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Convolution properties for subclasses of meromorphic univalent functions of complex order

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Abstract

Using the new linear operator

$$\mathcal{L}^m(\lambda,l)f(z)=\frac{1}{z}+\sum_{k=1}^{\infty}\left(\frac{l}{l+\lambda k}\right)^m a_k z^{k-1},\quad f\in\Sigma,$$

where $l>0$, $\lambda\geq 0$, and $m\in\mathbb{N}_0=\mathbb{N}\cup\{0\}$, we introduce two subclasses of meromorphic analytic functions, and we investigate several convolution properties, coefficient inequalities, and inclusion relations for these classes.

1 Introduction

Let Σ be the class of functions of the form

$$f(z)=\frac{1}{z}+\sum_{k=1}^{\infty}a_k z^{k-1},\tag{1}$$

which are analytic in the punctured unit disc $\mathrm{U}^*=\mathrm{U}\setminus\{0\}$, where $\mathrm{U}=\{z\in\mathbb{C}:|z|<1\}$. For the functions $f\in\Sigma$ of the form (1) and $g\in\Sigma$ given by $g(z)=\frac{1}{z}+\sum_{k=1}^{\infty}b_k z^{k-1}$, *the Hadamard (or convolution) product of f and g* is defined by

$$(f*g)(z)=\frac{1}{z}+\sum_{k=1}^{\infty}a_k b_k z^{k-1}.$$

For $\lambda\geq 0$, $l>0$, and $m\in\mathbb{N}_0=\mathbb{N}\cup\{0\}$, El-Ashwah [6] and El-Ashwah and Aouf (see [8] and [9]) defined the linear operator $\mathrm{I}^m(\lambda,l):\Sigma\to\Sigma$ by

$$\mathrm{I}^m(\lambda,l)f(z)=\frac{1}{z}+\sum_{k=1}^{\infty}\left(\frac{l+\lambda k}{l}\right)^m a_k z^{k-1},$$

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where f has the form (1). We note that $\mathrm{I}^0(\lambda,l)f(z) = f(z)$ and $\mathrm{I}^1(1,1)f(z) = \left(z^2 f(z)\right)'/z = 2f(z)+zf'(z)$, and by specializing the parameters λ, l, and m, we obtain the following operators studied by various authors:

(i) $\mathrm{I}^m(1,l)f(z) =: \mathrm{D}_l^m\, f(z)$, (see Cho et al. [3], [4]);

(ii) $\mathrm{I}^m(\lambda,1)f(z) =: \mathrm{D}_\lambda^m\, f(z)$, (see Al-Oboudi and Al-Zkeri [1]);

(iii) $\mathrm{I}^m(1,1)f(z) =: \mathrm{I}^m\, f(z)$, (see Uralegaddi and Somanatha [15]).

Definition 1. For $\lambda \geq 0$, $l > 0$, and $m \in \mathbb{N}_0$, we will define *the dual operator* $\mathcal{L}^m(\lambda,l) : \Sigma \to \Sigma$,

$$\mathcal{L}^m(\lambda,l)f(z) = \frac{1}{z} + \sum_{k=1}^{\infty} \left(\frac{l}{l+\lambda k}\right)^m a_k z^{k-1},$$

where f is given by (1).

Denoting by $\Psi^m(\lambda,l)(z) = \dfrac{1}{z} + \sum\limits_{k=1}^{\infty} \left(\dfrac{l}{l+\lambda k}\right)^m z^{k-1}$, it is easy to verify that

$$\mathcal{L}^m(\lambda,l)f(z) = \Psi^m(\lambda,l)(z) * f(z), \tag{2}$$

$$\lambda z\left(\mathcal{L}^{m+1}(\lambda,l)f(z)\right)' = l\mathcal{L}^m(\lambda,l)f(z) - (l+\lambda)\mathcal{L}^{m+1}(\lambda,l)f(z), \tag{3}$$

and

$$\mathcal{L}^m(\lambda,l)f(z) = \underbrace{\mathcal{L}^1(\lambda,l)\left(\tfrac{1}{z(1-z)}\right) * \ldots * \mathcal{L}^1(\lambda,l)\left(\tfrac{1}{z(1-z)}\right)}_{m \text{ times}} * f(z).$$

We note that $\mathcal{L}^\alpha(1,\beta)f(z) =: P_\beta^\alpha f(z)$, $\quad \alpha > 0$, $\beta > 0$ (see Lashin [10]).

If f and g are two analytic functions in U, we say that f *is subordinate to* g, written symbolically as $f(z) \prec g(z)$, if there exists a *Schwarz function* w, which (by definition) is analytic in U, with $w(0) = 0$, and $|w(z)| < 1$ for all $z \in \mathrm{U}$, such that $f(z) = g(w(z))$. Furthermore, if the function g is univalent in U, then we have the following equivalence, (cf., e.g., [11], see also [12, p. 4]):

$$f(z) \prec g(z) \Leftrightarrow f(0) = g(0) \text{ and } f(\mathrm{U}) \subset g(\mathrm{U}).$$

Definition 2. For $-1 \leq B < A \leq 1$, and $b \in \mathbb{C}^* = \mathbb{C} \setminus \{0\}$ we define two subclasses of the class Σ as follows:

$$\Sigma S^*[b;A,B] = \left\{ f \in \Sigma : 1 - \frac{1}{b}\left(1 + \frac{zf'(z)}{f(z)}\right) \prec \frac{1+Az}{1+Bz} \right\} \tag{4}$$

and

$$\Sigma K[b;A,B] = \left\{ f \in \Sigma : 1 - \frac{1}{b}\left(2 + \frac{zf''(z)}{f'(z)}\right) \prec \frac{1+Az}{1+Bz} \right\}.$$

We emphasize that in the above definitions, both of the functions that appeared in the left-hand side of the subordinations are regular in the point $z_0 = 0$. Also, it is easy to check the duality formula

$$f \in \Sigma K[b; A, B] \Leftrightarrow -zf'(z) \in \Sigma S^*[b; A, B], \tag{5}$$

while some special cases of these classes was studied by different authors:

(i) $\Sigma S^*[b; 1, -1] =: \Sigma S(b)$, with $b \in \mathbb{C}^*$, (see Aouf [2]);

(ii) $\Sigma K[b; 1, -1] =: \Sigma K(b)$, with $b \in \mathbb{C}^*$, (see Aouf [2]);

(iii) $\Sigma S^*[1; (1-2\alpha)\beta, -\beta] =: \Sigma S[\alpha, \beta]$, with $0 \le \alpha < 1$, $0 < \beta \le 1$, (see El-Ashwah and Aouf [7]);

(iv) $\Sigma K[1; (1-2\alpha)\beta, -\beta] =: \Sigma K[\alpha, \beta]$, with $0 \le \alpha < 1$, $0 < \beta \le 1$, (see El-Ashwah and Aouf [7]);

(v) $\Sigma S^* \left[(1-\alpha)e^{-i\mu}\cos\mu; 1, -1\right] =: \Sigma S^\mu(\alpha)$, with $\mu \in \mathbb{R}$, $|\mu| \le \pi/2$, $0 \le \alpha < 1$, (see [14] for $p = 1$);

(vi) $\Sigma K \left[(1-\alpha)e^{-i\mu}\cos\mu; 1, -1\right] =: \Sigma K^\mu(\alpha)$, with $\mu \in \mathbb{R}$, $|\mu| \le \pi/2$, $0 \le \alpha < 1$, (see [14] for $p = 1$).

Considering $\mu \in \mathbb{R}$ with $|\mu| \le \pi/2$, $0 \le \alpha < 1$, and $0 < \beta \le 1$, for the special cases $b = e^{-i\mu}\cos\mu$, $A = (1-2\alpha)\beta$, and $B = -\beta$ we will use the notations

$$\Sigma S^\mu[\alpha, \beta] := \Sigma S^* \left[e^{-i\mu}\cos\mu; (1-2\alpha)\beta, -\beta\right],$$
$$\Sigma K^\mu[\alpha, \beta] := \Sigma K \left[e^{-i\mu}\cos\mu; (1-2\alpha)\beta, -\beta\right].$$

Definition 3. For $\lambda \ge 0$, $l > 0$, $m \in \mathbb{N}_0$, and $-1 \le B < A \le 1$, using the linear operator $\mathcal{L}^m(\lambda, l)$ we define two subclasses of the class Σ as follows:

$$S^*_{\lambda,l}[m; b; A, B] = \left\{f \in \Sigma : \mathcal{L}^m(\lambda, l)f \in \Sigma S^*[b; A, B]\right\} \tag{6}$$

and

$$K_{\lambda,l}[m; b; A, B] = \left\{f \in \Sigma : \mathcal{L}^m(\lambda, l)f \in \Sigma K[b; A, B]\right\}. \tag{7}$$

Lemma 1. *The following duality formula between the above defined classes holds:*

$$f \in K_{\lambda,l}[m; b; A, B] \Leftrightarrow -zf'(z) \in S^*_{\lambda,l}[m; b; A, B]. \tag{8}$$

Proof. According to the definition formula (7), we have that $f \in K_{\lambda,l}[m; b; A, B]$ if and only if $\mathcal{L}^m(\lambda, l)f \in \Sigma K[b; A, B]$, and from (5) this last relation is equivalent to $-z\left(\mathcal{L}^m(\lambda, l)f(z)\right)' \in \Sigma S^*[b; A, B]$. Using the representation (2) we deduce the equalities

$$-z\left(\mathcal{L}^m(\lambda, l)f(z)\right)' = -z\left(\Psi^m(\lambda, l)(z) * f(z)\right)' =$$
$$\Psi^m(\lambda, l)(z) * \left(-zf'(z)\right) = \mathcal{L}^m(\lambda, l)\left(-zf'(z)\right),$$

hence $\mathcal{L}^m(\lambda, l)\left(-zf'(z)\right) \in \Sigma S^*[b; A, B]$, so the definition formula (6) yields that $-zf'(z) \in S^*_{\lambda,l}[m; b; A, B]$. $\square$

Supposing $\mu \in \mathbb{R}$ with $|\mu| \le \pi/2$, $0 \le \alpha < 1$, and $0 < \beta \le 1$, for the special cases $b = e^{-i\mu}\cos\mu$, $A = (1-2\alpha)\beta$, and $B = -\beta$ in (6) and (7), we will use the notations

$$S^*_{\lambda,l}(m;\mu;\alpha,\beta) := S^*_{\lambda,l}\left[m; e^{-i\mu}\cos\mu; (1-2\alpha)\beta, -\beta\right] = \\ \left\{f \in \Sigma : \mathcal{L}^m(\lambda,l)f \in \Sigma S^\mu[\alpha,\beta]\right\},$$

and

$$K_{\lambda,l}(m;\mu;\alpha,\beta) := K_{\lambda,l}\left[m; e^{-i\mu}\cos\mu; (1-2\alpha)\beta, -\beta\right] = \\ \left\{f \in \Sigma : \mathcal{L}^m(\lambda,l)f \in \Sigma K^\mu[\alpha,\beta]\right\}.$$

Note that many important properties of several subclasses of meromorphic univalent functions were studied by several authors. In this paper we will investigate convolution properties, coefficient inequalities, and inclusion relations for the subclasses we defined above.

2 Main results

We assume throughout this section that $0 \le \theta < 2\pi$, $b \in \mathbb{C}^*$, and $-1 \le B < A \le 1$.

Theorem 1. *If $f \in \Sigma$, then $f \in \Sigma S^*[b;A,B]$ if and only if*

$$z\left[f(z) * \frac{1+(C-1)z}{z(1-z)^2}\right] \neq 0,\ z \in \mathrm{U}, \tag{9}$$

for all $C = C_\theta = \dfrac{e^{-i\theta}+B}{(A-B)b}$, $\theta \in [0,2\pi)$, and also for $C = 0$.

Proof. It is easy to check that the relations

$$f(z) * \frac{1}{z(1-z)} = f(z),\ f(z) * \left[\frac{1}{z(1-z)^2} - \frac{2}{(1-z)^2}\right] = -zf'(z), \tag{10}$$

hold for all $z \in \mathrm{U}^*$, and for any function $f \in \Sigma$.

(i) To prove the first implication, if $f \in \Sigma S^*[b;A,B]$ is an arbitrary function, from (4) we have

$$-\frac{zf'(z)}{f(z)} \prec \frac{1+[B+(A-B)b]\,z}{1+Bz}. \tag{11}$$

Since the function from the left-hand side of the subordination is analytic in U, it follows that $f(z) \neq 0$ for all $z \in \mathrm{U}^*$, i.e. $zf(z) \neq 0$, $z \in \mathrm{U}$, and using the first identity of (10), this is equivalent to the fact that (9) holds for $C = 0$.

From (11), according to the definition of the subordination, there exists a function w analytic in U, with $w(0) = 0$, and $|w(z)| < 1$, $z \in \mathrm{U}$, such that

$$-\frac{zf'(z)}{f(z)} = \frac{1+[B+(A-B)b]\,w(z)}{1+Bw(z)},\ z \in \mathrm{U},$$

hence it follows

$$z\Big[-zf'(z)\left(1+Be^{i\theta}\right)-f(z)\Big[1+[B+(A-B)b]\,e^{i\theta}\Big]\Big]\neq 0, \tag{12}$$
$$z\in\mathrm{U},\ \theta\in[0,2\pi).$$

Using the formulas (10), the relation (12) is equivalent to

$$z\left[f(z)*\frac{1+\left[\dfrac{e^{-i\theta}+B}{(A-B)b}-1\right]z}{z(1-z)^2}(A-B)be^{i\theta}\right]\neq 0,\ z\in\mathrm{U},\ \theta\in[0,2\pi),$$

which leads to (9), and the first part of the Theorem 1 was proved.

(ii) Reversely, because the assumption (9) holds for $C=0$, it follows that $zf(z)\neq 0$ for all $z\in\mathrm{U}$, hence the function $\varphi(z)=-\dfrac{zf'(z)}{f(z)}$ is analytic in U (i.e. it is regular in $z_0=0$, with $\varphi(0)=1$).

Since it was shown in the first part of the proof that the assumption (9) is equivalent to (12), we obtain that

$$-\frac{zf'(z)}{f(z)}\neq\frac{1+[B+(A-B)b]\,e^{i\theta}}{1+Be^{i\theta}},\ z\in\mathrm{U},\ \theta\in[0,2\pi), \tag{13}$$

If we denote

$$\psi(z)=\frac{1+[B+(A-B)b]\,z}{1+Bz},$$

the relation (13) shows that $\varphi(\mathrm{U})\cap\psi(\partial\mathrm{U})=\emptyset$. Thus, the simply-connected domain $\varphi(\mathrm{U})$ is included in a connected component of $\mathbb{C}\setminus\psi(\partial\mathrm{U})$. From here, using the fact that $\varphi(0)=\psi(0)$ together with the univalence of the function ψ, it follows that $\varphi(z)\prec\psi(z)$, which represents in fact the subordination (11), i.e. $f\in\Sigma S^*[b;A,B]$. $\square$

Remark 1. (i) Taking in Theorem 1 the special case $b=1$, and $e^{i\theta}=x$, we obtain the result of Ponnusamy [13, Theorem 2.1];

(ii) Taking in Theorem 1 the special case $b=(1-\alpha)e^{-i\mu}\cos\mu$, where $\mu\in\mathbb{R}$, $|\mu|\leq\pi/2$, $0\leq\alpha<1$, and $e^{i\theta}=x$, we obtain the result of Ravichandran et al. [14, Theorem 1.2 with $p=1$].

Theorem 2. *If $f\in\Sigma$, then $f\in\Sigma K[b;A,B]$ if and only if*

$$z\left[f(z)*\frac{1-3z-2(C-1)z^2}{z(1-z)^3}\right]\neq 0,\ z\in\mathrm{U},$$

for all $C=C_\theta=\dfrac{e^{-i\theta}+B}{(A-B)b}$, $\theta\in[0,2\pi)$, and also for $C=0$.

Proof. If we let $g(z)=\dfrac{1+(C-1)z}{z(1-z)^2}$, then

$$zg'(z)=\frac{-1+3z+2(C-1)z^2}{z(1-z)^3}.$$

From the duality formula (5), using the identity

$$[-zf'(z)]*g(z)=f(z)*[-zg'(z)]\,,$$

the result follows from Theorem 1. □

Remark 2. Putting $b=1$ and $e^{i\theta}=x$ in Theorem 2, this special case will correct the result obtained by Ponnusamy [13, Theorem 2.2].

Theorem 3. *Let $\lambda\geq 0$, $l>0$, and $m\in\mathbb{N}_0$. If $f\in\Sigma$ is of the form (1), then $f\in S^*_{\lambda,l}[m;b;A,B]$ if and only if*

$$1+\sum_{k=1}^{\infty}\left(\frac{l}{l+\lambda k}\right)^m a_k z^k\neq 0,\ z\in \mathrm{U}, \tag{14}$$

and

$$1+\sum_{k=1}^{\infty}\frac{ke^{-i\theta}+kB+(A-B)\,b}{(A-B)\,b}\left(\frac{l}{l+\lambda k}\right)^m a_k z^k\neq 0,\ z\in \mathrm{U}, \tag{15}$$

for all $\theta\in[0,2\pi)$.

Proof. If $f\in\Sigma$, according to Theorem 1 we have $f\in S^*_{\lambda,l}[m;b;A,B]$ if and only if

$$z\left[\mathcal{L}^m(\lambda,l)f(z)*\frac{1+(C-1)z}{z(1-z)^2}\right]\neq 0,\ z\in \mathrm{U}, \tag{16}$$

for all $C=C_\theta=\dfrac{e^{-i\theta}+B}{(A-B)b}$, $\theta\in[0,2\pi)$, and also for $C=0$.

Using the first part of the identities (10), it is easy to see that the above relation holds for $C=0$ if and only if (14) is satisfied.

On the other hand, using the relation

$$\frac{1+(C-1)z}{z(1-z)^2}=\frac{1}{z}+\sum_{k=1}^{\infty}(1+Ck)z^{k-1},\ z\in \mathrm{U}^*,$$

we may easily check that (16) is equivalent to (15), which proves our result. □

Theorem 4. *Let $\lambda\geq 0$, $l>0$, and $m\in\mathbb{N}_0$. If $f\in\Sigma$ is of the form (1), then $f\in K_{\lambda,l}[m;b;A,B]$ if and only if*

$$1-\sum_{k=1}^{\infty}(k-1)\left(\frac{l}{l+\lambda k}\right)^m a_k z^k\neq 0,\ z\in \mathrm{U}, \tag{17}$$

and

$$1-\sum_{k=1}^{\infty}(k-1)\frac{ke^{-i\theta}+Bk+(A-B)b}{(A-B)\,b}\left(\frac{l}{l+\lambda k}\right)^{m}a_kz^k\neq 0,\ z\in\mathrm{U}, \tag{18}$$

for all $\theta\in[0,2\pi)$.

Proof. If $f\in\Sigma$, from Theorem 2 we have that $f\in K_{\lambda,l}[m;b;A,B]$ if and only if

$$z\left[\mathcal{L}^m(\lambda,l)f(z)*\frac{1-3z-2(C-1)z^2}{z(1-z)^3}\right]\neq 0,\ z\in\mathrm{U}, \tag{19}$$

for all $C=C_\theta=\dfrac{e^{-i\theta}+B}{(A-B)b}$, $\theta\in[0,2\pi)$, and also for $C=0$.

Using the relation

$$\frac{1}{z(1-z)^2}=\frac{1}{z}+\sum_{k=1}^{\infty}(k+1)z^{k-1},\ z\in\mathrm{U}^*,$$

it is easy to see that (19) holds for $C=0$ if and only if the assumption (17) is satisfied.

Now, from the formula

$$\frac{1}{z(1-z)^3}=\frac{1}{z}+\sum_{k=1}^{\infty}\frac{(k+1)(k+2)}{2}z^{k-1},\ z\in\mathrm{U}^*,$$

we may easily deduce that

$$\frac{1-3z-2(C-1)z^2}{z(1-z)^3}=\frac{1}{z}-\sum_{k=1}^{\infty}(k-1)(1+Ck)z^{k-1},\ z\in\mathrm{U}^*,$$

and a simple computation shows that (19) is equivalent to (18), hence the proof of the theorem is completed. □

Theorem 5. *Let* $\lambda\geq 0$, $l>0$, $m\in\mathbb{N}_0$, $-1\leq B<A\leq 1$ *and* $b\in\mathbb{C}^*$. *If* $f\in\Sigma$ *has the from* (1) *and satisfies the inequalities*

$$\sum_{k=1}^{\infty}\left(\frac{l}{l+\lambda k}\right)^{m}|a_k|<1, \tag{20}$$

and

$$\sum_{k=1}^{\infty}\left[k(1+B)+(A-B)\,|b|\right]\left(\frac{l}{l+\lambda k}\right)^{m}|a_k|<(A-B)\,|b|\,, \tag{21}$$

then $f\in S^*_{\lambda,l}[m;b;A,B]$.

Proof. According to (20), a simple computation shows that

$$\left|1+\sum_{k=1}^{\infty}\left(\frac{l}{l+\lambda k}\right)^{m}a_kz^k\right|\geq 1-\left|\sum_{k=1}^{\infty}\left(\frac{l}{l+\lambda k}\right)^{m}a_kz^k\right|\geq$$
$$1-\sum_{k=1}^{\infty}\left(\frac{l}{l+\lambda k}\right)^{m}|a_k|\left|z^k\right|\geq 1-\sum_{k=1}^{\infty}\left(\frac{l}{l+\lambda k}\right)^{m}|a_k|>0,\ z\in\mathrm{U},$$

hence the condition (14) is satisfied.

Using the inequality

$$\left|\frac{ke^{-i\theta}+Bk+(A-B)b}{(A-B)\,b}\right|\leq\frac{k(1+B)+(A-B)\,|b|}{(A-B)\,|b|}$$

together with the assumption (21), we may easily deduce

$$\left|1+\sum_{k=1}^{\infty}\left[\frac{ke^{-i\theta}+Bk+(A-B)b}{(A-B)\,b}\right]\left(\frac{l}{l+\lambda k}\right)^{m}a_kz^k\right|>$$
$$1-\sum_{k=1}^{\infty}\left|\frac{ke^{-i\theta}+Bk+(A-B)b}{b\,(A-B)}\right|\left(\frac{l}{l+\lambda k}\right)^{m}|a_k|\geq$$
$$1-\sum_{k=1}^{\infty}\frac{k(1+B)+(A-B)\,|b|}{(A-B)\,|b|}\left(\frac{l}{l+\lambda k}\right)^{m}|a_k|>0,\ z\in\mathrm{U},$$

which shows that (15) holds, hence our result follows from Theorem 3. □

Using Theorem 4, in the same way we may also prove the next result:

Theorem 6. *Let* $\lambda\geq 0$, $l>0$, $m\in\mathbb{N}_0$, $-1\leq B<A\leq 1$ *and* $b\in\mathbb{C}^*$. *If* $f\in\Sigma$ *has the from* (1) *and satisfies the inequalities*

$$\sum_{k=1}^{\infty}(k-1)\left(\frac{l}{l+\lambda k}\right)^{m}|a_k|<1,$$

and

$$\sum_{k=1}^{\infty}(k-1)\left[k(1+B)+(A-B)\,|b|\right]\left(\frac{l}{l+\lambda k}\right)^{m}|a_k|<(A-B)\,|b|\,,$$

then $f\in K_{\lambda,l}[m;b;A,B]$.

We will discuss two inclusion relations for the classes $S^*_{\lambda,l}[m;b;A,B]$ and $K_{\lambda,l}[m;b;A,B]$. To prove these results we shall require the following lemma:

Lemma 2. ([5]) *Let* h *be convex (univalent) in* U, *with* $\mathrm{Re}[\beta h(z)+\gamma]>0$ *for all* $z\in\mathrm{U}$. *If* p *is analytic in* U, *with* $p(0)=h(0)$, *then*

$$p(z)+\frac{zp'(z)}{\beta p(z)+\gamma}\prec h(z)\Rightarrow p(z)\prec h(z).$$

Theorem 7. *Let $\lambda > 0$, $l > 0$, and $m \in \mathbb{N}_0$. Suppose that $b \in \mathbb{C}^*$ and $-1 \leq B < A \leq 1$, such that*

$$\frac{\cos(\theta + \arg b) + B\cos(\arg b)}{1 + B^2 + 2B\cos\theta} \leq \frac{l}{|b|\lambda(A-B)},\ \theta \in [0, 2\pi). \tag{22}$$

If $f \in S^*_{\lambda,l}[m; b; A, B]$, with $\mathcal{L}^{m+1}(\lambda, l)f(z) \neq 0$ for all $z \in \mathrm{U}^*$, then $f \in S^*_{\lambda,l}[m+1; b; A, B]$.

Proof. Suppose that $f \in S^*_{\lambda,l}[m; b; A, B]$, and let define the function

$$p(z) = 1 - \frac{1}{b}\left(1 + \frac{z\big(\mathcal{L}^{m+1}(\lambda, l)f(z)\big)'}{\mathcal{L}^{m+1}(\lambda, l)f(z)}\right). \tag{23}$$

Then p is analytic in U with $p(0) = 1$, and using the relation (3), from (23) we obtain

$$-\frac{b\lambda}{l}\left(p(z) - 1\right) + 1 = \frac{\mathcal{L}^{m}(\lambda, l)f(z)}{\mathcal{L}^{m+1}(\lambda, l)f(z)}. \tag{24}$$

Differentiating logarithmically (24) and then using (23), we deduce that

$$p(z) + \frac{zp'(z)}{-bp(z) + \left(b + \frac{l}{\lambda}\right)} \prec \frac{1 + Az}{1 + Bz} =: h(z). \tag{25}$$

A simple calculus shows that the inequality $\operatorname{Re}\left[-bh(z) + \left(b + \frac{l}{\lambda}\right)\right] > 0$, $z \in \mathrm{U}$, may be written as

$$\operatorname{Re}\frac{bz}{1 + Bz} < \frac{l}{\lambda(A-B)},\ z \in \mathrm{U},$$

which is equivalent to (22). Since the function h is convex (univalent) in U, according to Lemma 2 the subordination (25) implies $p(z) \prec h(z)$, which proves that $f \in S^*_{\lambda,l}[m+1; b; A, B]$. $\square$

From the duality formula (8), and using the fact that

$$\mathcal{L}^{m+1}(\lambda, l)\left(-zf'(z)\right) = -z\left(\mathcal{L}^{m+1}(\lambda, l)f(z)\right)',$$

the above theorem yields the following inclusion:

Theorem 8. *Let $\lambda > 0$, $l > 0$, and $m \in \mathbb{N}_0$. Suppose that $b \in \mathbb{C}^*$ and $-1 \leq B < A \leq 1$, such that* (22) *holds.*

If $f \in K_{\lambda,l}[m; b; A, B]$, with $\left(\mathcal{L}^{m+1}(\lambda, l)f(z)\right)' \neq 0$ for all $z \in \mathrm{U}^*$, then $f \in K_{\lambda,l}[m+1; b; A, B]$.

Remark 3. (i) Putting in the above results $b = e^{-i\mu}\cos\mu$, $A = (1-2\alpha)\beta$ and $B = -\beta$, where $\mu \in \mathbb{R}$ with $|\mu| \leq \pi/2$, $0 \leq \alpha < 1$, and $0 < \beta \leq 1$, we obtain analogous results for the classes $\Sigma S^\mu[\alpha, \beta]$ and $\Sigma K^\mu[\alpha, \beta]$, respectively;

(ii) By specializing the parameters λ, l and m, we obtain various special cases for different operators defined in the introduction.

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