we finally obtain

$$(x,y)\in B\Leftrightarrow\left(\begin{pmatrix}x\\0\end{pmatrix},\begin{pmatrix}0\\y\end{pmatrix}\right)\in\mathcal{C}^{-1}\mathcal{A}\mathcal{C}\Leftrightarrow(x,y)\in C^{-1}AC.$$

Proposition 8. *Assume that $\pm A$ generate C-distribution semigroups $G_\pm$ and that A^2 is closed. Then $C^{-1}A^2C$ generates a (C-DCF) $\mathbf{G}$, which is given by $\mathbf{G}(\varphi) := \frac{1}{2}(G_+(\varphi)+G_-(\varphi))$, $\varphi\in\mathcal{D}$.*

Proof. Since $\pm A$ generate C-distribution semigroups, it follows that, for every $\tau>0$, there exists $n_\tau\in\mathbb{N}$ such that $\pm A$ generate local n_τ-times integrated C-semigroups $(S_{n,\pm}(t))_{t\in[0,\tau)}$. The closedness of A^2 taken together with [39, Proposition 2.1.17] imply that, for every $\tau>0$, the operator A^2 is a subgenerator of the local n_τ-times integrated C-cosine function $(\frac{1}{2}(S_{n,-}(t)+S_{n,-}(t)))_{t\in[0,\tau)}$. Keeping in mind Theorem 6(iii), the above ensures that the operator $C^{-1}A^2C$ is the generator of a $(C-DCF)$ $\mathbf{G}$.

Theorem 9.

(i) *Let A be the generator of a (C-DCF) $\mathbf{G}$. Then $\mathbf{G} \in \mathcal{D}_0'(L(E,[D(A)]))$,*

$$\mathbf{G} * P = \delta' \otimes C_{[D(A)]} \in \mathcal{D}_0'(L([D(A)])) \text{ and } P * \mathbf{G} = \delta' \otimes C \in \mathcal{D}_0'(L(E)), \quad (1)$$

where $P := \delta'' \otimes I - \delta \otimes A \in \mathcal{D}_0'(L([D(A)], E))$ and I denotes the inclusion $[D(A)] \hookrightarrow E$.

(ii) *Suppose A is a closed linear operator, $\mathbf{G} \in \mathcal{D}_0'(L(E,[D(A)]))$, $\mathbf{G}(\varphi)C = C\mathbf{G}(\varphi)$, $\varphi \in \mathcal{D}$ and (1) holds. Then $\mathbf{G}$ is a (C-DCF) generated by $C^{-1}AC$.*

(iii) *Let $\mathbf{G} \in \mathcal{D}_0'(L(E))$ and $\mathbf{G}(\varphi)C = C\mathbf{G}(\varphi)$, $\varphi \in \mathcal{D}$. Then $\mathbf{G}$ is a (C-DCF) in E generated by A iff $\mathcal{G}$ is a ($\mathcal{C}$-DS) in $E \oplus E$ generated by $\mathcal{A}$.*

Proof. Let $X = L(E,[D(A)])$, $Y = L([D(A)],E)$, $Z = L([D(A)])$ and let $b : X \times Y \to Z$ be defined by $b(B,D) := BD$, $B \in X$, $D \in Y$. The definition of $\mathbf{G} * P$ is given by Proposition 2; the convolution $P * \mathbf{G}$ can be understood similarly. Let $x \in D(A)$, $k \in \mathbb{N}_0$ and $\varphi \in \mathcal{D}$. Then it is obvious that $(\mathbf{G} * (\delta^{(k)} \otimes I))(\varphi)x = (-1)^k\mathbf{G}(\varphi^{(k)})x$, $(\mathbf{G} * (\delta^{(k)} \otimes A))(\varphi)x = (-1)^k\mathbf{G}(\varphi^{(k)})Ax$, $((\delta^{(k)} \otimes I) * \mathbf{G})(\varphi)x = (-1)^k\mathbf{G}(\varphi^{(k)})x$ and $((\delta^{(k)} \otimes A) * \mathbf{G})(\varphi)x = (-1)^k A\mathbf{G}(\varphi^{(k)})x$, $\varphi \in \mathcal{D}$, $x \in E$, $k \in \mathbb{N}_0$. Suppose that $\mathbf{G}$ is a $(C - DCF)$ generated by A and $x \in E$. Then an application of Theorem 6(i)(a) gives $A\mathbf{G}(\varphi)x = \mathbf{G}(\varphi'')x + \varphi'(0)Cx$, which implies $\mathbf{G} \in \mathcal{D}_0'(L(E,[D(A)]))$, $(P * \mathbf{G})(\varphi)x = \mathbf{G}(\varphi'')x - A\mathbf{G}(\varphi)x = -\varphi'(0)Cx$ and $P * \mathbf{G} = \delta' \otimes C$. We obtain $\mathbf{G} * P = \delta' \otimes C_{[D(A)]}$ along the same lines, which completes the proof of (i). In order to prove (ii), let us assume $\mathbf{G} \in \mathcal{D}_0'(L(E,[D(A)]))$, $\mathbf{G} * P = \delta' \otimes C_{[D(A)]}$ and $P * \mathbf{G} = \delta' \otimes C$. Since $\operatorname{supp}\mathbf{G} \subseteq [0,\infty)$, it follows that $\operatorname{supp}\mathbf{G}^{-1} \subseteq [0,\infty)$ and $\operatorname{supp}\mathcal{G} \subseteq [0,\infty)$. If $x \in E$, then the assumptions $\mathbf{G} * P = \delta' \otimes C_{[D(A)]}$ and $P * \mathbf{G} = \delta' \otimes C$ imply $\mathbf{G}(\varphi)Ax = \mathbf{G}(\psi'')x + \psi'(0)Cx$, $\varphi \in \mathcal{D}$, $x \in D(A)$, $A\mathbf{G}^{-1}(\varphi)x = -\mathbf{G}(\varphi')x - \varphi(0)Cx$, $\varphi \in \mathcal{D}$, $x \in E$ and $\mathbf{G}^{-1}(\varphi)Ax = -\mathbf{G}(\varphi')x - \varphi(0)Cx$, $\varphi \in \mathcal{D}$, $x \in E$. It is also clear that $\mathcal{G}$ commutes with $\mathcal{C}$. Then one can repeat literally the proof of [39, Theorem 3.1.7] with a view to obtain that, for every $\tau > 0$, there exists $n_\tau \in \mathbb{N}$ such that $\mathcal{A}$ is a subgenerator of a local $(n_\tau + 1)$-times integrated $\mathcal{C}$-semigroup $(S_{n_\tau+1}(t))_{t\in[0,\tau)}$ whose integral generator is $\mathcal{C}^{-1}\mathcal{A}\mathcal{C}$ and which satisfies $\mathcal{G}(\varphi)\binom{x}{y} = (-1)^{n_\tau+1}\int_0^\tau \varphi^{(n_\tau+1)}(t)S_{n_\tau+1}(t)\binom{x}{y}dt$, $x, y \in E$, $\varphi \in \mathcal{D}_{(-\infty,\tau)}$. By making use of [39, Theorem 2.1.11], we get that, for every $\tau > 0$, there exists $n_\tau \in \mathbb{N}$ such that A is a subgenerator of a local n_τ-times integrated C-cosine function $(C_{n_\tau}(t))_{t\in[0,\tau)}$ whose integral generator is $C^{-1}AC$. Furthermore, the next equality holds $S_{n_\tau+1}(t) = \begin{pmatrix} \int_0^t C_{n_\tau}(s)ds & \int_0^t (t-s)C_{n_\tau}(s)ds \\ C_{n_\tau}(t) - \frac{t^{n_\tau}}{n_\tau!}C & \int_0^t C_{n_\tau}(s)ds \end{pmatrix}$, $t \in [0,\tau)$. This implies that $C^{-1}AC$ is the generator of a $(C - DCF)$ $\mathbf{G}$ and the proof of (ii) is completed. The proof of (iii) can be obtained as in the case of distribution cosine functions.

Definition 10. A $(C-DCF)$ $\mathbf{G}$ is said to be an *exponential C-distribution cosine function*, $(E-CDCF)$ in short, if $\mathcal{G}$ is an $(E-\mathcal{C}DS)$ in $E\oplus E$.

Theorem 11. ([34], [39])

(i) *Let $\mathbf{G}$ be a (C-DCF). Then $\mathbf{G}$ is exponential iff there exists $\omega\in\mathbb{R}$ such that $e^{-\omega t}\mathbf{G}^{-1}\in\mathcal{S}_0'(L(E))$.*

(ii) *Let A be a closed operator. Then the following assertions are equivalent:*

  (a) *The operator A is the generator of an (E-CDCF) in E.*

  (b) *The operator $\mathcal{A}$ is the generator of an (E-$\mathcal{C}$DS) in $E\oplus E$.*

  (c) *There exists $n\in\mathbb{N}$ such that A is the generator of an exponentially bounded n-times integrated C-cosine function.*

  (d) *There exist $\omega>0,\ M>0$ and $k\in\mathbb{N}$ such that $\Pi_\omega=\{x+iy: x>\omega^2-\frac{y^2}{4\omega^2}\}\subseteq\rho(A)$, $\|(\lambda-A)^{-1}C\|\leq M|\lambda|^k$, $\lambda\in\Pi_\omega$ and that the mapping $\lambda\mapsto(\lambda-A)^{-1}C$, $\lambda\in\Pi_\omega$ is strongly continuous.*

(iii) *Let A be a densely defined operator and let $R(C)$ be dense in E. If A is the generator of an (exponential) (C-DCF) in E, then A^* is the generator of an (exponential) (C^*-DCF) in E^*.*

(iv) *If A is the generator of an (C-EDCF), then for every $z\in\mathbb{C}$ the operator $A+z$ is also the generator of an (C-EDCF).*

(v) *Suppose $\pm A$ generate exponential C-distribution semigroups and A^2 is closed. Then $C^{-1}A^2C$ is the generator of an (E-CDCF).*

Theorem 12. ([39])

(i) *Let $\mathbf{G}$ be a (C-DCF). Then for all $\binom{x}{y}\in\mathcal{R}(\mathcal{G})$ there exists a unique function $u\in C^1([0,\infty):E)$ satisfying $u(0)=Cx$, $u'(0)=Cy$ and*

$$\mathbf{G}(\psi)x+\mathbf{G}^{-1}(\psi)y=\int\limits_0^\infty\psi(t)u(t)dt,\ \psi\in\mathcal{D}.$$

(ii) *Let $\mathbf{G}$ be a (C-DCF) generated by A. Then for all $x,y\in D_\infty(A)$ there exists a unique function $u\in C^1([0,\infty):E)$ satisfying $u(0)=Cx$, $u'(0)=Cy$ and*

$$\mathbf{G}(\varphi)x+\mathbf{G}^{-1}(\varphi)y=\int\limits_0^\infty\varphi(t)u(t)dt,\ \varphi\in\mathcal{D}_0.$$

(iii) *Let $\mathbf{G}$ be a (C-DCF) generated by A. Then $C(D_\infty(A))\subseteq\overline{\mathcal{R}(\mathbf{G})}$.*

(iv) *Let $R(C)$ be dense in E and let $\mathbf{G}$ be a (C-DCF) generated by A. Then the following assertions are equivalent:*

(a) $\mathbf{G}$ *is dense.*

(b) A *is densely defined.*

(c) $\mathbf{G}^*$ *is a (C^*-DCF) in* E^*.

(d) $\mathcal{G}$ *is dense.*

(e) $\mathcal{A}$ *is densely defined.*

(f) $\mathcal{G}^*$ *is a ($\mathcal{C}^*$-DS) in* $(E \oplus E)^*$.

In order to complete the structural theory of C-distribution cosine functions ([49], [34]-[35], [39]), one has to consider global integrated C-cosine functions with corresponding growth order, cosine convolution products and almost C-distribution cosine functions. The following may be of some independent interest and will not be further investigated in sections 3-5. Assume that $\tau_0 : [0,\infty) \to [0,\infty)$ is a measurable function such that $\inf\limits_{t\geq 0} \tau_0(t) > 0$ and that there exists $C_0 > 0$ satisfying:

$$\tau_0(t+s) \leq C_0\tau_0(t)\tau_0(s),\ t,s \geq 0 \text{ and } \tau_0(t-s) \leq C_0\tau_0(t)\tau_0(s),\ 0 < s < t.$$

Then $(L^1([0,\infty) : \tau_0), ||\cdot||_{\tau_0})$ denotes the Banach space which consists of those measurable functions $f : [0,\infty) \to \mathbb{C}$ such that $||f||_{\tau_0} := \int\limits_0^\infty |f(t)|\tau_0(t)dt < \infty$. If $f,g \in L^1([0,\infty) : \tau_0)$, define $f \circ g(t) := \int\limits_t^\infty f(s-t)g(s)ds,\ t \geq 0$. Clearly, $f *_0 g \in L^1([0,\infty) : \tau_0)$ and $f \circ g \in L^1([0,\infty) : \tau_0)$. The *cosine convolution product* $f *_c g$ is defined by $f *_c g := \frac{1}{2}(f *_0 g + f \circ g + g \circ f)$; the *sine convolution product* by $f *_s g := \frac{1}{2}(f *_0 g - f \circ g - g \circ f)$ and the *sine-cosine convolution product* by $f *_{sc} g := \frac{1}{2}(f *_0 g - f \circ g + g \circ f)$. It is obvious that $f *_c g,\ f *_s g,\ f *_{sc} g \in L^1([0,\infty) : \tau_0)$, resp. $\mathcal{D}_+$, if $f,g \in L^1([0,\infty) : \tau_0)$, resp. $f,g \in \mathcal{D}_+$.

Proposition 13. ([35])

(i) *Let* $\mathbf{G}$ *be a (C-DCF) generated by* A. *Then the following holds:*

$$\mathbf{G}(\varphi *_0 \psi)Cx = \mathbf{G}(\varphi)\mathbf{G}(\psi)x + A\mathbf{G}^{-1}(\varphi)\mathbf{G}^{-1}(\psi)x,\ \varphi,\psi \in \mathcal{D},\ x \in E.$$

(ii) *Let* $\mathbf{G} \in \mathcal{D}_0'(L(E))$ *satisfy* $\mathbf{G}(\varphi)\mathbf{G}(\psi) = \mathbf{G}(\psi)\mathbf{G}(\varphi),\ \varphi,\psi \in \mathcal{D}$. *Then the following assertions are equivalent:*

  (a) $\mathbf{G}$ *is a pre-(C-DCF) and* $\mathbf{G}^{-1}(\Lambda(f \circ g - g \circ f))C = \mathbf{G}(\Lambda(f))\mathbf{G}^{-1}(\Lambda(g)) - \mathbf{G}^{-1}(\Lambda(f))\mathbf{G}(\Lambda(g)),\ f,g \in \mathcal{D}_+$.

  (b) $\mathbf{G}^{-1}(\Lambda(f *_{sc} g))C = \mathbf{G}^{-1}(\Lambda(f))\mathbf{G}(\Lambda(g)),\ f,g \in \mathcal{D}_+$.

Definition 14. An element $\mathrm{G} \in L(\mathcal{D}_+ : L(E))$ is called an *almost C-distribution cosine function*, $(A-CDCF)$ in short, if $\mathrm{G}(f)C = C\mathrm{G}(f),\ f \in \mathcal{D}_+$,

(i) $\mathrm{G}(f *_c g)C = \mathrm{G}(f)\mathrm{G}(g),\ f,g \in \mathcal{D}_+$, and

(ii) $\bigcap_{f\in\mathcal{D}_+}\mathrm{Kern}(\mathrm{G}(f))=\{0\}$.

The (*infinitesimal*) *generator* A of G is defined by

$$A:=\Big\{(x,y)\in E\oplus E:\mathrm{G}(f)y=\mathrm{G}(f'')x+f'(0)Cx\ \text{ for all } f\in\mathcal{D}_+\Big\}.$$

It can be straightforwardly proved that A is a closed linear operator which satisfies $\mathrm{G}(f)A\subseteq A\mathrm{G}(f)$, $\mathrm{G}(f)x\in D(A)$, $A\mathrm{G}(f)x=\mathrm{G}(f'')x+f'(0)Cx$, $f\in\mathcal{D}_+$, $x\in E$ and $C^{-1}AC=A$.

Theorem 15. ([35])

(i) *Let* $\mathbf{G}$ *be a (C-DCF) generated by A. Then* $\mathbf{G}\Lambda$ *is an (A-CDCF) generated by A.*

(ii) *Let* $\mathbf{G}$ *be a (C-DCF) generated by A. Then*

$$G(\Lambda(f*_s g))C=AG^{-1}(\Lambda(f))G^{-1}(\Lambda(g)),\ f,g\in\mathcal{D}_+.$$

(iii) *Let* G *be an (A-CDCF) generated by A. Then A is the generator of a (C-DCF)* $\mathbf{G}$, *which is given by* $\mathbf{G}(\varphi):=\mathrm{G}(\mathcal{K}(\varphi))$, $\varphi\in\mathcal{D}$.

(iv) *Every (almost) C-distribution cosine function is uniquely determined by its generator.*

(v) *Let A be a closed linear operator. Then A is the generator of a (C-DCF) iff A is the generator of an (A-CDCF).*

(vi) *Let* $\mathbf{G}$ *be a (C-DCF). Then* $\mathbf{G}(\varphi)\mathbf{G}(\psi)=\mathbf{G}(\psi)\mathbf{G}(\varphi)$, $\varphi,\psi\in\mathcal{D}$.

Let $f\in\mathcal{D}_+$. Then the *Weyl fractional integral of order* $\alpha>0$ is defined by $(W_+^{-\alpha}f)(t):=\int_t^{\infty}\frac{(s-t)^{\alpha-1}}{\Gamma(\alpha)}f(s)ds$, $f\in\mathcal{D}_+$, $t\geq 0$. It is well known that, for every $\alpha>0$, the mapping $W_+^{-\alpha}:\mathcal{D}_+\to\mathcal{D}_+$ is bijective. The inverse mapping of $W_+^{-\alpha}(\cdot)$, denoted by $W_+^{\alpha}(\cdot)$, is called the *Weyl fractional derivative of order* $\alpha>0$. If $\alpha\in\mathbb{N}$, then $W_+^{\alpha}f=(-1)^n f^{(n)}$, $f\in\mathcal{D}_+$. Furthermore, $W_+^{\alpha}W_+^{\beta}=W_+^{\alpha+\beta}$ for all $\alpha,\beta\in\mathbb{R}$, where we put $W_+^0:=I$. Let us recall ([49]) that the family of *Bochner-Riesz functions* (R_t^{θ}), $\theta>-1$, $t>0$, is defined by $R_t^{\theta}(s)=\frac{(t-s)^{\theta}}{\Gamma(\theta+1)}\chi_{(0,t)}$. The Weyl functional calculus can be applied to the functions which do not belong to the space $\mathcal{D}_+$; for example, in the case of Bochner-Riesz functions one has $W_+^{\alpha}R_t^{\theta}=R_t^{\theta-\alpha}$, $\theta+1>\alpha\geq 0$. Designate by Ω_α, $\alpha>0$ the set which consists of all nondecreasing continuous functions $\tau_\alpha(\cdot)$ on $(0,\infty)$ such that $\inf_{t>0}t^{-\alpha}u(t)>0$ and that there exists a constant $C_\alpha>0$ satisfying

$$\int\limits_{[0,t]\ \cap\ [s,s+t]} u^{\alpha-1}\tau_\alpha(t+s-u)du\leq C_\alpha\tau_\alpha(t)\tau_\alpha(s),\ 0<t\leq s.$$

The typical functions $\tau_\alpha(t) = t^\alpha;\ t^\beta(1+t)^\gamma\ (\beta \in [0,\alpha],\ \beta+\gamma \geq \alpha);\ t^\beta e^{\tau t}\ (\beta \in [0,\alpha],\ \tau > 0)$ belong to Ω_α. Suppose $\tau_\alpha \in \Omega_\alpha$ and $\nu > \alpha$; then the function $\tau_\nu = t^{\nu-\alpha}\tau_\alpha,\ t > 0$ belongs to Ω_ν. Designate by Ω_α^h the subset of $\Omega_\alpha,\ \alpha > 0$ which consists of all functions of the form $\tau_\alpha(t) = t^\alpha\omega_0(t),\ t > 0$, where the continuous nondecreasing function $\omega_0 : [0,\infty) \to [0,\infty)$ satisfies $\inf_{t>0} \omega_0(t) > 0$ and $\omega_0(t+s) \leq \omega_0(t)\omega_0(s),\ t, s > 0$. Suppose $\alpha > 0$, $\tau_\alpha \in \Omega_\alpha$ and define

$$q_{\tau_\alpha}(\varphi) := \int_0^\infty \frac{\tau_\alpha(t)}{\Gamma(\alpha+1)}\Big|W_+^\alpha\varphi(t)\Big|dt,\ \varphi \in \mathcal{D}_+.$$

Then $q_{\tau_\alpha}(\cdot)$ is a norm on $\mathcal{D}_+$ and there exists a constant $C_\alpha > 0$ such that $q_{\tau_\alpha}(\varphi *_c \phi) \leq C_\alpha q_{\tau_\alpha}(\varphi)q_{\tau_\alpha}(\phi),\ \varphi, \phi \in \mathcal{D}_+$ ([49]). Let $\mathfrak{T}_+^\alpha(\tau_\alpha,\ *_c)$ denote the completion of the normed space $(\mathcal{D}_+, q_{\tau_\alpha})$; then $\mathfrak{T}_+^\alpha(\tau_\alpha,\ *_c)$ is invariant under the cosine convolution cosine product $*_c$ and the following holds (cf. [49, Theorem 3]):

(i) $\mathfrak{T}_+^\alpha(\tau_\alpha,\ *_c) \hookrightarrow \mathfrak{T}_+^\alpha(t^\alpha,\ *_c) \hookrightarrow L^1([0,\infty),\ *_c)$, where $\hookrightarrow$ denotes the dense and continuous embedding,

(ii) $\mathfrak{T}_+^\beta(t^\beta,\ *_c) \hookrightarrow \mathfrak{T}_+^\alpha(t^\alpha,\ *_c)$, $\beta > \alpha > 0$,

(iii) $R_t^{\nu-1} \in \mathfrak{T}_+^\alpha(\tau_\alpha,\ *_c),\ \nu > \alpha,\ t > 0$ and there exists $C_{\nu,\alpha} > 0$ such that $q_{\tau_\alpha}(R_t^{\nu-1}) \leq C_{\nu,\alpha}t^{\nu-\alpha}\tau_\alpha(t),\ t > 0$.

An $(A - CDCF)$ G is said to be of *order* $\alpha > 0$ and *growth* $\tau_\alpha \in \Omega_\alpha$ if G can be extended to a continuous linear mapping from $\mathfrak{T}_+^\alpha(\tau_\alpha,\ *_c)$ into $L(E)$.

Theorem 16. ([49])

(i) *Let A be the generator of an α-times integrated C-cosine function $(C_\alpha(t))_{t\geq 0}$ and let $||C_\alpha(t)|| = O(\tau_\alpha(t))$, $t > 0$. Then the mapping $\mathrm{G} : \mathfrak{T}_+^\alpha(\tau_\alpha,\ *_c) \to L(E)$, given by*

$$\mathrm{G}(f)x := \int_0^\infty W_+^\alpha f(t)C_\alpha(t)xdt,\ f \in \mathfrak{T}_+^\alpha(\tau_\alpha,\ *_c),\ x \in E, \tag{2}$$

is a continuous algebra homomorphism satisfying:

$$\int_0^t \frac{(t-s)^{\nu-\alpha-1}}{\Gamma(\nu-\alpha)}C_\alpha(s)xds = \mathrm{G}(R_t^{\nu-1})x,\ \nu > \alpha,\ x \in E$$

and

$$\int_0^\infty W_+^\alpha f(t)C_\alpha(t)xdt = \int_0^\infty W_+^\nu f(t)\int_0^t \frac{(t-s)^{\nu-\alpha-1}}{\Gamma(\nu-\alpha)}C_\alpha(s)xdsdt,$$

for all $f \in {\mathfrak{T}_+}^\nu(t^{\nu-\alpha}\tau_\alpha,\ *_c),\ x \in E$. Furthermore, the restriction of G to $\mathcal{D}_+$ is an almost-distribution cosine function of order $\alpha > 0$ and growth τ_α with the generator A.

(ii) *Suppose A is the generator of an (A-CDCF)* G *of order* $\alpha > 0$ *and growth* $\tau_\alpha \in \Omega_\alpha$. *Then, for every* $\nu > \alpha$, A *generates a* ν*-times integrated* C*-cosine function* $(C_\nu(t))_{t\geq 0}$ *such that* $||C_\nu(t)|| \leq C_\nu t^{\nu-\alpha}\tau_\alpha(t),\ t > 0$ *and that*

$$\mathrm{G}(f)x = \int_0^\infty W_+^\nu f(t) \int_0^t \frac{(t-s)^{\nu-\alpha-1}}{\Gamma(\nu-\alpha)} C_\alpha(s)xdsdt,\ f \in \mathcal{D}_+,\ x \in E.$$

(iii) *Let* $\alpha > 0$, $\tau_\alpha \in \Omega_\alpha^h$ *and let* $D(A)$ *and* $R(C)$ *be dense in* E. *Then the following assertions are equivalent:*

(a) *The operator* A *is the generator of an* α*-times integrated* C*-cosine function* $(C_\alpha(t))_{t\geq 0}$ *such that* $||C_\alpha(t)|| = O(\tau_\alpha(t)),\ t > 0$.

(b) *The operator* A *is the generator of an (A-CDCF)* G *of order* $\alpha > 0$ *and growth* τ_α *such that* $\mathrm{G}(\mathcal{D}_+)$ *is dense in* E.

The following theorem will be useful in our further work.

Theorem 17. *Assume* $\alpha \geq 0$ *and* A *is a subgenerator of a global* α*-times integrated* C*-cosine function* $(C_\alpha(t))_{t\geq 0}$. *Then, for every* $\beta > \alpha$, *the operator* A *is a subgenerator of a global* β*-times integrated* C*-cosine function* $(C_\beta(t))_{t\geq 0}$, *which is given by* $C_\beta(t)x = \int_0^t \frac{(t-s)^{\beta-\alpha-1}}{\Gamma(\beta-\alpha)} C_\alpha(s)xds$, $x \in E$, $t \geq 0$. *Define*

$$\mathbf{G}(\varphi)x := \int_0^\infty W_+^\alpha(\mathcal{K}(\varphi))(t)C_\alpha(t)xdt,\ x \in E,\ \varphi \in \mathcal{D}. \tag{3}$$

Then $\mathbf{G}$ *is a (C-DCF) generated by* $C^{-1}AC$ *and the following equality holds:* $\mathbf{G}(\varphi)x = \int_0^\infty W_+^\beta(\mathcal{K}(\varphi))(t)C_\beta(t)xdt$, $x \in E$, $\varphi \in \mathcal{D}$.

A function $u(t)$ is said to be a *mild solution* of the abstract Cauchy problem

$$(ACP_1) : u'(t) = Au(t),\ t \geq 0,\ u(0) = x,\ \text{resp.},$$

$$(ACP_2) : u''(t) = Au(t),\ t \geq 0,\ u(0) = x,\ u'(0) = y,$$

if the mapping $t \mapsto u(t),\ t \geq 0$ is continuous, $\int_0^t u(s)ds \in D(A)$ and $A\int_0^t u(s)ds = u(t) - x,\ t \geq 0$, resp., if the mapping $t \mapsto u(t),\ t \geq 0$ is continuous, $\int_0^t (t-s)u(s)ds \in D(A)$ and $A\int_0^t (t-s)u(s)ds = u(t) - x - ty$, $t \geq 0$. It can be proved that there exists at most one mild solution of (ACP_1), resp. (ACP_2), provided that there exists $\alpha \geq 0$ such that A is a subgenerator of a local α-times integrated C-semigroup, resp., a local α-times integrated C-cosine function. If mild solutions of (ACP_1) are unique, then the solution space for A, denoted by $Z(A)$, is defined to be the set of all $x \in E$ for which there exists a unique mild solution of (ACP_1). In order not to put a strain on the exposition, and to stay consistent with previously given definitions of hypercyclicity and chaos of cosine functions ([11], [31]), we primarily consider mild solutions of (ACP_2) with $y = 0$. This, however, may not be the optimal choice and we refer the reader to [15] as well as Theorem 33, Theorem 35, Example 29, Example 36 and Remark 34 for further information in this direction. Denote by $Z_2(A)$ the set which consists of all $x \in E$ for which there exists such a solution. Let $\pi_1 : E \oplus E \to E$ and $\pi_2 : E \oplus E \to E$ be the projections and let G be a $(C-DCF)$ generated by A. Then $\mathcal{G}$ is a $(\mathcal{C}-DS)$ generated by $\mathcal{A}$ and the solution space $Z(\mathcal{A})$ can be characterized by means of [37, Lemma 6]: Denote by $D(\mathcal{G})$ the set of all $x \in \bigcap_{t\geq 0} D(\mathcal{G}_1(\delta_t))$ satisfying that the mapping $t \mapsto \mathcal{G}_1(\delta_t)x,\ t \geq 0$ is continuous; here $\mathcal{G}_1(\delta_t) = \Big\{ (\binom{x_1}{y_1}, \binom{x_2}{y_2}) : \mathcal{G}_1(\varphi(\cdot - t))\binom{x_1}{y_1} = \mathcal{G}_1(\varphi)\binom{x_2}{y_2},\ \varphi \in \mathcal{D}_0 \Big\}$. Then $Z(\mathcal{A}) = D(\mathcal{G})$ and the mild solution $u(\cdot; \binom{x}{y})$ of (ACP_1) with initial value $\binom{x}{y} \in Z(\mathcal{A})$ is given by $u(t; \binom{x}{y}) = \mathcal{G}_1(\delta_t)\binom{x}{y},\ t \geq 0$. Assume that, for every $\tau > 0$, A is the integral generator of a local n_τ-times integrated C-cosine function $(C_{n_\tau}(t))_{t\in[0,\tau)}$. Then it can be proved that the solution space $Z(\mathcal{A})$ consists of those pairs $\binom{x}{y}$ in $E\oplus E$ which fulfill that, for every $\tau > 0$, $C_{n_\tau}(t)x + \int_0^t C_{n_\tau}(s)yds \in \mathrm{R}(C)$, $t \in [0, \tau)$ and that the mapping $t \mapsto C^{-1}(C_{n_\tau}(t)x + \int_0^t C_{n_\tau}(s)yds)$, $t \in [0, \tau)$ is $(n_\tau + 1)$-times continuously differentiable. By prior arguments, one yields that $x \in Z_2(A)$ iff $\binom{0}{x} \in Z(\mathcal{A})$ iff $\binom{0}{x} \in D(\mathcal{G})$, and $u(t; x) = \pi_2(\mathcal{G}_1(\delta_t)\binom{0}{x})$, $t \geq 0$, where $u(\cdot; x)$ denotes the mild solution of (ACP_2) with $y = 0$. Define $G(\delta_t)x := \pi_2(\mathcal{G}_1(\delta_t)\binom{0}{x})$, $t \geq 0$, $x \in Z_2(A)$.

Proposition 18. ([37]) *Assume that, for every $\tau > 0$, there exists $n_\tau \in \mathbb{N}$ such that A is a subgenerator of a local n_τ-times integrated C-cosine function $(C_{n_\tau}(t))_{t\in[0,\tau)}$. Then the solution space $Z_2(A)$ consists exactly of those vectors $x \in E$ such that, for every $\tau > 0$, $C_{n_\tau}(t)x \in R(C)$ and that the mapping $t \mapsto C^{-1}C_{n_\tau}(t)x$, $t \in [0, \tau)$ is n_τ-times continuously differentiable. If $x \in Z_2(A)$ and $t \in [0, \tau)$, then $G(\delta_t)x = \frac{d^{n_\tau}}{dt^{n_\tau}} C^{-1}C_{n_\tau}(t)x$.*

Proposition 19. *Let A be the generator of a (C-DCF)* **G** *and let $x \in Z_2(A)$. Then $G(\delta_t)(Z_2(A)) \subseteq Z_2(A)$, $t \geq 0$, $2G(\delta_s)G(\delta_t)x = G(\delta_{t+s})x + G(\delta_{|t-s|})x$, $t, s \geq 0$ and* $\mathbf{G}(\varphi)x = \int_0^\infty \varphi(t)CG(\delta_t)xdt$, $\varphi \in \mathcal{D}_0$.

PROOF. We will only prove the d'Alambert formula $2G(\delta_s)G(\delta_t)x = G(\delta_{t+s})x + G(\delta_{|t-s|})x$, $t, s \geq 0$. Fix a $t \geq 0$ and define afterwards

$$u(s; G(\delta_t)x) := \frac{1}{2}\Big[G(\delta_{t+s})x + G(\delta_{|t-s|})x\Big],\ s \geq 0.$$

Then the mapping $s \mapsto u(s; G(\delta_t)x)$, $s \geq 0$ is continuous and, for every $s \in [0, t]$,

$\int\limits_0^s (s-r)u(r;G(\delta_t)x)dr = \int\limits_0^{t+s} (t+s-r)u(r;G(\delta_t)x)dr - \int\limits_0^t (t-r)u(r;G(\delta_t)x)dr + \int\limits_0^{t-s} (t-s-r)u(r;G(\delta_t)x)dr \in D(A)$ and $A\int\limits_0^s (s-r)u(r;G(\delta_t)x)dr = \frac{1}{2}[G(\delta_{t+s})x - x] - \frac{1}{2}[G(\delta_t)x - x] + \frac{1}{2}[G(\delta_{t-s})x - x] = \frac{1}{2}[G(\delta_{t+s})x - x] = u(s;G(\delta_t)x) - G(\delta_t)x$. One can similarly prove that $A\int\limits_0^s (s-r)u(r;G(\delta_t)x)dr = u(s;G(\delta_t)x) - G(\delta_t)x$, $s > t$, which completes the proof of theorem.

Assume $\mathbf{G}$ is a $(C-DCF)$ generated by A and $x \in Z_2(A)$. Then Proposition 19 implies $C(Z_2(A)) \subseteq \overline{\mathcal{R}(\mathcal{G})}$ and $\mathcal{G}(\varphi)x \in \mathrm{R}(C)$, $\varphi \in \mathcal{D}_0$. Further on, $C(Z_2(A)) \subseteq Z_2(A)$ and $G(\delta_t)Cx = CG(\delta_t)x$, $t \geq 0$.

Proposition 20.

(i) *Assume* $\mathbf{G}$ *is a (C-DCF) generated by* A*. Then* $\mathcal{R}(\mathbf{G}) \subseteq Z_2(A)$.

(ii) *Assume* A *is a closed linear operator,* $x \in Z(A) \cap Z(-A)$*,* $u_1(\cdot;x)$ *and* $u_2(\cdot;x)$ *are mild solutions of* (ACP_1) *for* A *and* $-A$*, respectively, and* $u(t;x) := \frac{1}{2}(u_1(t;x) + u_2(t;x))$*,* $t \geq 0$*. If* A^2 *is closed, then* $u(\cdot;x)$ *is a mild solution of* (ACP_2) *for* A^2.

Proof. We will prove only (i). Assume $x \in \mathcal{R}(\mathbf{G})$ and $x = \mathbf{G}(\varphi)y$ for some $\varphi \in \mathcal{D}_0$ and $y \in E$. Put

$$u(t;x) := \frac{1}{2}\Big[\mathbf{G}(\varphi(\cdot - t))y + \mathbf{G}(\varphi(\cdot + t))y + \mathbf{G}(\varphi(t - \cdot))y\Big],\ t \geq 0. \tag{4}$$

Using the continuity of $\mathbf{G}$, one gets that $u(\cdot;x) \in C([0,\infty) : E)$. Denote $f(t) := \mathbf{G}(\varphi(\cdot - t))y$, $g(t) := \mathbf{G}(\varphi(\cdot + t))y$ and $h(t) := \mathbf{G}(\varphi(t - \cdot))y$, $t \geq 0$. Then $f, g, h \in C^2([0,\infty) : E)$, $f'(t) = -\mathbf{G}(\varphi'(\cdot - t))y$, $f''(t) = \mathbf{G}(\varphi''(\cdot - t))y$, $g'(t) = \mathbf{G}(\varphi'(\cdot + t))y$, $g''(t) = \mathbf{G}(\varphi''(\cdot + t))y$, $h'(t) = -\mathbf{G}(\varphi'(t - \cdot))y$ and $h''(t) = \mathbf{G}(\varphi''(t - \cdot))y$, $t \geq 0$. The above equalities, the partial integration, the representation formula (4) and Theorem 6(ii)(a) taken together imply:

$$A\int\limits_0^t (t-s)u(s;x)ds$$
$$= \frac{1}{2}\int\limits_0^t (t-s)[\mathbf{G}(\varphi''(\cdot - s))y + \mathbf{G}(\varphi''(\cdot + s))y + \varphi'(s)Cy + \mathbf{G}(\varphi''(s - \cdot))y - \varphi'(s)Cy]ds$$
$$= \frac{1}{2}[-\int\limits_0^t \mathbf{G}(\varphi'(\cdot - s))y + \int\limits_0^t \mathbf{G}(\varphi'(\cdot + s))y - \int\limits_0^t \mathbf{G}(\varphi'(s - \cdot))y]$$
$$= u(t;x) - x,\ t \geq 0.$$

3 Hypercyclicity and chaos for C-distribution cosine functions and integrated C-cosine functions

Henceforth we assume that E is a separable infinite-dimensional complex Banach space and that S is a non-empty closed subset of $\mathbb{C}$ satisfying $\mathrm{S} \setminus \{0\} \neq \emptyset$.

Let $\mathbf{G}$ be a $(C-DCF)$. A closed linear subspace $\tilde{E}$ of E is said to be $\mathbf{G}$-*admissible* iff $G(\delta_t)(Z_2(A)\cap\tilde{E})\subseteq Z_2(A)\cap\tilde{E}$, $t\geq 0$. Define $\mathbf{G}_{wm}(\varphi)\binom{x}{y}:=\binom{\mathbf{G}(\varphi)x}{\mathbf{G}(\varphi)y}$, $x,y\in E$, $\varphi\in\mathcal{D}$. Then $\mathbf{G}_{wm}$ is a $(\mathcal{C}-DCF)$ in $E\oplus E$ generated by $A\oplus A$, $Z_2(A\oplus A)=Z_2(A)\oplus Z_2(A)$, and $\tilde{E}\oplus\tilde{E}$ is $\mathbf{G}_{wm}$-admissible provided that $\tilde{E}$ is $\mathbf{G}$-admissible.

Definition 21. Let $\mathbf{G}$ be a $(C-DCF)$ and let $\tilde{E}$ be $\mathbf{G}$-admissible. Then it is said that $\mathbf{G}$ is:

(i) $\tilde{E}$-*hypercyclic*, if there exists $x\in Z_2(A)\cap\tilde{E}$ such that the set $\{G(\delta_t)x:t\geq 0\}$ is dense in $\tilde{E}$,

(ii) $\tilde{E}$-*chaotic*, if $\mathbf{G}$ is $\tilde{E}$-hypercyclic and the set of $\tilde{E}$-periodic points of $\mathbf{G}$, $\mathbf{G}_{\tilde{E},per}$, defined by $\{x\in Z_2(A)\ \cap\ \tilde{E}:G(\delta_{t_0})x=x$ for some $t_0>0\}$, is dense in $\tilde{E}$,

(iii) $\tilde{E}$-*topologically transitive*, if for every $y,z\in\tilde{E}$ and $\varepsilon>0$, there exist $v\in Z_2(A)\ \cap\ \tilde{E}$ and $t\geq 0$ such that $||y-v||<\varepsilon$ and $||z-G(\delta_t)v||<\varepsilon$,

(iv) $\tilde{E}$-*topologically mixing*, if for every $y,z\in\tilde{E}$ and $\varepsilon>0$, there exists $t_0\geq 0$ such that, for every $t\geq t_0$, there exists $v_t\in Z_2(A)\ \cap\ \tilde{E}$ such that $||y-v_t||<\varepsilon$ and $||z-G(\delta_t)v_t||<\varepsilon$, $t\geq t_0$,

(v) $\tilde{E}$-*weakly mixing*, if $\mathbf{G}_{wm}$ is $(\tilde{E}\oplus\tilde{E})$-topologically transitive in $E\oplus E$,

(vi) $\tilde{E}$-*supercyclic*, if there exists $x\in Z_2(A)\ \cap\ \tilde{E}$ such that its projective orbit $\{cG(\delta_t)x:c\in\mathbb{C},\ t\geq 0\}$ is dense in $\tilde{E}$,

(vii) $\tilde{E}$-*positively supercyclic*, if there exists $x\in Z_2(A)\ \cap\ \tilde{E}$ such that its positive projective orbit $\{cG(\delta_t)x:c\geq 0,\ t\geq 0\}$ is dense in $\tilde{E}$,

(viii) $\tilde{E}_{S}$-*hypercyclic*, if there exists $x\in Z_2(A)\ \cap\ \tilde{E}$ such that its S-projective orbit $\{cG(\delta_t)x:c\in \mathrm{S},\ t\geq 0\}$ is dense in $\tilde{E}$; any element $x\in Z_2(A)\ \cap\ \tilde{E}$ which satisfies the above property is called a $\tilde{E}_{\mathrm{S}}$-hypercyclic vector of $\mathbf{G}$,

(ix) $\tilde{E}_{S}$-*topologically transitive*, if for every $y,z\in\tilde{E}$ and $\varepsilon>0$, there exist $v\in Z_2(A)\ \cap\ \tilde{E}$, $t\geq 0$ and $c\in\mathrm{S}$ such that $||y-v||<\varepsilon$ and $||z-cG(\delta_t)v||<\varepsilon$,

(x) *sub-chaotic*, if there exists a $\mathbf{G}$-admissible subset $\hat{E}$ such that $\mathbf{G}$ is $\hat{E}$-chaotic.

In what follows, we use the fact that the notion of $\tilde{E}$-periodic points and $\tilde{E}$-topological transitivity ($\tilde{E}_{\mathrm{S}}$-topological transitivity) of a $(C-DCF)$ $\mathbf{G}$ (or a $(C-DS)$ G, cf. [37] for the notion) can be defined even in the case that $\tilde{E}$ is not $\mathbf{G}$-admissible.

Assume that there exists $\alpha\geq 0$ such that A is the integral generator of an α-times integrated C-cosine function $(C_\alpha(t))_{t\geq 0}$. Put

$$\mathbf{G}_\alpha(\varphi)x:=\int\limits_0^\infty W_+^\alpha(\mathcal{K}(\varphi))(t)C_\alpha(t)x\,dt,\ x\in E,\ \varphi\in\mathcal{D}.$$

Then Theorem 17 implies that $\mathbf{G}_\alpha$ is a $(C-DCF)$ generated by A.

Definition 22. Let $\tilde{E}$ be a closed linear subspace of E. Then it is said that $\tilde{E}$ is $(C_\alpha(t))_{t\geq 0}$-*admissible* iff $\tilde{E}$ is $\mathbf{G}_\alpha$-admissible, and that $(C_\alpha(t))_{t\geq 0}$ is $\tilde{E}$-*hypercyclic* iff $\mathbf{G}_\alpha$ is; all other dynamical properties of $(C_\alpha(t))_{t\geq 0}$ are understood in the same sense. Let $\tilde{E}$ be $(C_\alpha(t))_{t\geq 0}$-admissible; then a point $x\in\tilde{E}$ is said to be a $\tilde{E}$-*periodic point* ($\tilde{E}_{\mathrm{S}}$-hypercyclic vector) of $(C_\alpha(t))_{t\geq 0}$ iff x is a $\tilde{E}$-periodic point ($\tilde{E}_{\mathrm{S}}$-hypercyclic vector) of $\mathbf{G}_\alpha$.

It is clear that the notion of $\tilde{E}_{\mathrm{S}}$-hypercyclicity generalizes the notions of (positive) $\tilde{E}$-supercyclicity and $\tilde{E}$-hypercyclicity. In the case $\tilde{E}=E$, it is also said that $\mathbf{G}$ ($(C_\alpha(t))_{t\geq 0}$) is hypercyclic, chaotic, ..., S-hypercyclic, S-topologically transitive, and we write $\mathbf{G}_{per}$ instead of $\mathbf{G}_{\tilde{E},per}$. Using Theorem 17 again, we get that a closed linear subspace $\tilde{E}$ of E is $(C_\alpha(t))_{t\geq 0}$-admissible iff $\tilde{E}$ is $(C_\beta(t))_{t\geq 0}$-admissible, and that $(C_\alpha(t))_{t\geq 0}$ is $\tilde{E}$-hypercyclic ($\tilde{E}$-chaotic, ..., sub-chaotic) iff $(C_\beta(t))_{t\geq 0}$ is; this is why we assume in the sequel that $\alpha\in\mathbb{N}_0$. Let $\mathbf{G}_i$ be a (C_i-DCF) generated by A, $i=1,2$. Then a closed linear subspace $\tilde{E}$ of E is $\mathbf{G}_1$-admissible iff $\tilde{E}$ is $\mathbf{G}_2$-admissible. Furthermore, it follows from Definition 21 that $\mathbf{G}_1$ and $\mathbf{G}_2$ share common dynamical properties, which can be simply reformulated in the case of global integrated C-cosine functions.

It is easily seen that $\tilde{E}_{\mathrm{S}}$-hypercyclicity ($\tilde{E}_{\mathrm{S}}$-topological transitivity) of $\mathbf{G}$ implies $\overline{\tilde{E}\cap Z_2(A)}=\tilde{E}$. By Proposition 19, the assumption $G(\delta_{t_0})x=x$ for some $t_0>0$ and $x\in Z_2(A)$ implies by induction $G(\delta_{t_0})^n x=G(\delta_{nt_0})x=x$, $n\in\mathbb{N}$, so that the notion of $\tilde{E}$-periodic points of $\mathbf{G}$ is meaningful in some sense.

Before going any further, we would like to make a general observation on infinitely regular S-hypercyclic vectors of cosine functions. Let $(C(t))_{t\geq 0}$ be an S-topologically transitive cosine function and let $\mathrm{HC}_{\mathrm{S}}(C(\cdot))$ denote the set which consists of all S-hypercyclic vectors of $(C(t))_{t\geq 0}$. Then one can prove by means of [28, Lemma 3.1, Theorem 3.2] that $\mathrm{HC}_{\mathrm{S}}(C(\cdot))\ \cap\ D_\infty(A)$ is a dense subset of E.

Given $t>0$ and $\sigma>0$, set

$$\Phi_{t,\sigma}:=\Bigl\{\varphi\in\mathcal{D}_0:\operatorname{supp}\varphi\subseteq(t-\sigma,t+\sigma),\ \varphi\geq 0,\ \int\varphi(s)ds=1\Bigr\}.$$

The following theorem can be proved by making use of Proposition 19 and the proof of [19, Theorem 4.6].

Theorem 23. ([19], [37])

(i) *Assume $n\in\mathbb{N}_0$, A is the integral generator of an n-times integrated C-cosine function $(C_n(t))_{t\geq 0}$, $\overline{C(\tilde{E})}=\tilde{E}$ and $\tilde{E}$ is $\mathbf{G}_n$-admissible. Then the following holds:*

(a) *$(C_n(t))_{t\geq 0}$ is $\tilde{E}_S$-hypercyclic iff there exists $x\in\tilde{E}$ such that the mapping $t\mapsto C_n(t)x$, $t\geq 0$ is n-times continuously differentiable and that the set $\{c\frac{d^n}{dt^n}C_n(t)x : c\in S,\ t\geq 0\}$ is dense in $\tilde{E}$.*

(b) $(C_n(t))_{t\geq 0}$ *is* $\tilde{E}_S$*-topologically transitive iff for every* $y, z \in \tilde{E}$ *and* $\varepsilon > 0$, *there exist* $v \in \tilde{E}$, $t_0 \geq 0$ *and* $c \in S$ *such that the mapping* $t \mapsto C_n(t)v$, $t \geq 0$ *is* n*-times continuously differentiable and that* $||y - v|| < \varepsilon$ *as well as* $||z - c(\frac{d^n}{dt^n}C_n(t)v)_{t=t_0}|| < \varepsilon$.

(c) $(C_n(t))_{t\geq 0}$ *is* $\tilde{E}$*-chaotic iff* $(C_n(t))_{t\geq 0}$ *is* $\tilde{E}$*-hypercyclic and there exists a dense subset of* $\tilde{E}$ *consisting of those vectors* $x \in \tilde{E}$ *for which there exists* $t_0 > 0$ *such that the mapping* $t \mapsto C_n(t)x$, $t \geq 0$ *is* n*-times continuously differentiable and that* $(\frac{d^n}{dt^n}C_n(t)x)_{t=t_0} = Cx$.

(ii) *Let* A *be the generator of a (C-DCF)* $\mathbf{G}$ *and let* $\tilde{E}$ *be* $\mathbf{G}$*-admissible. Then:*

(a) $\mathbf{G}$ *is* $\tilde{E}_S$*-hypercyclic iff there exists* $x_0 \in Z_2(A) \cap \tilde{E}$ *such that, for every* $x \in \tilde{E}$ *and* $\varepsilon > 0$, *there exist* $t_0 > 0$, $c \in S$ *and* $\sigma > 0$ *such that*

$$||cC^{-1}\mathbf{G}(\varphi)x_0 - x|| < \varepsilon,\ \varphi \in \Phi_{t_0,\sigma}.$$

(b) $\mathbf{G}$ *is* $\tilde{E}_S$*-topologically transitive iff for every* $y, z \in \tilde{E}$ *and* $\varepsilon > 0$, *there exist* $t_0 > 0$, $c \in S$, $\sigma > 0$ *and* $v \in Z_2(A) \cap \tilde{E}$ *such that, for every* $\varphi \in \Phi_{t_0,\sigma}$,

$$||y - v|| < \varepsilon \text{ and } ||z - cC^{-1}\mathbf{G}(\varphi)v|| < \varepsilon.$$

(c) $\mathbf{G}$ *is* $\tilde{E}$*-chaotic iff* $\mathbf{G}$ *is* $\tilde{E}$*-hypercyclic and if there exists a dense set in* $\tilde{E}$ *of vectors* $x \in Z_2(A) \cap \tilde{E}$ *for which there exists* $\tau > 0$ *such that, for every* $\varepsilon > 0$, *there exists* $\sigma > 0$ *satisfying*

$$||C^{-1}\mathbf{G}(\varphi)x - x|| < \varepsilon,\ \varphi \in \Phi_{\tau,\sigma}.$$

Corollary 24. *Let* A *be the generator of a (C-DCF)* $\mathbf{G}$. *Assume* $\tilde{E}$ *is* $\mathbf{G}$*-admissible and* $\mathbf{G}$ *is* $\tilde{E}_S$*-hypercyclic (*$\tilde{E}_S$*-topologically transitive). Then* $\overline{C(\tilde{E})} \subseteq \overline{\mathcal{R}(\mathbf{G})} \subseteq \overline{D_\infty(A)}$.

The proof of following theorem follows from Proposition 19 and the fact that the continuity of a single operator $C(t)$ ($t \geq 0$) is not used in the proofs of [11, Theorem 1.2, Corollary 1.3, Theorem 1.4].

Theorem 25. *Let* $\mathbf{G}$ *be a (C-DCF) and let* $\tilde{E}$ *be* $\mathbf{G}$*-admissible.*

(i) *Assume that there exists a sequence* (t_n) *of non-negative real numbers such that*

$$X_{0,\tilde{E}} := \Big\{x \in Z_2(A) \cap \tilde{E} : \lim_{n\to\infty} G(\delta_{t_n})x = 0\Big\}$$

and

$$X_{\infty,\tilde{E}} := \Big\{y \in \tilde{E} : \text{ there exists a zero sequence } (u_n) \text{ in } Z_2(A) \cap \tilde{E} \text{ and } c \in S \setminus \{0\} \text{ such that } \lim_{n\to\infty} G(\delta_{t_n})cu_n = y\Big\}$$

are dense subsets of $\tilde{E}$. *Then* $\mathbf{G}$ *is* $\tilde{E}_S$*-topologically transitive.*

(ii) *Assume that there exists a sequence (t_n) of non-negative real numbers such that the set*

$$X_{1,\tilde{E}} := \Big\{ x \in Z_2(A) \ \cap \ \tilde{E} : \lim_{n\to\infty} G(\delta_{t_n})x = \lim_{n\to\infty} G(\delta_{2t_n})x = 0 \Big\}$$

is dense in $\tilde{E}$. Then $\mathbf{G}$ *is $\tilde{E}$-topologically transitive.*

(iii) *Assume that the set*

$$X_{\tilde{E}} := \Big\{ x \in Z_2(A) \ \cap \ \tilde{E} : \lim_{t\to\infty} G(\delta_t)x = 0 \Big\}$$

is dense in $\tilde{E}$. Then $\mathbf{G}$ *is $\tilde{E}$-topologically mixing.*

Remark 26.

(i) Assume $x, y \in E$, $\lambda_1, \lambda_2 \in \mathbb{C}$, $Ax = \lambda_1 x$ and $Ay = \lambda_2 y$. Then $x \in Z(A) \cap Z_2(A)$, the mild solution of (ACP_1) is given by $u(t; x) = e^{\lambda_1 t}x$, $t \geq 0$ and the mild solution of (ACP_2) is given by $u(t; x, y) = \sum_{n=0}^{\infty} \frac{t^{2n}}{(2n)!}\lambda_1^n x + \sum_{n=0}^{\infty} \frac{t^{2n+1}}{(2n+1)!}\lambda_2^n y$, $t \geq 0$. This implies that the condition $f(\lambda) \in Z(A)$, $\lambda \in \Omega$ stated in the formulation of [37, Theorem 11] automatically holds and that the proof of [37, Theorem 13] can be simplified.

(ii) Let $t_0 > 0$. By the proof of Theorem 27 (cf. also [16]), we obtain that C-distribution semigroups appearing in the formulation of [37, Theorem 11] are (subspace) topologically mixing. With a little abuse of notation, we have that every single operator $G_1(\delta_{t_0})$ in [37, Theorem 11(i)] is topologically mixing and has a dense set of periodic points in E, resp. the part of the operator $G_1(\delta_{t_0})$ in the Banach space $\tilde{E}$ appearing in the formulation of [37, Theorem 11(ii)] is topologically mixing in $\tilde{E}$ and the set of $\tilde{E}$-periodic points of such an operator is dense in $\tilde{E}$.

The following theorem is an important extension of [50, Theorem 2.1], [14, Proposition 2.1], [15, Theorem 1.1] and [37, Theorem 11(i)].

Theorem 27.

(i) *Assume G is a (C-DS) generated by A, $\omega_1, \omega_2 \in \mathbb{R} \cup \{-\infty, \infty\}$, $\omega_1 < \omega_2$ and $t_0 > 0$. If $\sigma_p(A) \cap i\mathbb{R} \supseteq (i\omega_1, i\omega_2) \cap \frac{2\pi i\mathbb{Q}}{t_0}$, $k \in \mathbb{N}$ and $g_j : (\omega_1, \omega_2) \cap \frac{2\pi\mathbb{Q}}{t_0} \to E$ is a function which satisfies that, for every $j = 1, \cdots, k$, $Ag_j(s) = isg_j(s)$, $s \in (\omega_1, \omega_2) \cap \frac{2\pi\mathbb{Q}}{t_0}$, then every point in span$\{g_j(s) : s \in (\omega_1, \omega_2) \cap \frac{2\pi\mathbb{Q}}{t_0},\ 1 \leq j \leq k\}$ is a periodic point of $G_1(\delta_{t_0})$. Assume now that $f_j : (\omega_1, \omega_2) \to E$ is a Bochner integrable function which satisfies that, for every $j = 1, \cdots, k$, $Af_j(s) = isf_j(s)$ for a.e. $s \in (\omega_1, \omega_2)$. Put $\psi_{r,j} := \int_{\omega_1}^{\omega_2} e^{irs} f_j(s)\,ds$, $r \in \mathbb{R}$, $1 \leq j \leq k$.*

(a) *Assume* $span\{f_j(s) : s \in (\omega_1, \omega_2) \setminus \Omega,\ 1 \leq j \leq k\}$ *is dense in* E *for every subset* Ω *of* (ω_1, ω_2) *with zero measure. Then* G *is topologically mixing and* $G_1(\delta_{t_0})$ *is topologically mixing.*

(b) *Put* $\tilde{E} := \overline{span\{\psi_{r,j} : r \in \mathbb{R},\ 1 \leq j \leq k\}}$. *Then* G *is* $\tilde{E}$*-topologically mixing and the part of* $G_1(\delta_{t_0})$ *in* $\tilde{E}$ *is topologically mixing in the Banach space* $\tilde{E}$.

(ii) *Assume* G *is a (C-DS) generated by* A, $t_0 > 0$, $\tilde{E}$ *is a closed linear subspace of* E, $E_0 := span\{x \in Z(A) : \exists \lambda \in \mathbb{C},\ \Re\lambda < 0,\ G_1(\delta_t)x = e^{\lambda t}x,\ t \geq 0\}$, $E_\infty := span\{x \in Z(A) : \exists \lambda \in \mathbb{C},\ \Re\lambda > 0,\ G_1(\delta_t)x = e^{\lambda t}x,\ t \geq 0\}$ *and* $E_{per} := span\{x \in Z(A) : \exists \lambda \in \mathbb{Q},\ G_1(\delta_t)x = e^{\pi\lambda i t}x,\ t \geq 0\}$. *Then the following holds:*

(a) *If* $E_0 \cap \tilde{E}$ *is dense in* $\tilde{E}$ *and if* E_∞ *is a dense subspace of* $\tilde{E}$, *then* G *is* $\tilde{E}$*-topologically mixing; if* $G_1(\delta_t)(E_0 \cap \tilde{E}) \subseteq \tilde{E}$, $t \geq 0$, *then the part of* $G_1(\delta_{t_0})$ *in* $\tilde{E}$ *is topologically mixing in the Banach space* $\tilde{E}$.

(b) *If* $E_{per} \cap \tilde{E}$ *is dense in* $\tilde{E}$, *then the set of* $\tilde{E}$*-periodic points of* G *is dense in* $\tilde{E}$; *if, additionally,* E_{per} *is a dense subspace of* $\tilde{E}$, *then the set of all periodic points of the part of the operator* $G_1(\delta_{t_0})$ *in* $\tilde{E}$ *is dense in* $\tilde{E}$.

Proof. We will prove the assertion (i)(a). By Riemann-Lebesgue lemma and the dominated convergence theorem, we have that $\lim_{|r|\to\infty} \psi_{r,j} = 0$ and that the mapping $r \mapsto \psi_{r,j}$, $r \in \mathbb{R}$ is continuous ($1 \leq j \leq k$). By Remark 26 and [37, Lemma 6(i)], we obtain $G_1(\delta_t)f_j(s) = e^{its}f_j(s)$ for a.e. $s \in (\omega_1, \omega_2)$, $G_1(\delta_t)\psi_{r,j} = \psi_{r+t,j}$, $t \geq 0$, $r \in \mathbb{R}$, $1 \leq j \leq k$ and $\mathrm{span}\{\psi_{r,j} : r \in \mathbb{R},\ 1 \leq j \leq k\} \subseteq D(G)$. Using the proof of [50, Theorem 2.1], it can be easily seen that $\mathrm{span}\{\psi_{r,j} : r \in \mathbb{R},\ 1 \leq j \leq k\}$ is dense in E. So, it suffices to show that, given $y, z \in \mathrm{span}\{\psi_{r,j} : r \in \mathbb{R},\ 1 \leq j \leq k\}$ and $\varepsilon > 0$ in advance, there exists $t_0 \geq 0$ such that, for every $t \geq t_0$, there exists $x_t \in Z(A) = D(G)$ such that:

$$||y - x_t|| < \varepsilon \text{ and } ||z - G_1(\delta_t)x_t|| < \varepsilon. \tag{5}$$

Let $y = \sum_{l=1}^{m} \alpha_l \psi_{r_l, i_l}$ and $z = \sum_{l=1}^{n} \beta_l \psi_{\tilde{r}_l, \tilde{i}_l}$ for some $\alpha_l, \beta_l \in \mathbb{C}$, $r_l, \tilde{r}_l \in \mathbb{R}$ and $1 \leq i_l, \tilde{i}_l \leq k$. Then there exists $t_0(\varepsilon) > 0$ such that $||\sum_{l=1}^{n} \beta_l \psi_{\tilde{r}_l - t, \tilde{i}_l}|| < \varepsilon$ and $G_1(\delta_t)\sum_{l=1}^{n} \beta_l \psi_{\tilde{r}_l - t, \tilde{i}_l} = z$, $t \geq t_0(\varepsilon)$. Furthermore, there exists $t_1(\varepsilon) > 0$ such that $||G_1(\delta_t)y|| = ||\sum_{l=1}^{m} \alpha_l \psi_{r_l + t, i_l}|| < \varepsilon$, $t \geq t_1(\varepsilon)$. Then (5) holds with $t_0 = \max(t_0(\varepsilon), t_1(\varepsilon))$ and $x_t = \sum_{l=1}^{n} \beta_l \psi_{\tilde{r}_l - t, \tilde{i}_l} + y$, $t \geq t_0$. The operator $G_1(\delta_{t_0})$ is obviously topologically mixing, which completes the proof.

Remark 28.

(i) Assume the function $f_j : (\omega_1, \omega_2) \to E$ is weakly continuous for every $j = 1, \cdot \cdot \cdot, k,\ t_0 > 0$ and Ω is a subset of (ω_1, ω_2) with zero measure. Then $\overline{\operatorname{span}\{f_j(s) : s \in (\omega_1, \omega_2) \cap \frac{2\pi\mathbb{Q}}{t_0},\ 1 \leq j \leq k\}}$
$= \overline{\operatorname{span}\{f_j(s) : s \in (\omega_1, \omega_2),\ 1 \leq j \leq k\}}$
$= \overline{\operatorname{span} \bigcup_{j=1}^{k} \{f_j(s) : s \in (\omega_1, \omega_2) \setminus \Omega\}}.$

(ii) Let Ω be a subset of (ω_1, ω_2) with zero measure, let $r \in \mathbb{R}$ and let $1 \leq j \leq k$. Then $\psi_{r,j} = \int_{\omega_1}^{\omega_2} e^{irs} f_j(s)\,ds \in \overline{\operatorname{span}\{f_j(s) : s \in (\omega_1, \omega_2) \setminus \Omega\}}$.

(iii) Assume that the mapping $r \mapsto \psi_{r,j}$, $r \in \mathbb{R}$ is an element of the space $L^1(\mathbb{R} : E)$ for every $j = 1, \cdot \cdot \cdot, k$. Then the inversion theorem for the Fourier transform implies that there exists a subset Ω of (ω_1, ω_2) with zero measure such that

$$\overline{\operatorname{span}\{f_j(s) : s \in (\omega_1, \omega_2) \setminus \Omega,\ 1 \leq j \leq k\}} = \overline{\operatorname{span}\{\psi_{r,j} : r \in \mathbb{R},\ 1 \leq j \leq k\}}.$$

(iv) By multiplying with an appropriate scalar-valued function, we may assume that, for every $j = 1, \cdot \cdot \cdot, k$, the function $f_j(\cdot)$ is strongly measurable (cf. also [50, Remark 2.4]).

The following example illustrates an application of Theorem 27(i) and can be formulated in a more general setting.

Example 29. Assume $\alpha > 0$, $\tau \in i\mathbb{R} \setminus \{0\}$ and $E :=$BUC$(\mathbb{R})$. After the usual matrix reduction to a first order system, the equation $\tau u_{tt} + u_t = \alpha u_{xx}$ becomes

$$\frac{d}{dt}\vec{u}(t) = P(D)\vec{u}(t),\ t \geq 0,$$

where $D \equiv -i\frac{d}{dx}$, $P(x) \equiv \begin{bmatrix} 0 & 1 \\ -\frac{\alpha}{\tau}x^2 & -\frac{1}{\tau} \end{bmatrix}$ and $P(D)$ acts on $E \oplus E$ with its maximal distributional domain. The polynomial matrix $P(x)$ is not Petrovskii correct and [17, Theorem 14.1] implies that there exists an injective operator $C \in L(E \oplus E)$ such that $P(D)$ generates an entire C-regularized group $(T(z))_{z \in \mathbb{C}}$, with R$(C)$ dense. Put $\omega_1 = -\infty$ and $\omega_2 = 0$, resp. $\omega_1 = 0$ and $\omega_2 = +\infty$, if $\Im\tau > 0$, resp. $\Im\tau < 0$. Then $\frac{-\tau s^2 + is}{\alpha} \in (-\infty, 0)$, $s \in (\omega_1, \omega_2)$. Let $h_1(s) := \cos(\cdot(\frac{\tau s^2 - is}{\alpha})^{1/2})$, $h_2(s) := \sin(\cdot(\frac{\tau s^2 - is}{\alpha})^{1/2})$, $s \in (\omega_1, \omega_2)$ and let $f \in C^\infty((0, \infty))$ be such that the mapping $s \mapsto f_j(s) := (f(s)h_j(s), isf(s)h_j(s))^T$, $s > 0$ is Bochner integrable and that the mapping $s \mapsto \begin{cases} f_j(s), & s \in (\omega_1, \omega_2) \\ 0, & s \notin (\omega_1, \omega_2) \end{cases}$ belongs to the space $H^1(\mathbb{R})$ for $j = 1, 2$. Put $\psi_{r,j} = \int_{\omega_1}^{\omega_2} e^{irs} f_j(s)\,ds$, $r \in \mathbb{R}$, $j = 1, 2$ and $\tilde{E} = \overline{\operatorname{span}\{\psi_{r,j} : r \in \mathbb{R},\ j = 1, 2\}}$. By

Bernstein lemma [1, Lemma 8.2.1, p. 429], Theorem 27(i)(b) and Remark 28(i)-(iii), one gets that $(T(t))_{t\geq 0}$ is $\tilde{E}$-topologically mixing as well as that for each $t_0 > 0$ the part of the operator $C^{-1}T(t_0)$ in $\tilde{E}$ is topologically mixing in $\tilde{E}$ and that the set of $\tilde{E}$-periodic points of such an operator is dense in $\tilde{E}$.

Theorem 30. *Let $\pm A$ be the generators of C-distribution semigroups $G_\pm$, let A^2 be closed and let $\mathbf{G}(\varphi) = \frac{1}{2}(G_+(\varphi) + G_-(\varphi))$, $\varphi \in \mathcal{D}$. Assume $\omega_1, \omega_2 \in \mathbb{R} \cup \{-\infty, \infty\}$, $\omega_1 < \omega_2$, $t_0 > 0$, $\sigma_p(A) \supseteq (i\omega_1, i\omega_2) \cap \frac{2\pi i\mathbb{Q}}{t_0}$, $k \in \mathbb{N}$ and $f_j : (\omega_1, \omega_2) \cap \frac{2\pi\mathbb{Q}}{t_0} \to E$ satisfies $Af_j(s) = isf_j(s)$, $s \in (\omega_1, \omega_2) \cap \frac{2\pi\mathbb{Q}}{t_0}$ $(1 \leq j \leq k)$. Then $\mathbf{G}$ is a (C-DCF) generated by $C^{-1}A^2C$ and, for every $x \in span\{f_j(s) : s \in (\omega_1, \omega_2) \cap \frac{2\pi\mathbb{Q}}{t_0},\ 1 \leq j \leq k\}$, there exists $n \in \mathbb{N}$ such that x is a fixed point of $G(\delta_{nt_0})$.*

Proof. Clearly, Proposition 8 implies that $\mathbf{G}$ is a $(C - DCF)$ generated by $C^{-1}A^2C$. By Remark 26 and [37, Lemma 6(i)], one has $G_{\pm,1}(\delta_t)f_j(s) = e^{\pm ist}f_j(s)$, $t \geq 0$, $s \in (\omega_1, \omega_2) \cap \frac{2\pi\mathbb{Q}}{t_0}$. Now it is straightforward to see that, for every $x \in \mathrm{span}\{f_j(s) : s \in (\omega_1, \omega_2) \cap \frac{2\pi\mathbb{Q}}{t_0},\ 1 \leq j \leq k\}$, there exists $n \in \mathbb{N}$ such that $G_{\pm,1}(\delta_{t_0})^n x = x$. Then, by Proposition 20(ii),

$$\begin{aligned} G(\delta_{nt_0})x &= \frac{1}{2}(G_{+,1}(\delta_{nt_0})x + G_{-,1}(\delta_{nt_0})x) \\ &= \frac{1}{2}(G_{+,1}(\delta_{t_0})^n x + G_{-,1}(\delta_{t_0})^n x) = \frac{1}{2}(x + x) = x. \end{aligned}$$

Remark 31. Assume Ω is an open connected subset of $\mathbb{C}$, which satisfies $\sigma_p(A) \supseteq \Omega$ and intersects the imaginary axis, $f : \Omega \to E$ is an analytic mapping with $f(\lambda) \in \mathrm{Kern}(A - \lambda)$, $\lambda \in \Omega$, $E_0 = \mathrm{span}\{f(\lambda) : \lambda \in \Omega\}$, $k = 1$ and $f_1(s) = f(is)$, $s \in (\omega_1, \omega_2) \cap \frac{2\pi\mathbb{Q}}{t_0}$, where $\omega_1, \omega_2 \in \mathbb{R}$ and $(i\omega_1, i\omega_2) \subseteq \Omega$. Then [5, Lemma 2.4] implies that $\mathrm{span}\{f_1(s) : s \in (\omega_1, \omega_2) \cap \frac{2\pi\mathbb{Q}}{t_0}\}$ is dense in $\tilde{E}$.

Lemma 32. *Let $\lambda \in \mathbb{C}$. Then $\lambda \in \sigma_p(\mathcal{A})$ iff $\lambda^2 \in \sigma_p(A)$; if $f(\lambda^2)$ an eigenvector of A with the eigenvalue λ^2, then $F(\lambda) = (f(\lambda^2), \lambda f(\lambda^2))^T$ is an eigenvector of $\mathcal{A}$ with the eigenvalue λ.*

The proof of the first part of the following theorem follows immediately from Lemma 32 and Theorem 27 while the proof of the second part of the theorem follows from Lemma 32, [37, Theorem 11(ii)] and Remark 26.

Theorem 33.

(i) *Assume A is the generator of a (C-DCF) $\mathbf{G}$, $t_0 > 0$, $\omega_1, \omega_2 \in \mathbb{R} \cup \{-\infty, \infty\}$, $\omega_1 < \omega_2$, $k \in \mathbb{N}$ and $\Psi(\omega_1, \omega_2, t_0) := \{-s^2 : s \in (\omega_1, \omega_2) \cap \frac{2\pi\mathbb{Q}}{t_0}\}$. Then the existence of functions $g_j : \Psi(\omega_1, \omega_2, t_0) \to E$ which satisfy that, for every $j = 1, \cdots, k$, $Ag_j(-s^2) = -s^2 g_j(-s^2)$, $s \in (\omega_1, \omega_2) \cap \frac{2\pi\mathbb{Q}}{t_0}$, implies that every $x \in span\{(g_j(-s^2), isg_j(-s^2))^T : s \in (\omega_1, \omega_2) \cap \frac{2\pi\mathbb{Q}}{t_0},\ 1 \leq j \leq k\}$ is a periodic point of $\mathcal{G}_1(\delta_{t_0})$. Let $f_j : (-\omega_2^2, -\omega_1^2) \to E$ be a measurable function which satisfies that, for every $j = 1, \cdots, k$, $Af_j(-s^2) = -s^2 f_j(-s^2)$ for a.e. $s \in$*

(ω_1,ω_2). Put $F_j(s) := (f_j(-s^2), isf_j(-s^2))^T$, $s \in (\omega_1,\omega_2)$, $1 \leq j \leq k$. Let the mapping $F_j : (\omega_1,\omega_2) \to E \oplus E$ be Bochner integrable provided $1 \leq j \leq k$ and let $\zeta_{r,j} := \int_{\omega_1}^{\omega_2} e^{irs} F_j(s)ds$, $r \in \mathbb{R}$, $1 \leq j \leq k$.

(a) *Assume $span\{F_j(s) : s \in (\omega_1,\omega_2) \setminus \Omega,\ 1 \leq j \leq k\}$ is dense in $E \oplus E$ for every subset Ω of (ω_1,ω_2) with zero measure. Then $\mathcal{G}$ is topologically mixing and $\mathcal{G}_1(\delta_{t_0})$ is topologically mixing.*

(b) *Let $\hat{E} = \overline{span\{\zeta_{r,j} : r \in \mathbb{R},\ 1 \leq j \leq k\}}$. Then $\mathcal{G}$ is $\hat{E}$-topologically mixing and the part of $\mathcal{G}_1(\delta_{t_0})$ in $\hat{E}$ is topologically mixing in the Banach space $\hat{E}$.*

(ii) *Assume A is the generator of a (C-DCF) $\mathbf{G}$, there exists an open connected subset Ω of $\mathbb{C}$ which satisfies $\sigma_p(A) \supseteq \{\lambda^2 : \lambda \in \Omega\}$ and $\Omega \cap i\mathbb{R} \neq \emptyset$. Let $f : \{\lambda^2 : \lambda \in \Omega\} \to E$ be an analytic mapping satisfying $f(\lambda^2) \in Kern(A-\lambda^2) \setminus \{0\}$, $\lambda \in \Omega$, let $F(\lambda) := (f(\lambda^2), \lambda f(\lambda^2))^T$, $\lambda \in \Omega$ and let $\hat{E} = \overline{span\{F(\lambda) : \lambda \in \Omega\}}$. Then $\mathcal{G}$ is $\hat{E}$-topologically mixing, the part of the operator $\mathcal{G}_1(\delta_{t_0})$ in $\hat{E}$ is topologically mixing in the Banach space $\hat{E}$, the set $\mathcal{G}_{\hat{E},per}$ is dense in $\hat{E}$ and the set of all $\hat{E}$-periodic points of the part of the operator $\mathcal{G}_1(\delta_{t_0})$ in $\hat{E}$ is dense in $\hat{E}$.*

Remark 34.

(i) Assume $\mathbf{G}$ is a $(C-DCF)$ generated by A. Then one can prove with the help of [39, Theorem 2.1.11], Proposition 18 and [37, Lemma 6] that x is a periodic point of $\mathbf{G}$, resp. a hypercyclic vector of $\mathbf{G}$, if $\binom{x}{0}$ ($\binom{0}{x}$) is a periodic point of $\mathcal{G}$, resp. a hypercyclic vector of $\mathcal{G}$. Moreover, the $\mathbf{G}$-admissability of a closed linear subspace $\tilde{E}$ of E implies $\mathcal{G}_1(\delta_t)(\{0\} \oplus \tilde{E}) \subseteq \tilde{E} \oplus \tilde{E}$.

(ii) Assume now $\hat{E}$ is $\mathcal{G}$-admissible and $\binom{x}{y}$ is a $\hat{E}_{\mathrm{S}}$-hypercyclic vector for $\mathcal{G}$. Then $\mathcal{G}_1(\delta_t)\binom{x}{y} = \Big(\pi_1(\mathcal{G}_1(\delta_t)\binom{x}{y}), \frac{d}{dt}\pi_1(\mathcal{G}_1(\delta_t)\binom{x}{y})\Big)^T$, $t \geq 0$, and $u(t) = \pi_1\Big(\mathcal{G}_1(\delta_t)\binom{x}{y}\Big)$, $t \geq 0$ is a mild solution of (ACP_2). Then $\{cu(t) : c \in \mathrm{S},\ t \geq 0\}$ and $\{cu'(t) : c \in \mathrm{S},\ t \geq 0\}$ are dense subsets of $\hat{E}$, which can be simply reformulated in any of considered hypercyclic properties.

The following theorem can be rearranged by assuming that there exists $\alpha \geq 0$ such that $-A$ generates an exponentially bounded, analytic α-times integrated semigroup of angle $\theta \in (0, \frac{\pi}{2})$ and that $\sigma_p(-A)$ strictly lies on the imaginary axis (cf. Theorem 27).

Theorem 35. *Let $\theta \in (0, \frac{\pi}{2})$ and let $-A$ generate an analytic strongly continuous semigroup of angle θ. Assume $n \in \mathbb{N}$, $a_n > 0$, $a_{n-i} \in \mathbb{C}$, $1 \leq i \leq n$, $D(p(A)) = D(A^n)$, $p(A) = \sum_{i=0}^{n} a_i A^i$ and $n(\frac{\pi}{2} - \theta) < \frac{\pi}{2}$. Then there exists $\omega \in \mathbb{R}$ such that, for* *every* $\alpha \in (1, \frac{\pi}{n\pi - 2n\theta})$, $p(A)$ *generates an entire* $C \equiv e^{-(p(A)-\omega)^{\alpha}}$*-regularized group* $(T(t))_{t\in\mathbb{C}}$. *Put* $C(z) := \frac{1}{2}(T(z) + T(-z))$, $z \in \mathbb{C}$. *Then* $(C(t))_{t\geq 0}$ *is a* C*-regularized cosine function generated by* $p^2(A)$ *and the mapping* $z \mapsto C(z)$, $z \in \mathbb{C}$ *is entire.*

(i) *Assume that there exists an open connected subset* Ω *of* $\mathbb{C}$, *which satisfies* $\sigma_p(-A) \supseteq \Omega$, $p(-\Omega) \cap i\mathbb{R} \neq \emptyset$, *and let* $f : \Omega \to E$ *be an analytic mapping satisfying* $f(\lambda) \in Kern(-A - \lambda) \setminus \{0\}$, $\lambda \in \Omega$.

  (a) *Assume that* $\langle x^*, f(\lambda)\rangle = 0$, $\lambda \in \Omega$, *for some* $x^* \in E^*$, *implies* $x^* = 0$. *Then there exists a dense subspace* C_{per} *of* E *which satisfies* $C_{per} \subseteq Z_2(A)$ *and that, for every* $t_0 > 0$ *and* $x \in C_{per}$, *there exists* $n_0 \in \mathbb{N}$ *such that* $C^{-1}C(nn_0t_0)x = x$, $n \in \mathbb{N}$. *In particular, the set of all periodic points of* $(C(t))_{t\geq 0}$ *is dense in* E.

  (b) *Assume that the supposition* $\langle x^*, f(\lambda)\rangle + \langle y^*, p(-\lambda)f(\lambda)\rangle = 0$, $\lambda \in \Omega$, *for some* $x^*, y^* \in E^*$, *implies* $x^* = y^* = 0$. *Let*

$$S_0(z) := \begin{pmatrix} C(z) & \int_0^z C(s)ds \\ \frac{d}{dz}C(z) & C(z) \end{pmatrix}, \ z \in \mathbb{C}.$$

  Then $(S_0(z))_{z\in\mathbb{C}}$ *is an entire* $\mathcal{C}$*-regularized group generated by the operator* $\begin{pmatrix} 0 & I \\ p^2(A) & 0 \end{pmatrix}$, $(S_0(t))_{t\geq 0}$ *is both topologically mixing and chaotic, and for every* $t > 0$, *the operator* $\mathcal{C}^{-1}S_0(t) \oplus \mathcal{C}^{-1}S_0(t)$ *is chaotic and topologically mixing.*

(ii) *Assume that there exists an open connected subset* Ω *of* $\mathbb{C}$, *which satisfies* $\sigma_p(-A) \supseteq \Omega$ *and* $p(-\Omega) \cap i\mathbb{R} \neq \emptyset$. *Let* $f : \Omega \to E$ *be an analytic mapping satisfying* $f(\lambda) \in Kern(-A - \lambda) \setminus \{0\}$, $\lambda \in \Omega$. *Set*

$$\hat{E} := \overline{\operatorname{span}\{(f(\lambda), p(-\lambda)f(\lambda))^T : \lambda \in \Omega\}} \text{ and } \tilde{E} := \overline{\{f(\lambda) : \lambda \in \Omega\}}.$$

  (a) *Then there exists a dense subspace* C_{per} *of* $\tilde{E}$ *which satisfies* $C_{per} \subseteq Z_2(A)$ *and that, for every* $t_0 > 0$ *and* $x \in C_{per}$, *there exists* $n_0 \in \mathbb{N}$ *such that* $C^{-1}C(nn_0t_0)x = x$, $n \in \mathbb{N}$. *In particular, the set of all* $\tilde{E}$*-periodic points of* $(C(t))_{t\geq 0}$ *is dense in* $\tilde{E}$.

  (b) *Let* $(S_0(z))_{z\in\mathbb{C}}$ *be as in (i). Then* $(S_0(z))_{z\in\mathbb{C}}$ *is an entire* $\mathcal{C}$*-regularized group generated by* $\begin{pmatrix} 0 & I \\ p^2(A) & 0 \end{pmatrix}$, $(S_0(t))_{t\geq 0}$ *is* $\hat{E}$*-topologically mixing, the set of* $\hat{E}$*-periodic points of* $(S_0(t))_{t\geq 0}$ *is dense in* $\hat{E}$, *and* $R(\mathcal{C}_{\hat{E}})$ *is dense in the Banach space* $\hat{E}$. *Let* $t > 0$ *be fixed and let* $T_0(t)$ *be the part of* $\mathcal{C}^{-1}S_0(t)$ *in* $\hat{E}$. *Then the operator* $T(t) := T_0(t) \oplus T_0(t)$ *is chaotic and topologically mixing in the Banach space* $\hat{E} \oplus \hat{E}$.

Proof. We will only prove the part (b) of (ii). Notice that the mapping $z \mapsto p(z)$, $z \in \mathbb{C}$ is open and that the set $p(-\Omega)$ is open and connected. Put $S_1(s) :=$

$\begin{pmatrix} \int_0^s C(r)dr & \int_0^s (s-r)C(r)dr \\ C(s)-C & \int_0^s C(r)dr \end{pmatrix}$, $s \geq 0$. By [39, Theorem 2.1.11], $(S_1(s))_{s\geq 0}$ is a once integrated $\mathcal{C}$-semigroup generated by $\begin{pmatrix} 0 & I \\ p^2(A) & 0 \end{pmatrix}$. On the other hand, it is clear that the mapping $s \mapsto S_1(s)$, $s \geq 0$ can be analytically extended to the whole complex plane, which simply implies that $(S_0(z))_{z\in\mathbb{C}}$ is an entire $\mathcal{C}$-regularized group generated by $\begin{pmatrix} 0 & I \\ p^2(A) & 0 \end{pmatrix}$. In order to prove that $(S_0(s))_{s\geq 0}$ is $\hat{E}$-topologically mixing and that the set of $\hat{E}$-periodic points of $(S_0(s))_{s\geq 0}$ is dense in $\hat{E}$, one can use the equalities $p^2(A)f(\lambda) = p^2(-\lambda)f(\lambda)$, $\lambda \in \Omega$, $\begin{pmatrix} 0 & I \\ p^2(A) & 0 \end{pmatrix} \binom{f(\lambda)}{p(-\lambda)f(\lambda)} = p(-\lambda)\binom{f(\lambda)}{p(-\lambda)f(\lambda)}$, $\lambda \in \Omega$ as well as Remark 26 and [37, Theorem 11(ii)]. Moreover, the same argumentation shows that the single operator $T(t)$, considered as an unbounded linear operator in the Banach space $\hat{E} \oplus \hat{E}$, is topologically mixing and that the set of $\hat{E} \oplus \hat{E}$-periodic points of $T(t)$ is dense in $\hat{E} \oplus \hat{E}$. By [37, Remark 14(ii)], $\mathrm{R}(\mathcal{C}_{\hat{E}})$ is dense in $\hat{E}$. Therefore, it remains to be shown that the operator $T(t)$ is hypercyclic in the Banach space $\hat{E} \oplus \hat{E}$. Towards this end, put $X_0 := \operatorname{span}\{(f(\lambda), p(-\lambda)f(\lambda))^T : \lambda \in \Omega, \Re p(-\lambda) < 0\}$, $X_\infty := \operatorname{span}\{(f(\lambda), p(-\lambda)f(\lambda))^T : \lambda \in \Omega, \Re p(-\lambda) > 0\}$, $Y_1 := X_0 \oplus X_0$, $Y_2 := X_\infty \oplus X_\infty$,
$S\left(\sum_{i=1}^k \alpha_i \binom{f(\lambda_i)}{p(-\lambda_i)f(\lambda_i)}, \sum_{i=1}^l \beta_i \binom{f(z_i)}{p(-z_i)f(z_i)}\right)$
$:= \left(\sum_{i=1}^k \alpha_i e^{-p(-\lambda_i)} \binom{f(\lambda_i)}{p(-\lambda_i)f(\lambda_i)}, \sum_{i=1}^l \beta_i e^{-p(-z_i)} \binom{f(z_i)}{p(-z_i)f(z_i)}\right)$, $k, l \in \mathbb{N}$, $\alpha_i \in \mathbb{C}$, $\Re(p(-\lambda_i)) < 0$, $1 \leq i \leq k$, $\beta_i \in \mathbb{C}$, $\Re(p(-z_i)) < 0$, $1 \leq i \leq l$. Then it follows from [19, Theorem 2.3] (with $\mathcal{C}_{\hat{E}}$) that the operator $T(t)$ is hypercyclic in $\hat{E} \oplus \hat{E}$, as required.

In the following instructive example, we consider a class of abstract second order differential equations which cannot be treated by integrated cosine functions.

Example 36.

(i) ([21, Example 4.12], [20, Example 2.4], [37, Example 15]) Let $a, b, c > 0$ and $c < \frac{b^2}{2a} < 1$. Consider the equation

$$\begin{cases} u_t = au_{xx} + bu_x + cu := -Au, \\ u(0,t) = 0, \ t \geq 0, \\ u(x,0) = u_0(x), \ x \geq 0. \end{cases}$$

Then the operator $-A$, with domain $D(-A) = \{f \in W^{2,2}([0,\infty)) : f(0) = 0\}$, generates an analytic strongly continuous semigroup of angle $\frac{\pi}{2}$ in the space $E = L^2([0,\infty))$; the same assertion holds in the case when the operator $-A$

acts on $E = L^1([0,\infty))$ with domain $D(-A) = \{f \in W^{2,1}([0,\infty)) : f(0) = 0\}$. Put

$$\Omega := \left\{ \lambda \in \mathbb{C} : |\lambda - (c - \frac{b^2}{4a})| \leq \frac{b^2}{4a},\ \Im(\lambda) \neq 0 \text{ if } \Re(\lambda) \leq c - \frac{b^2}{4a} \right\}$$

and assume that $p(x) = \sum_{i=0}^{n} a_i x^i$ is a nonconstant polynomial such that $a_n > 0$ and that $p(-\Omega) \cap i\mathbb{R} \neq \emptyset$ (this, in particular, holds if $a_0 \in i\mathbb{R}$). An application of Theorem 35(i) gives that there exists an injective operator $C \in L(E)$ such that $p^2(A)$ generates a global C-regularized cosine function $(C(t))_{t\geq 0}$ satisfying that the set of periodic points of $(C(t))_{t\geq 0}$ is dense in E. Let $\hat{E} := \overline{\{(f_\lambda(\cdot), p(-\lambda)f_\lambda(\cdot))^T : \lambda \in \Omega\}}$, where the function f_λ is defined in [21, Example 4.12]. By Theorem 35(ii), we get that the induced entire $\mathcal{C}$-regularized semigroup $(S_0(t))_{t\geq 0}$ generated by $\begin{pmatrix} 0 & I \\ p^2(A) & 0 \end{pmatrix}$ is $\hat{E}$-topologically mixing and that the set of all $\hat{E}$-periodic points of $(S_0(t))_{t\geq 0}$ is dense in $\hat{E}$. Herein it is worth noting that every single operator $T(t)$ (cf. the formulation of Theorem 35) is chaotic and topologically mixing in the Banach space $\hat{E} \oplus \hat{E}$. Using the composition property of regularized semigroups, it simply follows that there exist $x, y \in \hat{E}$ such that the set $\{\mathcal{C}^{-1}S_0(nt)\binom{x}{y} : n \in \mathbb{N}_0\}$ is a dense subset of $\hat{E}$. Since $\mathrm{R}(\mathcal{C}_{\hat{E}})$ is dense in $\hat{E}$, one gets that $\{S_0(nt)\binom{x}{y} : n \in \mathbb{N}_0\}$ is also a dense subset of $\hat{E}$. This implies that $(S_0(t))_{t\geq 0}$ is $\hat{E}$-hypercyclic in the sense of [37, Remark 14(i)], which remains true in examples given in (ii) and (iv).

(ii) ([18]-[19]) Assume that ω_1, ω_2, V_{ω_2,ω_1}, Q, $Q(B)$, N, h_μ and E possess the same meaning as in [19, Section 5] and that $Q(\mathrm{int}(V_{\omega_2,\omega_1})) \cap i\mathbb{R} \neq \emptyset$. Then $\pm Q(B)h_\mu = \pm Q(\mu)h_\mu$, $e^{-(-B^2)^N}h_\mu = e^{-(-\mu^2)^N}h_\mu$, $\mu \in \mathrm{int}(V_{\omega_2,\omega_1})$ and $h_\mu \in (\mathrm{Kern}(Q(B)) \setminus \{0\})$, provided $\Re\mu \in (\omega_2, \omega_1)$. Let

$$\hat{E} = \overline{\mathrm{span}\{(h_\mu, Q(\mu)h_\mu)^T : \mu \in \mathrm{int}(V_{\omega_2,\omega_1})\}}.$$

By Theorem 30, one yields that $Q^2(B)$ is the integral generator of a global $(e^{-(-z^2)^N})(B)$-regularized cosine function $((\cosh(tQ(z))e^{-(-z^2)^N})(B))_{t\geq 0}$ which has dense set of periodic points and satisfies that the mapping $t \mapsto (\cosh(tQ(z))e^{-(-z^2)^N})(B)$, $t \geq 0$ can be analytically extended to the whole complex plane. It is readily seen that the mapping $\mu \mapsto h_\mu$, $\mu \in \mathrm{int}(V_{\omega_2,\omega_1})$ is analytic. Owing to [37, Theorem 11(ii)], the induced entire $\begin{pmatrix} (e^{-(-z^2)^N})(B) & 0 \\ 0 & (e^{-(-z^2)^N})(B) \end{pmatrix}$-regularized semigroup $(S_0(t))_{t\geq 0}$ generated by $\begin{pmatrix} 0 & I \\ Q^2(B) & 0 \end{pmatrix}$ is $\hat{E}$-topologically mixing and the set of all $\hat{E}$-periodic points of $(S_0(t))_{t\geq 0}$ is dense in $\hat{E}$. Furthermore, the analysis given in [19, Theorem 5.8] can serve one to construct important examples of regular ultradistribution semigroups of Beurling class ([39]).

(iii) In this example we deal with the space $L^p_\rho(\Omega,\mathbb{C})$, where Ω is an open non-empty subset of $\mathbb{R}^n$, $\rho:\Omega\to(0,\infty)$ is a locally integrable function, m_n is the Lebesgue measure in $\mathbb{R}^n$ and the norm of an element $f\in L^p_\rho(\Omega,\mathbb{C})$ is given by $||f||_p:=\Big(\int\limits_\Omega |f(\cdot)|^p\rho(\cdot)dm_n\Big)^{1/p}$ (cf. also Section 4). In the sequel, we use the Euclidean norm $|x|=(x_1^2+\cdots+x_n^2)^{1/2}$, $x=(x_1,\cdots,x_n)\in\mathbb{R}^n$. Let $1\leq p<\infty$, $\varphi(t,x):=e^tx$, $t\in\mathbb{R}$, $x\in\mathbb{R}^n$, $\alpha>0$, $\rho(x):=e^{-|x|^\alpha}$, $x\in\mathbb{R}^n$, $E:=L^p_\rho(\mathbb{R}^n,\mathbb{C})$, and $(T_\varphi(t)f)(x):=f(\varphi(t,x))$, $t\in\mathbb{R}$, $x\in\mathbb{R}^n$, $f\in E$. Owing to [29, Theorem 3.2], $(T_\varphi(t))_{t\geq 0}$ is not a strongly continuous semigroup. Put now

$$(T(t)f)(x):=e^{-(|e^tx|^{2\alpha}+1)}f(e^tx),\ t\in\mathbb{R},\ x\in\mathbb{R}^n$$

and $C(t):=\frac{1}{2}(T(t)+T(-t)),t\geq 0$. Then it is straightforward to see that $(T(t))_{t\in\mathbb{R}}$ is an exponentially bounded $T(0)$-regularized group generated by the closure A of the operator $f\mapsto\sum\limits_{i=1}^n x_i\frac{\partial f}{\partial x_i}$, $f\in C^1_c(\mathbb{R}^n:E)$ ([28]) and that $(C(t))_{t\geq 0}$ is an exponentially bounded $C(0)$-regularized cosine function generated by A^2. Denote by $\mathbf{G}$ the induced $(C-DCF)$ generated by A^2. Then $(G(\delta_t)\psi)(x)=\frac{1}{2}(\psi(\varphi(t,x))+\psi(\varphi(-t,x))),t\geq 0,\psi\in\mathcal{D},\lim_{t\to+\infty}\psi(\varphi(t,\cdot))=0$, $\psi\in C_c(\mathbb{R}^n)$ and, by the dominated convergence theorem, one has $\lim_{t\to-\infty}\psi(\varphi(t,\cdot))=0,\psi\in C_c(\mathbb{R}^n)$. By Theorem 25(iii), we get that $(C(t))_{t\geq 0}$ is topologically mixing. In order to prove that $(C(t))_{t\geq 0}$ has a dense set of periodic points, one can assume without loss of generality that $n=1$. Let $a>0$, $\psi\in C_c(\mathbb{R})$, $\mathrm{supp}\psi\subseteq[-a,a]$ and $\varepsilon>0$. Since $\mathrm{R}(C(0))$ is dense in E, it is enough to prove that there exists a sufficiently large $P>0$ such that the sequence $v_P(\cdot):=\sum\limits_{n\in\mathbb{Z}}\psi(\cdot e^{nP})$ is absolutely convergent in E and that $||\psi-v_P||<\varepsilon$. Towards this end, notice that an elementary computation shows that there exists $P_0>0$ such that, for every $P\geq P_0$ and $n\in\mathbb{N}$, $\int\limits_{-a}^a|\psi(x)|^pe^{np+nP-|x|^\alpha e^{-n\alpha P}}dx<\varepsilon^p$. This implies

$$\begin{aligned}
&||\psi-v_P||\\
&\leq\sum_{n\in\mathbb{N}}\Big[(\int\limits_{-a}^a|\psi(x)|^pe^{-|x|^\alpha e^{-n\alpha P}-nP}dx)^{1/p}+(\int\limits_{-a}^a|\psi(x)|^pe^{-|x|^\alpha e^{n\alpha P}+nP}dx)^{1/p}\Big]\\
&\leq(\int\limits_{-a}^a|\psi(x)|^pdx)^{1/p}\sum_{n\in\mathbb{N}}e^{-nP/p}+\sum_{n\in\mathbb{N}}\tfrac{\varepsilon}{e^n}\\
&\leq(\int\limits_{-a}^a|\psi(x)|^pdx)^{1/p}/(e^{P/p}-1)+\tfrac{\varepsilon}{e-\varepsilon}<\varepsilon,\ P\to+\infty.
\end{aligned}$$

By Proposition 37 given below, we have that $(C(t))_{t\geq 0}$ is hypercyclic and that the set $\mathrm{HC}(C(\cdot))$ is dense in E. Proceeding in a similar way, we are in a position to consider topologically mixing properties of perturbed wave equation $u_{tt}=u_{xx}+2au_x+a^2u$ $(a\in\mathbb{R})$ in the space $E=L^p_\rho(\mathbb{R},\mathbb{C})$, where $\rho(t)=e^{-|t|^\alpha}$, $t\in\mathbb{R}$

and $\alpha > 1$ (cf. also [31, Example]). Put $(Cf)(x) := e^{-|x|^{\alpha/p}}f(x)$, $f \in E$, $x \in \mathbb{R}$. Then the operator $A \equiv \frac{d}{dx} + a$, considered with its maximal distributional domain, is the integral generator of an exponentially bounded C-regularized group $(T(t))_{t\in\mathbb{R}}$, which is given by $(T(t)f)(x) := e^{at}e^{-|x+t|^b}f(x+t)$, $f \in E$, $t, x \in \mathbb{R}$. It can be simply proved that the induced C-regularized cosine function $(C(t))_{t\geq 0}$ is both topologically mixing and hypercyclic, and that the set of periodic points of $(C(t))_{t\geq 0}$ is dense in E.

(iv) ([27]) It is clear that Theorem 30, Theorem 33 and Theorem 35 can be applied to the operators considered by L. Ji and A. Weber in [27, Theorem 3.1(a), Theorem 3.2, Corollary 3.3]. For example, if X is a symmetric space of non-compact type (of rank one) and $p > 2$, then there exist a closed linear subspace $\tilde{X}$ of X (X, if the rank of X is one), a number $c_p > 0$ and an injective operator $C \in L(L^p_\natural(X))$ such that for any $c > c_p$ the operator $(-\Delta^\natural_{X,p} + c)^2$ generates a global C-regularized cosine function $(C(t))_{t\geq 0}$ in $L^p_\natural(X)$ which satisfies that the set of $\tilde{X}$-periodic points of $(C(t))_{t\geq 0}$ is dense in $\tilde{X}$. By Theorem 35(ii), we infer that there exists a closed linear subspace $\hat{X}$ of $X \oplus X$ such that the induced entire $\mathcal{C}$-regularized semigroup $(S_0(t))_{t\geq 0}$ generated by $\begin{pmatrix} 0 & I \\ (-\Delta^\natural_{X,p} + c)^2 & 0 \end{pmatrix}$ is $\hat{X}$-topologically mixing and that the set of all $\hat{X}$-periodic points of $(S_0(t))_{t\geq 0}$ is dense in $\hat{X}$.

Let $(O_n)_{n\in\mathbb{N}}$ be an open base of the topology of E and let $O_n \neq \emptyset$ for every $n \in \mathbb{N}$. We need the following simple proposition.

Proposition 37. *Suppose A is the integral generator of a global C-regularized cosine function $(C(t))_{t\geq 0}$ and $R(C)$ is dense in E. Put*

$$\mathcal{T} := \bigcap_{n\in\mathbb{N}} \bigcup_{t\geq 0} C(t)^{-1}(O_n).$$

Then

$$\mathcal{T} = \Big\{x \in E : \textit{the set } \{C(t)x : t \geq 0\} \textit{ is dense in } E\Big\} \tag{6}$$

and the following holds:

(i) *Let $(C(t))_{t\geq 0}$ be topologically transitive. Then $\mathcal{T}$ is a dense G_δ-subset of E and $C(\mathcal{T}) \subseteq HC(C(\cdot))$. In particular, $(C(t))_{t\geq 0}$ is hypercyclic and the set $HC(C(\cdot))$ is dense in E.*

(ii) *Let $(C(t))_{t\geq 0}$ be hypercyclic and $x \in HC(C(\cdot))$. Then $x \in \mathcal{T}$.*

PROOF. The proof of (6) is trivial and the proof of (ii) follows from the definition of hypercyclic vectors of $(C(t))_{t\geq 0}$, the denseness of $\mathrm{R}(C)$ in E and (6). Assume now that $(C(t))_{t\geq 0}$ is topologically transitive. Let U and V be arbitrary open subsets of E and let $y, z \in E$ and $\varepsilon > 0$ be such that $\{x \in E : ||x - y|| \leq \varepsilon\} =: B(y, \varepsilon) \subseteq U$

and that $B(Cz,\varepsilon) \subseteq V$. Then there exists $x \in Z_2(A)$ such that $||y-x|| < \varepsilon$ and $||z - C^{-1}C(t)x|| < \varepsilon/||C||$, which implies $||y-x|| < \varepsilon$, $||Cz - C(t)x|| < \varepsilon$, and $C(t)U \cap V \neq \emptyset$. Consequently, $\bigcup_{t\geq 0} C(t)^{-1}(O_n)$ is a dense open subset of E for every $n \in \mathbb{N}$ and $\mathcal{T}$ is a dense G_δ-subset of E. The inclusion $C(\mathcal{T}) \subseteq HC(C(\cdot))$ is trivial, which completes the proof of (i).

The next example is a continuation of [37, Example 4].

Example 38. Let $n \in \mathbb{N}$, $\rho(t) := \frac{1}{t^{2n}+1}$, $t \in \mathbb{R}$, $Af := f'$, $D(A) := \{f \in C_{0,\rho}(\mathbb{R}) : f' \in C_{0,\rho}(\mathbb{R})\}$, $E_n := (C_{0,\rho}(\mathbb{R}))^{n+1}$, $D(A_n) := D(A)^{n+1}$ and $A_n(f_1,\cdot\cdot\cdot,f_{n+1}) := (Af_1 + Af_2, Af_2 + Af_3, \cdot\cdot\cdot, Af_n + Af_{n+1}, Af_{n+1})$, $(f_1,\cdot\cdot\cdot,f_{n+1}) \in D(A_n)$. By the proof of [51, Proposition 2.4] (cf. also [45, Example 8.1, 8.2]) we have that $\pm A_n$ generate global polynomially bounded n-times integrated semigroups $(S_{n,\pm}(t))_{t\geq 0}$ and that neither A_n nor $-A_n$ generates a local $(n-1)$-times integrated semigroup. Denote by $G_\pm$ distribution semigroups generated by $\pm A$. Then it can be easily proved that for every $\varphi_1,\cdot\cdot\cdot,\varphi_{n+1} \in \mathcal{D}$:

$$G_\pm(\delta_t)(\varphi_1,\cdot\cdot\cdot,\varphi_{n+1})^T = (\psi_1,\cdot\cdot\cdot,\psi_{n+1})^T,$$

where $\psi_i(\cdot) = \sum_{j=0}^{n+1-i} \frac{(\pm t)^j}{j!}\varphi_{i+j}^{(j)}(\cdot \pm t)$, $1 \leq i \leq n+1$. This immediately implies the concrete representation formula for $(S_{n,\pm}(t))_{t\geq 0}$. Denote by $\mathbf{G}_n$ and $(C_n(t))_{t\geq 0}$ the (DCF) and global polynomially bounded n-times integrated cosine function generated by A_n^2. By Proposition 20(ii), we get that $\mathbf{G}_n(\delta_t)(\varphi_1,\cdot\cdot\cdot,\varphi_{n+1})^T = \frac{1}{2}[G_+(\delta_t)(\varphi_1,\cdot\cdot\cdot,\varphi_{n+1})^T + G_-(\delta_t)(\varphi_1,\cdot\cdot\cdot,\varphi_{n+1})^T]$, $t \geq 0$, $\varphi_1,\cdot\cdot\cdot,\varphi_{n+1} \in \mathcal{D}$. It is clear that the assumptions $0 \leq i \leq n$, $\varphi \in \mathcal{D}$ and $\mathrm{supp}\varphi \subseteq [a,b]$ imply $t^i \sup_{x\in\mathbb{R}} |\varphi(x \pm t)|\rho(x) \leq t^i \sup_{x\in[a\mp t,b\mp t]} \frac{1}{x^{2n}+1} \leq t^i(\frac{1}{(a-t)^{2n}+1} + \frac{1}{(a+t)^{2n}+1} + \frac{1}{(b-t)^{2n}+1} + \frac{1}{(b+t)^{2n}+1}) \to 0$, $|t| \to \infty$. Keeping this and Theorem 25(iii) in mind, it follows that $\mathbf{G}_n$ and $(C_n(t))_{t\geq 0}$ are topologically mixing. Arguing in the same way, we infer that $\mathbf{G}_n \oplus \mathbf{G}_n$ is also topologically mixing, which clearly implies that $\mathbf{G}_n$ and $(C_n(t))_{t\geq 0}$ are weakly mixing. Herein it is worthwhile to note that, for every $t > 0$, the operators $G_\pm(\delta_t) \oplus G_\pm(\delta_t)$ are hypercyclic in $\hat{E_n} \equiv E_n \oplus E_n$ ([19], [37]). Before proceeding further, we would like to observe that, for every $\tau > 0$, the mapping $t \mapsto C_n(t)$, $t \in [0,\tau)$ is not strongly differentiable and that A_n^2 cannot be the generator of any local $(n-1)$-times integrated cosine function. The existence of a positive real number λ_0 which belongs to the set $\rho(A_n) \cap \rho(-A_n)$ is obvious and the use of [39, Proposition 2.3.13] gives that $\pm A_n$ are the integral generators of global exponentially bounded $(\lambda_0 \mp A_n)^{-n}$-regularized semigroups $(S_{0,\pm}(t))_{t\geq 0}$ which fulfill the equalities $S_{n,\pm}(t)x = (\lambda_0 \mp A_n)^n \int_0^t \frac{(t-s)^{n-1}}{(n-1)!} S_{0,\pm}(s)x\,ds$, $t \geq 0$, $x \in E_n$. This implies that A_n^2 is the integral generator of a topologically mixing $((\lambda_0 - A_n)^{-n}(\lambda_0 + A_n)^{-n})$-regularized cosine function $(C_0(t))_{t\geq 0}$, where $C_0(t) = \frac{1}{2}(S_{0,+}(t)(\lambda_0 + A_n)^{-n} + S_{0,-}(t)(\lambda_0 - A_n)^{-n})$, $t \geq 0$. Using Proposition 37, one yields that $\mathbf{G}_n$ and $(C_n(t))_{t\geq 0}$ are hypercyclic. Put $C_n := I \oplus (\lambda_0 - A_n)^{-n}(\lambda_0 + A_n)^{-n}$. Then $\frac{d^n}{dt^n} S_{n,\pm}(t)(\varphi_1,\cdot\cdot\cdot,\varphi_{n+1})^T = (\lambda_0 \mp A_n)^n S_{0,\pm}(t)(\varphi_1,\cdot\cdot\cdot,\varphi_{n+1})^T$, $t \geq 0$,

$\varphi_1,\cdots,\varphi_{n+1}\in\mathcal{D}$, and an application of Theorem 25(iii) yields that $A_n^2\oplus A_n^2$ is the generator of a global topologically mixing n-times integrated C_n-cosine function $\Bigl(\overline{C_n}(t):=C_n(t)\oplus\int\limits_0^t\frac{(t-s)^{n-1}}{(n-1)!}C_0(s)\,ds\Bigr)_{t\geq 0}$. By [39, Proposition 2.3.12] and Proposition 37, $(\overline{C_n}(t))_{t\geq 0}$ is also hypercyclic. Finally, $A_n^2\oplus A_n^2$ cannot be the generator of any local $(n-1)$-times integrated C_n-cosine function in $\hat{E_n}$.

4 Hypercyclic and chaotic properties of cosine functions

Henceforth we consider hypercyclic and chaotic properties of various types of cosine functions in the space $L^p(\Omega,\mu,\mathbb{C})$, resp. $C_{0,\rho}(\Omega,\mathbb{C})$, where Ω is an open non-empty subset of $\mathbb{R}^d$, $p\in[1,\infty)$ and μ is a locally finite Borel measure on Ω, resp. $\rho:\Omega\to(0,\infty)$ is an upper semicontinuous function ([28]). Recall that the space $L^p(\Omega,\mu,\mathbb{C})$ consists of those measurable functions $f:\Omega\to\mathbb{C}$ for which $||f||_p:=\Bigl(\int\limits_\Omega f\,d\mu\Bigr)^{1/p}<\infty$. The space $C_{0,\rho}(\Omega,\mathbb{C})$ consists of all continuous functions $f:\Omega\to\mathbb{C}$ satisfying that, for every $\epsilon>0$, $\{x\in\Omega:|f(x)|\rho(x)\geq\epsilon\}$ is a compact subset of Ω; equipped with the norm $||f||:=\sup\limits_{x\in\Omega}|f(x)|\rho(x)$, $C_{0,\rho}(\Omega,\mathbb{C})$ becomes a Banach space. The space $C_c(\Omega,\mathbb{C})$ which consists of all continuous functions $f:\Omega\to\mathbb{C}$ whose support is a compact subset of Ω is dense in $L^p(\Omega,\mu,\mathbb{C})$ and $C_{0,\rho}(\Omega,\mathbb{C})$. Since no confusion seems likely, the above spaces are also denoted by $L^p(\Omega,\mu)$, $C_{0,\rho}(\Omega)$ and $C_c(\Omega)$. A continuous mapping $\varphi:\mathbb{R}\times\Omega\to\Omega$ is called a semiflow ([28]-[29]) iff $\varphi(0,x)=x,\ x\in\Omega,\ \varphi(t+s,x)=\varphi(t,\varphi(s,x)),\ t,s\in\mathbb{R},\ x\in\Omega$ and the mapping $x\mapsto\varphi(t,x),\ x\in\Omega$ is injective for all $t\in\mathbb{R}$. Denote by $\varphi(t,\cdot)^{-1}$ the inverse mapping of $\varphi(t,\cdot)$, i.e., $y=\varphi(t,x)^{-1}$ iff $x=\varphi(t,y),\ t\in\mathbb{R}$. In the sequel we assume that, for every $t\in\mathbb{R}$, the mapping $x\mapsto\varphi(t,x)$, $x\in\Omega$ is a homeomorphism of Ω.

Let $h:\Omega\to\mathbb{R}$ be a continuous function. A locally finite Borel measure μ on Ω is said to be p-admissible for φ and h iff $T(t)f:=e^{\int\limits_0^t h(\varphi(r,x))\,dr}f(\varphi(t,x))$, $t\in\mathbb{R},\ x\in\Omega$ defines a strongly continuous group on $L^p(\Omega,\mu)$. The C_0-admissibility of $(T(t))_{t\in\mathbb{R}}$ and the integral generator of cosine function $(C(t))_{t\geq 0}$, where $C(t):=\frac{1}{2}(T(t)+T(-t))$, $t\geq 0$, are precisely characterized in [31, Theorem 4(d)-(e)]. Using [25, Theorem 1, Proposition 1] and the proof of [31, Corollary 2], one gets that $(C(t))_{t\geq 0}$ is S-topologically transitive iff $(C(t))_{t\geq 0}$ is S-hypercyclic. Given a number $t\in\mathbb{R}$, we define $h_t(x):=e^{\int\limits_0^t h(\varphi(r,x))\,dr}$ and the Borel measures $\nu_{p,t}(B):=\int\limits_{\varphi(-t,B)}h_t^p\,d\mu$, $t\in\mathbb{R}$, $B\subseteq\Omega$ measurable.

The following theorem slightly improves [31, Theorem 5, Theorem 9].

Theorem 39.

(i) *Let $E = L^p(\Omega,\mu)$ and let μ be p-admissible for φ and h. Then (a) $\Rightarrow$ (b) $\Rightarrow$ (c) $\Rightarrow$ (d) $\Rightarrow$ (e) $\Rightarrow$ (f), where:*

(a) *For every compact set $K \subseteq \Omega$ there exist sequences (L_n^+) and (L_n^-) of Borel measurable subsets of K and a sequence of positive real numbers (t_n) such that for $L_n := L_n^+ \cup L_n^-$ one has*

$$\lim_{n\to\infty} \mu(K \setminus L_n) = \lim_{n\to\infty} v_{p,t_n}(L_n) = \lim_{n\to\infty} v_{p,-t_n}(L_n) = 0 \tag{7}$$

and

$$\lim_{n\to\infty} v_{p,2t_n}(L_n^+) = \lim_{n\to\infty} v_{p,-2t_n}(L_n^-) = 0. \tag{8}$$

(b) *$(C(t))_{t\geq 0}$ is weakly mixing.*

(c) *$(C(t))_{t\geq 0}$ is hypercyclic.*

(d) *$(C(t))_{t\geq 0}$ is S-hypercyclic for every closed subset S of $\mathbb{C}$ which satisfies $S \setminus \{0\} \neq \emptyset$.*

(e) *$(C(t))_{t\geq 0}$ is S-hypercyclic for every (some) bounded closed subset S of $[0,\infty)$ which satisfies $\inf S > 0$.*

Furthermore, if for every compact subset K of Ω one has $\lim_{|t|\to\infty} \varphi(K \cap \varphi(t,K)) = 0$, the above are equivalent.

(ii) *Let ρ be C_0-admissible for φ and h. Then (a) $\Rightarrow$ (b) $\Rightarrow$ (c) $\Rightarrow$ (d) $\Rightarrow$ (e) $\Rightarrow$ (f), where:*

(a) *For every compact set $K \subseteq \Omega$ there exist sequences of positive real numbers (t_n) and open subsets (U_n^+) and (U_n^-) of Ω such that $K \subseteq U_n^+ \cup U_n^-$,*

$$\lim_{n\to\infty} \sup_{x\in K} \frac{\rho(\varphi(-t_n,x))}{h_{-t_n}(x)} = \lim_{n\to\infty} \sup_{x\in K} \frac{\rho(\varphi(t_n,x))}{h_{t_n}(x)} = 0 \tag{9}$$

and

$$\lim_{n\to\infty} \sup_{K\cap U_n^-} \frac{\rho(\varphi(-2t_n,x))}{h_{-2t_n}(x)} = \lim_{n\to\infty} \sup_{K\cap U_n^+} \frac{\rho(\varphi(2t_n,x))}{h_{2t_n}(x)} = 0. \tag{10}$$

(b) *$(C(t))_{t\geq 0}$ is weakly mixing on $C_{0,\rho}(\Omega)$.*

(c) *$(C(t))_{t\geq 0}$ is hypercyclic on $C_{0,\rho}(\Omega)$.*

(d) *$(C(t))_{t\geq 0}$ is S-hypercyclic on $C_{0,\rho}(\Omega)$ for every closed subset S of $\mathbb{C}$ which satisfies $S \setminus \{0\} \neq \emptyset$.*

(e) *$(C(t))_{t\geq 0}$ is S-hypercyclic on $C_{0,\rho}(\Omega)$ for every (some) bounded closed subset S of $[0,\infty)$ which satisfies $\inf S > 0$.*

Furthermore, if for every compact subset K of Ω one has

$$\lim_{|t|\to\infty} \sup_{x\in\varphi(K\cap\varphi(t,K))} \rho(x) = 0 \text{ and } \inf_{x\in K} \rho(x) > 0,$$

the above are equivalent.

Proof. The implications (a) $\Rightarrow$ (b) $\Rightarrow$ (c) in (i) are consequences of [31, Theorem 5] and the implications (c) $\Rightarrow$ (d) $\Rightarrow$ (e) are trivial. Therefore, it suffices to show that the preassumption (e) combined with the additional condition $\lim_{|t|\to\infty} \varphi(K \cap \varphi(t,K)) = 0$ for each compact subset K of Ω implies (7)-(8). This can be obtained by an insignificant modification of the proof of the aforementioned theorem. Let $\Omega \supseteq K$ be compact and let S be a bounded subset of $[0,\infty)$ which satisfies that $\inf \mathrm{S} > 0$ and that $(C(t))_{t\geq 0}$ is S-hypercyclic. In what follows, we consider only the non-trivial case $\int_K d\mu > 0$. If the previous inequality holds, then there do not exist $c \in$ S and $t \geq 0$ such that $-\chi_K = cC(t)\chi_K$, which implies by the proof of [38, Lemma 3] that for given $\varepsilon \in (0,1)$ in advance, there exist $c_\varepsilon \in \mathrm{S}\setminus\{0\}$, $t_\varepsilon > 0$ and $v_\varepsilon \in L^p(\Omega,\mu)$ such that $||v_\varepsilon - \chi_K|| < \varepsilon^{2/p}$, $||c_\varepsilon C(t_\varepsilon)v_\varepsilon + \chi_K|| < \varepsilon^{2/p}$, $\mu(K \cap \varphi(2t_\varepsilon, K)) < \varepsilon^2$ and $\mu(K \cap \varphi(-2t_\varepsilon, K)) < \varepsilon^2$. Set $L_\varepsilon := K \cap \{|1 - v_\varepsilon|^p \leq \varepsilon\} \cap \{|1 + c_\varepsilon C(t_\varepsilon)v_\varepsilon|^p \leq \varepsilon\}$, $L_\varepsilon^- := \{x \in L : (c_\varepsilon T(t_\varepsilon)v_\varepsilon)(x) \leq \varepsilon^{1/p} - 1\}$ and $L_\varepsilon^+ := L_\varepsilon \setminus L_\varepsilon^-$. Then it is obvious that $\mu(K \setminus L_\varepsilon) < 2\varepsilon$, $v_{|L_\varepsilon} \geq 1 - \varepsilon^{1/p}$, $(c_\varepsilon C(t)v_\varepsilon)_{|L_\varepsilon} \leq \varepsilon^{1/p} - 1$. Employing the same notation as in [31], it follows that for every measurable subsets A, B of Ω : $||v_\varepsilon^- \chi_B|| < \varepsilon^{2/p}$ and

$$\begin{aligned} ||c_\varepsilon(C(t_\varepsilon)(v_\varepsilon^+\chi_B))\chi_A|| &\leq ||c_\varepsilon(C(t_\varepsilon)v_\varepsilon - c_\varepsilon^{-1}(-\chi_K) + c_\varepsilon^{-1}(-\chi_K))^+|| \\ &\leq ||c_\varepsilon(C(t_\varepsilon)v_\varepsilon + c_\varepsilon^{-1}\chi_K)|| \leq ||c_\varepsilon C(t_\varepsilon)v_\varepsilon + \chi_K|| < \varepsilon^{2/p}. \end{aligned}$$

This yields $\varepsilon^2 \geq 2^{-p}c_\varepsilon^p(\nu_{p,t_\varepsilon}(L_\varepsilon) + \nu_{p,-t_\varepsilon}(L_\varepsilon))$, which by arbitrariness of ε implies (7). Furthermore, $|c_\varepsilon|\frac{v_\varepsilon^-(x)}{h_{-t_\varepsilon}(x)} \geq 1 - \varepsilon^{1/p}$, $x \in \varphi(t_\varepsilon, L_\varepsilon^-)$, $|c_\varepsilon|\frac{v_\varepsilon^-(x)}{h_{t_\varepsilon}(x)} \geq 1 - \varepsilon^{1/p}$, $x \in \varphi(-t_\varepsilon, L_\varepsilon^+)$, and the following holds:

$$\begin{aligned} &(1-\varepsilon^{1/p})^p\nu_{p,2t_\varepsilon}(L_\varepsilon^+) \\ &\leq 2^{p+1} \int_{\varphi(-2t_\varepsilon, L_\varepsilon^+)} (c_\varepsilon C(t_\varepsilon^-))^p(x)\,d\mu(x) \\ &= 2^{p+1}||c_\varepsilon(C(t_\varepsilon)v_\varepsilon^+)\chi_{\varphi(-2t_\varepsilon,L_\varepsilon^+)} - (c_\varepsilon C(t_\varepsilon)v + \chi_K)\chi_{\varphi(-2t_\varepsilon,L_\varepsilon^+)} + \chi_{K\cap\varphi(-2t_\varepsilon,L_\varepsilon^+)}|| \\ &\leq 2^{3p+1}(2\varepsilon^2 + \mu(K \cap \varphi(-2t_\varepsilon, L_\varepsilon^+))) \leq 2^{3(p+1)}\varepsilon^2. \end{aligned}$$

The estimate $(1-\varepsilon^{1/p})^p\nu_{p,-2t_\varepsilon}(L_\varepsilon^-) \leq 2^{3(p+1)}\varepsilon^2$ can be proved analogously, which completes the proof of (i). The proof of (ii) is similar and therefore omitted.

Remark 40.

(i) The careful examination of the proof of [31, Theorem 5] implies that the condition $\lim_{|t|\to\infty} \mu(K \cap \varphi(t,K)) = 0$, for every compact subset K of Ω, can be neglected from the formulation of [31, Corollary 8]. Assume now that, for every compact subset K of Ω, one has $\inf_{x\in K} \rho(x) > 0$; then we get by means of the proofs of [31, Theorem 9] and [29, Theorem 4.11] that the hypercyclicity of cosine function $(C_\varphi(t))_{t\geq 0}$ in $C_{0,\rho}(\Omega)$ implies the hypercyclicity of $(T_\varphi(t))_{t\geq 0}$ in $C_{0,\rho}(\Omega)$.

(ii) It is well known that there exists a non-hypercyclic strongly continuous semigroup $(T_\varphi(t))_{t\geq 0}$ induced by semiflow which additionally satisfies that $(T_\varphi(t))_{t\geq 0}$ is positively supercyclic (cf. [29] and [40] for further information). Therefore, Theorem 39 might be surprising.

The characterizations of hypercyclicity and mixing can be simplified in the case that $\Omega \subseteq \mathbb{R}$. More precisely, for every $x_0 \in \Omega$, the semiflow $\varphi(\cdot, x_0)$ can be given as the unique solution of the initial value problem $\dot{x} = F(x)$, $x(0) = x_0$, where F is locally Lipschitz continuous vector field on Ω. For the sake of simplicity, we focus our attention to the case when F is continuously differentiable, which implies that the mapping $x \mapsto \varphi(t, x)$, $x \in \Omega$ is continuously differentiable for every fixed $t \in \mathbb{R}$. By the proofs of [31, Theorem 12, Theorem 15], we have the following.

Theorem 41. *Let $\Omega \subseteq \mathbb{R}$, let F be continuously differentiable and let the locally finite p-admissible measure μ has a positive Lebesgue density ρ, resp., let ρ be a positive function C_0-admissible for F and h. Then the following assertions are equivalent:*

(a) *$(C(t))_{t\geq 0}$ is hypercyclic on $L^p(\Omega, \mu)$, resp. $C_{0,\rho}(\Omega)$.*

(b) *$(C(t))_{t\geq 0}$ is S-hypercyclic on $L^p(\Omega, \mu)$, resp. $C_{0,\rho}(\Omega)$, for every closed subset S of $\mathbb{C}$ which satisfies $S \setminus \{0\} \neq \emptyset$.*

(c) *$(C(t))_{t\geq 0}$ is S-hypercyclic on $L^p(\Omega, \mu)$, resp. $C_{0,\rho}(\Omega)$, for every (some) bounded closed subset S of $[0, \infty)$ which satisfies* $\inf S > 0$.

In the subsequent theorems we analyze chaoticity of cosine functions on weighted function spaces.

Theorem 42. *Assume $\Omega \subseteq \mathbb{R}^d$ is open and ρ is a positive function on Ω that is C_0-admissible for φ and h. Assume further that, for every compact set K of Ω, there exists $t_K > 0$ such that $\varphi(t, K) \cap K = \emptyset$, $t \geq t_K$ and $\inf_{x\in K} \rho(x) > 0$. Then, the following are equivalent:*

(a) *$(C(t))_{t\geq 0}$ is chaotic on $C_{0,\rho}(\Omega)$.*

(b) *The set of periodic points of $(C(t))_{t\geq 0}$ is dense in $C_{0,\rho}(\Omega)$.*

(c) *For every every compact set K there exists $P > 0$ such that*

$$\lim_{n\to\infty} \sup_{x\in\varphi(nP,K)} \frac{\rho(x)}{h_{nP}(\varphi(-nP, x))} = \lim_{n\to\infty} \sup_{x\in\varphi(-nP,K)} h_{nP}(x)\rho(x) = 0.$$

(d) *$(T(t))_{t\geq 0}$ is chaotic on $C_{0,\rho}(\Omega)$.*

(e) $(T(-t))_{t\geq 0}$ *is chaotic on* $C_{0,\rho}(\Omega)$.

Proof. The equivalence relation (d) $\Leftrightarrow$ (e) follows from [21, Theorem 2.5] and the fact that $(T(t))_{t\geq 0}$ and $(T(-t))_{t\geq 0}$ have the same set of periodic points, while the equivalence of (c), (d) and (e) follows from [29, Theorem 5.7]; notice also that (c) implies the hypercyclicity of $(C(t))_{t\geq 0}$ in (a) since the assertion (i) of [31, Theorem 9] holds with $t_n = nP$ and $U_n^+ = U_n^- = \Omega$. Since every periodic point of $(T(t))_{t\geq 0}$ is also a periodic point of $(C(t))_{t\geq 0}$, we obtain that (c) and (d) together imply (a). The implication (a) $\Rightarrow$ (b) is trivial and it remains to be proved the implication (b) $\Rightarrow$ (c). Towards this end, assume K is a compact subset of Ω, U_K is a relatively compact, open neighborhood of K, $t > 0$ and, for every $s \geq t$, $\varphi(s, \overline{U_K}) \cap \overline{U_K} = \emptyset$. Let $f \in C_c(\Omega)$ be such that $f(x) = 1$, $x \in K$, $f(x) \geq 0$, $x \in \Omega$ and $f(x) = 0$, $x \in \Omega \setminus U_K$. Let $\varepsilon \in (0, \inf_{x\in K} \rho(x)/2)$ and let v be a real-valued P-periodic point of $(C(t))_{t\geq 0}$ with $\varepsilon > ||f - v||$. Then $v(x) \geq 1/2$, $x \in K$. Using induction and the composition property of cosine functions, one gets that $C(nP)v = v$, $n \in \mathbb{N}$ so that one can assume $P > t$. Taking into account the equalities

$$2v(x) = h_{nP}(x)v(\varphi(nP, x)) + h_{-nP}(x)v(\varphi(-nP, x)),\ n \in \mathbb{N},\ x \in \Omega \tag{11}$$

and $h_t(x)h_s(\varphi(t, x)) = h_{t+s}(x)$, $x \in \Omega$, $t, s \in \mathbb{R}$, it follows inductively that, for every $x \in \Omega$ and $n \in \mathbb{Z}$:

$$h_{nP}(x)v(\varphi(nP, x)) = nh_P(x)v(\varphi(P, x)) - (n - 1)v(x), \tag{12}$$

$$h_{-nP}(x)v(\varphi(-nP, x)) = -nh_P(x)v(\varphi(P, x)) + (n + 1)v(x), \tag{13}$$

and

$$h_{-nP}(x)v(\varphi(-nP, x)) = nh_{-P}(x)v(\varphi(-P, x)) - (n - 1)v(x). \tag{14}$$

Put $a_n := \sup_{x\in K} h_{nP}(x)|v(\varphi(nP, x))|$, $n \in \mathbb{Z}$. Without loss of generality, we may assume that $\max(a_1, a_{-1}) = a_1$. Clearly, (11) implies $a_1 \geq a_0$. By taking supremum on both sides of (12), we get

$$a_n \geq na_1 - (n - 1)a_0 \geq a_1 \geq a_0 \geq \frac{1}{2},\ n \in \mathbb{N}. \tag{15}$$

There exist two possibilities. The first one is $a_1 = a_0$, which implies by (14): $a_{-n} \geq na_{-1} - (n - 1)a_0 \geq a_{-1} \geq a_0 \geq \frac{1}{2}$, $n \in \mathbb{N}$; then $\sup_{x\in\varphi(-nP,\overline{U_K})} |v(x)|\rho(x) \geq \sup_{x\in\varphi(-nP,K)} |v(x)|\rho(x) \geq \frac{1}{2} \sup_{x\in K} \frac{\rho(\varphi(-nP,x))}{h_{-nP}(x)} = \frac{1}{2} \sup_{x\in K} h_{nP}(\varphi(-nP, x))\rho(\varphi(-nP, x))$, $\sup_{x\in\varphi(nP,\overline{U_K})} |v(x)|\rho(x) \geq \sup_{x\in K} |v(\varphi(nP, x))|\rho(\varphi(nP, x)) \geq \frac{1}{2} \sup_{x\in K} \frac{\rho(\varphi(nP,x))}{h_{nP}(x)}$ and the proof in this case completes an application of [29, Lemma 5.6]. The second one is $a_1 > a_0$ and the proof in this case is quite similar; as a matter of fact, (15) implies $a_n \geq \frac{1}{2}$, $n \in \mathbb{N}$ and we obtain from (13) that $a_{-n} \geq na_1 - (n+1)a_0 \to +\infty$, $n \to \infty$ and that there exists $n_0(K) \in \mathbb{N}$ such that $a_{-n} \geq \frac{1}{2}$, $n \geq n_0(K)$. This completes the proof.

Concerning the chaoticity of $(C(t))_{t\geq 0}$ in $L^p(\Omega,\mu)$, we have the following simple observation. Assume that there exists a closed μ-zero subset N of Ω such that $\varphi(t,N)=N$, $t>0$ and that for every compact subset K of $\Omega\setminus N$ and sufficiently large t, one has $\varphi(t,K)\cap K=\emptyset$. By [29, Theorem 5.3, Remark 5.4] and the proof of [31, Theorem 5], it follows that the chaoticity of $(T(t))_{t\geq 0}$ implies the chaoticity of $(C(t))_{t\geq 0}$. It is not clear whether the converse statement holds.

Let the spaces $L^p_\rho(\mathbb{R})$ and $C_{0,\rho}(\mathbb{R})$ possess the same meaning as in [21]. Arguing in a similar way, we get that the condition $\lim\limits_{|t|\to\infty}\rho(t)=0$ is equivalent to say that the cosine function $(C(t))_{t\geq 0}$, given by $(C(t)f)(x):=\frac{1}{2}(f(x+t)+f(x-t))$, $f\in C_{0,\rho}(\mathbb{R})$, $t\geq 0$, $x\in\mathbb{R}$, is chaotic in $C_{0,\rho}(\mathbb{R})$. This enables one to simply construct an example of hypercyclic cosine function $(C(t))_{t\geq 0}$ that is not chaotic. Put, for example, $\rho(t):=e^{-(|t|+1)\cos(\ln(|t|+1))+1}$, $t\in\mathbb{R}$ and notice that ρ is an admissible weight function which satisfies $\lim\limits_{|t|\to\infty}\rho(t)\neq 0$ ([53]). Hence, $(C(t))_{t\geq 0}$ is not chaotic in $C_{0,\rho}(\mathbb{R})$. The hypercyclicity of $(C(t))_{t\geq 0}$ follows from the fact that there exists a sequence (t_n) of positive real numbers which satisfies

$$\lim_{n\to\infty}\rho(t_n)=\lim_{n\to\infty}\rho(-t_n)=\lim_{n\to\infty}\rho(2t_n)=\lim_{n\to\infty}\rho(-2t_n)=0.$$

In the following theorem we consider necessary and sufficient conditions for the chaoticity of cosine function $(C(t))_{t\geq 0}$, $(C(t)f)(x)=\frac{1}{2}(f(x+t)+f(x-t))$, $t\geq 0$, $x\in\mathbb{R}$, in the space $L^p_\rho(\mathbb{R})$.

Theorem 43. *Assume that $\rho:\mathbb{R}\to(0,\infty)$ is measurable and that there exist $M\geq 1$ and $\omega\in\mathbb{R}$ such that $\rho(x)\leq Me^{\omega|t|}\rho(x+t)$, $x,t\in\mathbb{R}$. Then $(T(t))_{t\in\mathbb{R}}$ is a C_0-group in $L^p_\rho(\mathbb{R})$ and the following assertions are equivalent.*

(a) *$(C(t))_{t\geq 0}$ is chaotic on $L^p_\rho(\mathbb{R})$.*

(b) *The set of periodic points of $(C(t))_{t\geq 0}$ is dense in $L^p_\rho(\mathbb{R})$.*

(c) *For every $\varepsilon>0$ there exists $P>0$ such that*

$$\sum_{n\in\mathbb{Z}\setminus\{0\}}\rho(nP)<\varepsilon. \tag{16}$$

(d) *$(T(t))_{t\geq 0}$ is chaotic on $L^p_\rho(\mathbb{R})$.*

(e) *$(T(-t))_{t\geq 0}$ is chaotic on $L^p_\rho(\mathbb{R})$.*

Proof. The implication (a) $\Rightarrow$ (b) is trivial, the equivalence of (c) and (d) follows from [39, Theorem 2], and the equivalence of (d) and (e) follows from [21, Theorem 2.5] and the fact that $(T(t))_{t\geq 0}$ and $(T(-t))_{t\geq 0}$ have the same set of periodic points. Since every periodic point of $(T(t))_{t\geq 0}$ is also a periodic point of $(C(t))_{t\geq 0}$, (c) and (d) taken together imply (a) by [11, Theorem 1.1, 2.2]. Therefore, it remains to be proved the implication (b) $\Rightarrow$ (c). Let $\varepsilon>0$ be fixed, let $\theta>0$ and let $z\in C_c(\mathbb{R})$

be such that $||z|| = (\int_{-\infty}^{\infty} |z(x)|^p \rho(x)dx)^{1/p} = 1$, $z \geq 0$ and $\operatorname{supp}(z) \subseteq [-\theta, \theta]$. Then there exist $P > 0$ and a real-valued P-periodic point v of $(C(t))_{t\geq 0}$ such that $||z - v|| < \varepsilon$. Denote $a_n := \int_{-\theta+nP}^{\theta+nP} |2v(x)|^p dx$, $n \in \mathbb{Z}$. By the proof of Theorem 42, we have $C(nP)v = v$ and $2v(x + nP) = v(x + (n + 1)P) + v(x + (n - 1)P)$, $x \in \mathbb{R}$, $n \in \mathbb{Z}$, which implies

$$2\int_{-\theta}^{\theta} |2v(x+nP)|^p dx \leq \int_{-\theta}^{\theta} |2v(x+(n+1)P)|^p dx + \int_{-\theta}^{\theta} |2v(x+(n-1)P)|^p dx, \ n \in \mathbb{Z},$$

i.e.,

$$2a_n \leq a_{n+1} + a_{n-1}, \ n \in \mathbb{Z}. \tag{17}$$

We may assume without loss of generality that $P > 2\theta$ and $a_1 = \max(a_1, a_{-1})$. Then $a_1 \geq a_0$ and an induction argument combined with (17) shows that:

$$a_{n+1} \geq (n+1)a_1 - na_0, \ a_n - a_0 \geq n(a_1 - a_0) \text{ and } a_{n+1} \geq a_n, \ n \in \mathbb{N}_0. \tag{18}$$

We first consider the case $a_1 = a_0$. Then $a_{-1} \geq a_0$ and, by (17), $a_{-(n+1)} \geq a_{-n} \geq a_{-1}$, $n \in \mathbb{N}$. Since $\int_{-\theta}^{\theta} [2^{1-p}|2z(x)|^p - |2v(x)|^p]\rho(x)dx \leq \int_{-\theta}^{\theta} |2z(x) - 2v(x)|^p \rho(x)dx < (2\varepsilon)^p$, we get from [21, Lemma 4.2] that there exist $m_1 > 0$ and $M_1 > 0$ such that, for every $\sigma \in \mathbb{R}$, $m_1\rho(\sigma - \theta) \leq \rho(t) \leq M_1\rho(\sigma + \theta)$, and that $a_0 = \int_{-\theta}^{\theta} |2v(x)|^p dx \geq \frac{1}{M_1\rho(\theta)} \int_{-\theta}^{\theta} |2v(x)|^p \rho(x)dx \geq \frac{1}{M_1\rho(\theta)} 2(1 - \varepsilon^p)$. Therefore, we have the following:

$$(2\varepsilon)^p > \sum_{n\in\mathbb{Z}\setminus\{0\}} \int_{-\theta+nP}^{\theta+nP} |2v(x)|^p \rho(x)dx \geq \sum_{n\in\mathbb{Z}\setminus\{0\}} m_1\rho(-\theta + nP)a_n$$
$$\geq \sum_{n\in\mathbb{Z}\setminus\{0\}} m_1\rho(-\theta + nP)a_0 \geq \sum_{n\in\mathbb{Z}\setminus\{0\}} m_1\rho(-\theta + nP)\frac{1}{M_1\rho(\theta)}2(1 - \varepsilon^p),$$

which immediately implies by a straightforward computation (16). Assume now $a_1 > a_0$. Then (18) implies $\lim_{n\to+\infty} a_n = +\infty$ and the existence of an integer $n_0 \in \mathbb{N}$ such that $2^{1-p}a_{n_0} > a_0$. Using again the proof of Theorem 42, we obtain $v(x - nn_0P) + (n + 1)v(x) = nv(x + n_0P)$, $x \in \mathbb{R}$, $n \in \mathbb{Z}$,

$$2^{p-1}(|2v(x - nn_0P)|^p + (n + 1)^p|2v(x)|^p) \geq n^p|2v(x + n_0P)|^p$$

and after integration

$$a_{-nn_0} \geq 2^{1-p}n^p a_{n_0} - (n + 1)^p a_0 \to +\infty, \ n \to +\infty.$$

This enables one to conclude that there exists $n_1 \in \mathbb{N}$ such that $a_{-nn_0} \geq a_0 \geq \frac{1}{M_1\rho(\theta)}2(1 - \varepsilon^p)$, $n \geq n_1$. Hence,

$$(2\varepsilon)^p > \sum_{n\in\mathbb{Z}\setminus\{0\}} \int_{-\theta+nn_0n_1P}^{\theta+nn_0n_1P} |2v(x)|^p\rho(x)dx \geq \sum_{n\in\mathbb{Z}\setminus\{0\}} m_1\rho(-\theta+nn_0n_1P)a_{nn_0n_1}$$
$$\geq \sum_{n\in\mathbb{Z}\setminus\{0\}} m_1\rho(-\theta+nn_0n_1P)a_0 \geq \sum_{n\in\mathbb{Z}\setminus\{0\}} m_1\rho(-\theta+nn_0n_1P)\frac{1}{M_1\rho(\theta)}2(1-\varepsilon^p).$$

By choosing appropriate constants, the above estimate yields (16) with n_0n_1P, which complets the proof of theorem.

Let $h:\mathbb{R}\to\mathbb{C}$ be bounded and continuous. Then it is well known that the semigroup solution of the equation $u_t = u_x + h(x)u,\ u(0,x) = f(x),\ t\geq 0$ is given by $(T(t)f)(x) := e^{\int_x^{x+t} h(s)ds} f(x+t)$, $t\geq 0$, $x\in\mathbb{R}$. If the solution can be extended to the whole real axis, then one can consider hypercyclic properties of the equation $u_{tt} = u_{xx} + h(x)u_x + \frac{\partial}{\partial x}(h(x)u) + h^2(x)u$, $u(0,x) = f_1(x)$, $u_t(0,x) = f_2(x)$, $t\geq 0$, $x\in\mathbb{R}$. Assume, more generally, $g:\mathbb{R}\times\mathbb{R}\to\mathbb{C}$ is continuous, there exist $M\geq 1$ and $\omega\in\mathbb{R}$ such that $|g(x,t)|\leq Me^{\omega|t|}$, $x,t\in\mathbb{R}$ and the following conditions hold:

(H1) $g(x,t+s) = g(x,t)g(x+t,s),\ x,t,s\in\mathbb{R}$,

(H2) $g(x,0) = 1,\ x\in\mathbb{R}$,

(H3) $g(x,t)\neq 0,\ x,t\in\mathbb{R}$ and

(H4) $g(x,t) = \frac{g(0,x+t)}{g(0,x)},\ x,t\in\mathbb{R}$.

Put $\rho(t) := \frac{1}{|g(0,t)|},\ t\in\mathbb{R}$, $\rho_i(t) := \rho^i(t)$, $t\in\mathbb{R}$, $i\in\mathbb{N}_0$ and $g_j(x,t) := g^j(x,t)$, $x,t\in\mathbb{R}$, $j\in\mathbb{N}_0$. Let E_i be either $L^p_{\rho_i}(\mathbb{R})$ or $C_{0,\rho_i}(\mathbb{R})$ and let $(T_j(t)f)(x) := g_j(x,t)f(x+t)$, $x,t\in\mathbb{R}$, $f\in E_i$. By [38, Lemma 21, Theorem 23] we have that, for every $i\in\mathbb{N}_0$, ρ_i is an admissible weight function and that $(T_j(t))_{t\geq 0}$ is a strongly continuous group in E_i. By the proof of [38, Theorem 23], the cosine function $(C_j(t))_{t\geq 0}$, given by $(C_j(t)f)(x) := \frac{1}{2}(g_j(x,t)f(x+t) + g_j(x,-t)f(x-t))$, $x,t\in\mathbb{R}$, $f\in E_i$ is chaotic in E_i iff the cosine function $(\tilde{C}_j(t))_{t\geq 0}$, given by $(\tilde{C}_j(t)f)(x) := \frac{1}{2}(f(x+t)+f(x-t))$, $x,t\in\mathbb{R}$, $f\in E_{i+j}$ is chaotic in E_{i+j}. Assume $i+j>0$; then Theorem 42-Theorem 43 imply that the chaoticity of $(C_j(t))_{t\geq 0}$ in $C_{0,\rho_i}(\mathbb{R})$ is equivalent with $\lim_{|t|\to\infty}|g(0,t)| = \infty$ and that the chaoticity of $(C_j(t))_{t\geq 0}$ in $L^p_{\rho_i}(\mathbb{R})$ is equivalent to say that for every $\varepsilon>0$ there exists $P>0$ such that $\sum_{n\in\mathbb{Z}\setminus\{0\}}|g(0,nP)|^{-i-j}<\varepsilon$.

5 Disjoint hypercyclicity of C-distribution cosine functions

In the following definition we intend to limit ourselves specifically to the analysis of disjoint hypercyclicity and disjoint topological transitivity of C-distribution cosine functions ([8], [10], [40]).

Definition 44. Let $n\in\mathbb{N},\ n\geq 2$ and let $\mathbf{G}_i$ be a $(C_i - DCF)$ generated by A_i, $i=1,2,\cdots,n$. Then it is said that $\mathbf{G}_i,\ i=1,2,\cdots,n$ are:

(i) *disjoint hypercyclic*, in short d-hypercyclic, if there exists $x \in Z_2(A_1) \cap \cdots \cap Z_2(A_n)$ such that $\overline{\{(G_1(\delta_t)x, \cdots, G_n(\delta_t)x) : t \geq 0\}} = E^n$. An element $x \in E$ which satisfies the above equality is called a d-hypercyclic vector associated to $\mathbf{G}_1,\ \mathbf{G}_2, \cdots,\ \mathbf{G}_n$;

(ii) *disjoint topologically transitive*, in short d-topologically transitive, if for any open non-empty subsets $V_0, V_1, \cdots, V_n$ of E, there exist $t \geq 0$ and $x \in Z_2(A_1) \cap \cdots \cap Z_2(A_n)$ such that $x \in V_0 \cap G_1(\delta_t)^{-1}(V_1) \cap \cdots \cap G_n(\delta_t)^{-1}(V_n)$.

It is clear that the preceding definition can be reformulated in the case of fractionally integrated C-cosine functions in Banach spaces and that d-hypercyclicity (d-topological transitivity) of $(C_i - DCF)$'s $\mathbf{G}_i,\ i = 1, 2, \cdots, n$ implies that, for every $i, j \in \{1, 2, \cdots, n\}$ with $i \neq j$, there exists $t > 0$ such that $G_i(\delta_t) \neq G_j(\delta_t)$. If $(C_i(t))_{t\geq 0},\ i = 1, 2, \cdots, n$ are cosine functions, then the proof of [10, Proposition 2.3] yields that d-topological transitivity of $(C_i(t))_{t\geq 0},\ i = 1, 2, \cdots, n$ implies d-hypercyclicity of $(C_i(t))_{t\geq 0},\ 1 \leq i \leq n$ and that the set of all d-hypercyclic vectors associated to $(C_i(t))_{t\geq 0}$, $1 \leq i \leq n$ is a dense G_δ-subset of E.

The main objective in the subsequent theorem is to clarify sufficient conditions for d-topological transitivity of cosine functions on a class of weighted function spaces. An alternative proof of this theorem can be obtained by using d-Hypercyclicity Criterion from [10].

Theorem 45. *Suppose $\Omega \subseteq \mathbb{R}^d$ is open, $p \in [1, \infty)$, $n \in \mathbb{N} \setminus \{1\}$, $\varphi_i : [0, \infty) \times \Omega \to \Omega$ is a semiflow for all $i = 1, 2, \cdots, n$, $\rho : \Omega \to (0, \infty)$ is an upper semicontinuous function and μ is a locally finite Borel measure on Ω.*

(i) *Suppose $E = C_{0,\rho}(\Omega)$, ρ is C_0-admissible for φ_i and $h_{\cdot,i}$, $1 \leq i \leq n$ and*

$$(C_{\varphi_i}(t)f)(\cdot) = \frac{1}{2}\Big(h_{t,i}(\cdot)f(\varphi_i(t,\cdot)) + h_{-t,i}(\cdot)f(\varphi_i(-t,\cdot))\Big)$$

for any $t \geq 0,\ f \in E$ and $1 \leq i \leq n$. If for every compact subset $K \subseteq \Omega$ there exist a sequence (t_k) of non-negative real numbers and sequences $(U^+_{k,i})$ and $(U^-_{k,i})$ of open subsets of Ω such that, for every $i \in \{1, \cdots, n\}$ and $k \in \mathbb{N}$, $K \subseteq U^+_{k,i}\ \cup\ U^-_{k,i}$ and:

(a)

$$\lim_{k\to\infty} \sup_{x\in K} \frac{\rho(\varphi_i(-t_k, x))}{h_{-t_k,i}(x)} = \lim_{k\to\infty} \sup_{x\in K} \frac{\rho(\varphi_i(t_k, x))}{h_{t_k,i}(x)} = 0,$$

(b)

$$\lim_{k\to\infty} \sup_{x\in K\cap U^-_{k,i}} \frac{\rho(\varphi_i(-2t_k, x))}{h_{-2t_k,i}(x)} = \lim_{k\to\infty} \sup_{x\in K\cap U^+_{k,i}} \frac{\rho(\varphi_i(2t_k, x))}{h_{2t_k,i}(x)} = 0,$$

(c) *For every $i, j \in \{1, \cdots, n\}$ with $i \neq j$:*

$$\lim_{k\to\infty} (A_{ijk} + B_{ijk} + C_{ijk} + D_{ijk}) = 0,$$

where:

$$A_{ijk} := \sup_{x \in K \cap U^-_{k,j}} \frac{h_{t_k,i}(\varphi_i(-t_k,\varphi_j(-t_k,x)))\rho(\varphi_i(-t_k,\varphi_j(-t_k,x)))}{h_{-t_k,j}(x)},$$

$$B_{ijk} := \sup_{x \in K \cap U^-_{k,j}} \frac{h_{-t_k,i}(\varphi_i(t_k,\varphi_j(-t_k,x)))\rho(\varphi_i(t_k,\varphi_j(-t_k,x)))}{h_{-t_k,j}(x)},$$

$$C_{ijk} := \sup_{x \in K \cap U^+_{k,j}} \frac{h_{t_k,i}(\varphi_i(-t_k,\varphi_j(t_k,x)))\rho(\varphi_i(-t_k,\varphi_j(t_k,x)))}{h_{t_k,j}(x)}$$

$$D_{ijk} := \sup_{x \in K \cap U^+_{k,j}} \frac{h_{-t_k,i}(\varphi_i(t_k,\varphi_j(t_k,x)))\rho(\varphi_i(t_k,\varphi_j(t_k,x)))}{h_{t_k,j}(x)},$$

then the cosine functions $(C_{\varphi_i}(t))_{t\geq 0},\ i = 1, 2, \cdots, n$ *are d-topologically transitive.*

(ii) *Suppose* $X = L^p(\Omega, \mu)$ *and* μ *is p-admissible for* φ_i *and* $h_{\cdot,i}$, $1 \leq i \leq n$. *If for every compact subset* $K \subseteq \Omega$ *there exist a sequence* (t_k) *od non-negative real numbers and sequences of Borel measurable subsets* $(L^+_{k,i})$ *and* $(L^-_{k,i})$ *of* K *such that for* $L_{k,i} := L^+_{k,i} \cup L^-_{k,i}$ *the following holds:*

(a) $\lim_{k\to\infty} \mu(K \setminus L_{k,i}) = \lim_{k\to\infty} \nu_{p,t_k}(L_{k,i}) = \lim_{k\to\infty} \nu_{p,-t_k}(L_{k,i}) = 0,\ 1 \leq i \leq n,$

(b) $\lim_{k\to\infty} \nu_{p,2t_k}(L^+_{k,i}) = \lim_{k\to\infty} \nu_{p,-2t_k}(L^-_{k,i}) = 0,\ 1 \leq i \leq n,$ *and*

(c) *For every* $i, j \in \{1, \cdots, n\}$ *with* $i \neq j$:

$$\lim_{k\to\infty} \int_{\varphi_i(-t_k,\varphi_j(-t_k,L^+_{k,j}))} h^p_{t_k,i}(x)h^p_{t_k,j}(\varphi_i(t_k,x))d\mu = 0,$$

$$\lim_{k\to\infty} \int_{\varphi_i(-t_k,\varphi_j(t_k,L^-_{k,j}))} h^p_{t_k,i}(x)h^p_{-t_k,j}(\varphi_i(t_k,x))d\mu = 0,$$

$$\lim_{k\to\infty} \int_{\varphi_i(t_k,\varphi_j(-t_k,L^+_{k,j}))} h^p_{-t_k,i}(x)h^p_{t_k,j}(\varphi_i(-t_k,x))d\mu = 0,$$

$$\lim_{k\to\infty} \int_{\varphi_i(t_k,\varphi_j(t_k,L^-_{k,j}))} h^p_{-t_k,i}(x)h^p_{-t_k,j}(\varphi_i(-t_k,x))d\mu = 0,$$

then the cosine functions $(C_{\varphi_i}(t))_{t\geq 0},\ i = 1, 2, \cdots, n$ *are d-topologically transitive.*

Proof. We will prove only the first part of theorem. Let $\varepsilon > 0$, $u, v_1, \cdots, v_n \in C_c(\Omega)$ and $K = \mathrm{supp}u \cup \mathrm{supp}v_1 \cup \cdots \cup \mathrm{supp}v_n$. Then there exist a sequence (t_k) of non-negative real numbers and sequences $(U^+_{k,i})$ and $(U^-_{k,i})$ of open subsets of Ω satisfying that, for every $i \in \{1, \cdots, n\}$, $K \subseteq U^+_{k,i} \cup U^-_{k,i}$ and that (a)-(c) hold. Further on, for every $i \in \{1, \cdots, n\}$ and $k \in \mathbb{N}$, there exist non-negative C^∞-functions $\psi^\pm_{k,i}$ such that $\mathrm{supp}\psi^+_{k,i} \subseteq U^+_{k,i}$, $\mathrm{supp}\psi^-_{k,i} \subseteq U^-_{k,i}$ and $\psi^-_{k,i}(x) + \psi^-_{k,i}(x) = 2$, $x \in K$. Define, for every $k \in \mathbb{N}$, a function $\omega_k : \Omega \to \mathbb{C}$ by setting

$$\omega_k := u + \sum_{i=1}^n \Big[h_{t_k,i}(\cdot)v_i(\varphi_i(t_k,\cdot))\psi^-_{k,i}(\varphi_i(t_k,\cdot)) + h_{-t_k,i}(\cdot)v_i(\varphi_i(-t_k,\cdot))\psi^+_{k,i}(\varphi_i(-t_k,\cdot)) \Big].$$

Clearly, $\omega_k \in C_c(\Omega),\ k \in \mathbb{N}$ and it is enough to prove that there exists $k \in \mathbb{N}$ such that:

$$\max(||\omega_k - u||,\ ||C_{\varphi_1}(t_k)\omega_k - v_1||, \cdots,\ ||C_{\varphi_n}(t_k)\omega_k - v_n||) < \varepsilon. \tag{19}$$

By definition of ω_k, we easily infer that:

$$\begin{aligned} ||\omega_k - u|| &\leq \sum_{i=1}^{n} 2||v_i||_\infty \Bigg[\sup_{x \in \varphi_i(-t_k,K)} h_{t_k,i}(x)\rho(x) + \sup_{x \in \varphi_i(t_k,K)} h_{-t_k,i}(x)\rho(x) \Bigg] \\ &= \sum_{i=1}^{n} 2||v_i||_\infty \Bigg[\sup_{x \in K} \frac{\rho(\varphi_i(-t_k,x))}{h_{-t_k,i}(x)} + \sup_{x \in K} \frac{\rho(\varphi_i(t_k,x))}{h_{t_k,i}(x)} \Bigg],\ k \in \mathbb{N}. \end{aligned} \tag{20}$$

Set, for every $x \in \Omega$, $k \in \mathbb{N}$ and $1 \leq i, j \leq n$:
$a_{ijk}(x) := \varphi_j(t_k, \varphi_i(t_k, x))$,
$b_{ijk}(x) := \varphi_j(t_k, \varphi_i(-t_k, x))$,
$c_{ijk}(x) := \varphi_j(-t_k, \varphi_i(t_k, x))$
$d_{ijk}(x) := \varphi_j(-t_k, \varphi_i(-t_k, x))$,
$A_{ik}(x) := \sum_{\substack{1 \leq j \leq n \\ j \neq i}} [h_{t_k,i}(x)h_{t_k,j}(\varphi_i(t_k,x))v_j(a_{ijk}(x))\psi^-_{j,i}(a_{ijk}(x))]$,
$B_{ik}(x) := \sum_{\substack{1 \leq j \leq n \\ j \neq i}} [h_{-t_k,i}(x)h_{t_k,j}(\varphi_i(-t_k,x))v_j(b_{ijk}(x))\psi^-_{j,i}(b_{ijk}(x))]$,
$C_{ik}(x) := \sum_{\substack{1 \leq j \leq n \\ j \neq i}} [h_{t_k,i}(x)h_{-t_k,j}(\varphi_i(t_k,x))v_j(c_{ijk}(x))\psi^+_{j,i}(c_{ijk}(x))]$ and
$D_{ik}(x) := \sum_{\substack{1 \leq j \leq n \\ j \neq i}} [h_{-t_k,i}(x)h_{-t_k,j}(\varphi_i(-t_k,x))v_j(d_{ijk}(x))\psi^+_{j,i}(d_{ijk}(x))]$.
A straightforward computation shows that, for every $x \in \Omega$, $k \in \mathbb{N}$ and $1 \leq i \leq n$:
$2(C_{\varphi_i}(t_k)\omega_k - v_i)(x)$
$= [h_{t_k,i}(x)u(\varphi_i(t_k,x)) + h_{-t_k,i}(x)u(\varphi_i(-t_k,x))]$
$+[h_{2t_k,i}(x)v_i(\varphi_i(2t_k,x))\psi^-_{k,i}(\varphi_i(2t_k,x))$
$+h_{-2t_k,i}(x)v_i(\varphi_i(-2t_k,x))\psi^+_{k,i}(\varphi_i(-2t_k,x))]$
$+A_{ik}(x) + B_{ik}(x) + C_{ik}(x) + D_{ik}(x)$.
By virtue of (a)-(b), we get the following estimates:

$$\begin{aligned} &\sup_{x \in \Omega} \Big| h_{t_k,i}(x)u(\varphi_i(t_k,x)) + h_{-t_k,i}(x)u(\varphi_i(-t_k,x)) \Big| \rho(x) \\ &\leq ||u||_\infty \Big[\sup_{x \in K} \frac{\rho(\varphi_i(-t_k,x))}{h_{-t_k,i}(x)} + \sup_{x \in K} \frac{\rho(\varphi_i(-t_k,x))}{h_{-t_k,i}(x)} \Big],\ k \in \mathbb{N}, \end{aligned} \tag{21}$$

and

$$\begin{aligned} &\Big| h_{2t_k,i}(y)v_i(\varphi_i(2t_k,y))\psi^-_{k,i}(\varphi_i(2t_k,y)) + h_{-2t_k,i}(y)v_i(\varphi_i(-2t_k,y))\psi^+_{k,i}(\varphi_i(-2t_k,y)) \Big| \\ &\leq \frac{2||v_i||_\infty}{\rho(y)} \Bigg[\sup_{x \in K \cap U^-_{k,i}} \frac{\rho(\varphi_i(-2t_k,x))}{h_{-2t_k,i}(x)} + \sup_{x \in K \cap U^+_{k,i}} \frac{\rho(\varphi_i(2t_k,x))}{h_{2t_k,i}(x)} \Bigg],\ k \in \mathbb{N},\ y \in \Omega. \end{aligned} \tag{22}$$

Since $0 \leq \psi^\pm_{k,i} \leq 2$ on K we obtain that for every $x \in \Omega$, $k \in \mathbb{N}$ and $1 \leq i \leq n$:

$$|A_{ik}(x)| + |B_{ik}(x)| + |C_{ik}(x)| + |D_{ik}(x)| \leq \sum_{\substack{1 \leq j \leq n \\ j \neq i}} \frac{2||v_j||_\infty}{\rho(x)} \Big(A_{ijk} + B_{ijk} + C_{ijk} + D_{ijk} \Big). \tag{23}$$

Taking into account (23) and (c), we get that for $1 \leq i \leq n$:

$$\lim_{k\to\infty} \sup_{x\in\Omega} \Big(|A_{ik}(x)| + |B_{ik}(x)| + |C_{ik}(x)| + |D_{ik}(x)| \Big) \rho(x) = 0. \tag{24}$$

The proof of theorem now follows from (20)-(22) and (24).

Example 46. ([40]) Suppose $a_{ij} > 0$, $1 \leq i \leq n$, $1 \leq j \leq m$, and for every $i, j \in \{1, 2, \cdot\cdot\cdot, n\}$ with $i \neq j$ there exists $l \in \{1, \cdot\cdot\cdot, m\}$ such that $a_{il} \neq a_{jl}$. Let $p \geq 1$, $q > \frac{m}{2}$, $\Omega = (0, \infty)^m$, resp. $\Omega = \mathbb{R}^m$, and $h_t(x) = 1$, $t \in \mathbb{R}$, $x \in \Omega$. Define $\varphi_i : \mathbb{R} \times \Omega \to \Omega$, $i = 1, 2, \cdot\cdot\cdot, n$ and $\rho : \Omega \to (0, \infty)$ by:

$$\varphi_i(t, x_1, \cdot\cdot\cdot, x_m) := (e^{a_{i1}t} x_1, \cdot\cdot\cdot, e^{a_{im}t} x_m) \text{ and}$$

$$\rho(x_1, \cdot\cdot\cdot, x_m) := \frac{1}{(1 + |x|^2)^q},\ t \in \mathbb{R},\ x = (x_1, \cdot\cdot\cdot, x_m) \in \Omega.$$

Let μ be the measure on Ω with Lebesgue density ρ. Then one can simply verify with the help of [31, Theorem 4] that $(T_{\varphi_i}(t))_{t\in\mathbb{R}}$ is a strongly continuous group in $L^p(\Omega, \mu)$ $(C_{0,\rho}(\Omega))$, $1 \leq i \leq n$. Suppose first $\Omega = (0, \infty)^m$. Let $K = [a_1, b_1] \times \cdot\cdot\cdot \times [a_m, b_m]$ be a compact subset of Ω, let $L^+_{k,i} = L^-_{k,i} = K$, $k \in \mathbb{N}$ and let (t_k) be a sequence of positive real numbers such that $\lim_{k\to\infty} t_k = \infty$. Proceeding as in [40, Example 3(iii)], one can simply check that the conditions (a)-(c) stated in the formulation of Theorem 45(ii) hold, which implies that the induced cosine functions $(C_{\varphi_1}(t))_{t\geq0}$, $(C_{\varphi_2}(t))_{t\geq0}, \cdot\cdot\cdot, (C_{\varphi_n}(t))_{t\geq0}$, are d-topologically transitive in $L^p(\Omega, \mu)$. The above assertion remains true in the case that $\Omega = \mathbb{R}^m$, which follows from Theorem 45(ii) by choosing an appropriate sequence $(L^+_{k,i} = L^-_{k,i} = L_k)$ of measurable subsets of K satisfying $0 \notin L^\circ_k$, $k \in \mathbb{N}$. By [29, Theorem 3.7], we have that, for every $i = 1, 2, \cdot\cdot\cdot, n$, $(T_{\varphi_i}(t))_{t\geq0}$ is a non-hypercyclic strongly continuous semigroup in $C_{0,\rho}(\Omega)$. With Remark 40(i) in view, we obtain that $(C_{\varphi_i}(t))_{t\geq0}$ is a non-hypercyclic cosine function in $C_{0,\rho}(\Omega)$, which implies that $(C_{\varphi_i}(t))_{t\geq0}$, $1 \leq i \leq n$, cannot be d-hypercyclic in $C_{0,\rho}(\Omega)$.

We conclude the paper with the observation that the analysis given in Example 36(iii) and [40, Example 3(i)] may be applied in construction of d-topologically transitive C-regularized semigroups and C-regularized cosine functions.

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