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The structure of ν-isologic pairs of groups

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Abstract

In [2], the first two authors generalized the concept of n-isoclinism to the class of all pairs of groups. In this paper, we extend that notion to ν-isologism and study the details of this notion. In addition, it is shown that every pair of groups is ν-isologic with a quotient irreducible pair of groups. Finally, as an application, we drive some inequalities for the Baer-invariant of a pair of groups.

1 Introduction

In 1940, Hall [1] introduced the notion of isoclinism and then he extend it to the notion of ν-isologism with respect to a given variety ν . Hekster [3] considered the variety of nilpotent groups of class at most n and arose the concept of n-isoclinism. In [2], the first two authors generalized the concept of n-isoclinism to the class of all pairs of groups. In this paper, the notion of ν -isologism is extended for pairs of groups and some of its properties are presented. Our results are useful for studying pairs of groups. This is illustrated in section 5. If ν is the variety of abelian groups or nilpotent groups of class at most n, then ν-isologism coincides with isoclinism and n-isoclinism between pairs of groups, as defined in [2] and [3].

Let F_∞ be the free group freely generated by a countable set $X = \{x_1, x_2, \dots\}$. Let V be a nonempty subset of F_∞ and ν be a variety of groups defined by the set of laws V. Then the verbal subgroup and the marginal subgroup of G associated with the variety are denoted by $V(G)$ and $V^*(G)$, respectively. See H. Neumann [7] for more information regarding varieties of groups.

2 Notation and preparatory results

Definition 1. Let (G, M) be a pair of groups in which M is a normal subgroup of G. If ν is a variety of groups with the set of laws V, we define

$$V(M,G) = \langle v(g_1, ..., g_i m, ..., g_r)v(g_1, ...g_r)^{-1} | v \in V, m \in M, g_i \in G, 1 \le i \le r\rangle,$$
$$V^*(M,G) = \{m \in M | v(g_1, ..., g_i m, ..., g_r) = v(g_1, ...g_r), \forall v \in V, g_i \in G, 1 \le i \le r\}.$$

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If $M = G$, then $V(M,G) = V(G)$ and $V^*(M,G) = V^*(G)$ are ordinary verbal and marginal subgroups of G. If ν is the variety of abelian groups, then $V^*(M,G) = Z(M,G) = \{m \in M | g^{-1}mg = m, \forall g \in G\}$ and $V(M,G) = [M,G] = \{[m,g] | m \in M, g \in G\}$. In addition, if ν is the variety of nilpotent groups of class at most n, then $V^*(M,G) = Z_n(M,G)$ and $V(M,G) = \gamma_{n+1}(M,G)$ (see [2]). A variety ν is nilpotent, if ν is contained in the variety of nilpotent groups of class at most n, for some $n \geq 1$. If ν is a nilpotent variety, then there exists $n \geq 1$ such that $V^*(M,G) \subseteq Z_n(M,G)$.

In the following Lemma, we present some basic properties of $V(M,G)$ and $V^*(M,G)$, which is useful in our investigations.

Lemma 1. *If (G,M) is a pair of groups and $N \trianglelefteq G$ with $N \leq M$. Then*

(a) $V(\frac{M}{N}, \frac{G}{N}) = \frac{V(M,G)N}{N}$;

(b) $N \subseteq V^*(M,G)$ *if and only if* $V(N,G) = 1$;

(c) $V^*(M,G) = M$ *if and only if* $V(M,G) = 1$;

(d) $V^*(\frac{M}{N}, \frac{G}{N}) \supseteq \frac{V^*(M,G)N}{N}$;

(e) $V(N) \subseteq V(N,G) \subseteq N \cap V(M,G)$;

(f) *If* $[G,N] \subseteq V^*(M,G)$, *then* $[V(M,G),N] = 1$. *In particular* $[V(M,G),V^*(M,G)] = 1$;

(g) *If* $N \cap V(M,G) = 1$, *then* $N \subseteq V^*(M,G)$ *and* $V^*(\frac{G}{N}, \frac{M}{N}) = \frac{V^*(M,G)N}{N}$.

Proof. (a), (b), (c) and (d) immediately follow from definition.

(e) Let $n_1, ..., n_r \in N$ and $v \in V$. Then

$$v(n_1,...,n_r) = v(n_1,...,n_r)v(1,n_2,...,n_r)^{-1}v(1,n_2,...,n_r)v(1,1,n_3,...,n_r)^{-1},...,$$
$$v(1,...,1,n_{r-1},n_r)v(1,...,1,n_r)^{-1}v(1,...,1,n_r) \in V(N,G).$$

Therefore $V(N) \subseteq V(N,G)$. Another inclusion follows from definition.

(f) Let $m \in M$, $g_1,...,g_r \in G$ and $n \in N$. Then for every $v \in V$,

$$\begin{aligned}
& [v(g_1,...,g_im,...,g_r)v(g_1,...,g_r)^{-1}, n] \\
= \;& v(g_1,...,g_r)v(g_1,...,g_im,...,g_r)^{-1}n^{-1}v(g_1,...,g_im,...,g_r)v(g_1,...,g_r)^{-1}n \\
= \;& v(g_1,...,g_r)v(g_1,...,g_im,...,g_r)^{-1}v(n^{-1}g_1n,...,n^{-1}g_imn,...,n^{-1}g_rn) \\
& v(n^{-1}g_1n,...,n^{-1}g_rn)^{-1} = v(g_1,...,g_r)v(g_1,...,g_im,...,g_r)^{-1} \\
& v(g_1[g_1,n],...,g_im[g_im,n],...,g_r[g_r,n])v(g_1[g_1,n],...,g_r[g_r,n])^{-1} \\
= \;& v(g_1,...,g_r)^{-1}v(g_1,...,g_im,...,g_r)v(g_1,...,g_im,...,g_r)v(g_1,...,g_r)^{-1} \\
= \;& 1.
\end{aligned}$$

So $[V(M,G),N] = 1$.

(g) As $N \cap V(M,G) = 1$, part (e) shows that $V(N,G) = 1$. Therefore (b) implies that $N \subseteq V^*(M,G)$. Now by (d), $V^*(\frac{M}{N}, \frac{G}{N}) \supseteq \frac{V^*(M,G)N}{N}$. If K is a subgroup of G such that $V^*(\frac{M}{N}, \frac{G}{N}) = \frac{K}{N}$, then $K \trianglelefteq G$ and $V^*(M,G) \subseteq K$. So by (e), $V^*(K,G) \subseteq V^*(M,G)$ and by assumption $V(K,G) = 1$. Thus again (b) implies that $K \subseteq V(M,G)$ and so $V^*(\frac{M}{N}, \frac{G}{N}) \subseteq \frac{V^*(M,G)N}{N}$. $\square$

In the next Theorem, we compute verbal subgroup of a group G, when G is given as a product of its subgroups.

Theorem 1. *If (G,M) is a pair of groups and $H \leq G$ such that $G = HM$. Then*

$$V(G) = V(H)V(M,G).$$

Proof. As $V(M,G) \subseteq V(G)$ and $V(H) \subseteq V(G)$, we have $V(H) \cap V(M,G) \subseteq V(G)$. Let $m \in M$ and $x \in V(H)$, then $m^{-1}xm = x[x,m] \in V(H)[V(H),M] \subseteq V(H)V(M,G)$. Thus M normalizes $V(H)V(M,G)$. As $G = HM$, $V(M,G) \trianglelefteq G$ and $V(H) \trianglelefteq H$, we conclude that $V(H)V(M,G)$ is a normal subgroup of G. In addition, as $V(G) = V(G,G)$, for every $v \in V$ and $g, g_i \in G$, $i = 1, ..., r$, we can write $g = hm$ and $g_i = h_i m_i$ with $h, h_i \in H$ and $m, m_i \in M$. If bar $\bar{.}$ denotes reduction modulo $V(H)V(M,G)$, then

$$\begin{aligned} v(\bar{g_1}, ..., \bar{g_i}\bar{g}, ..., \bar{g_r})v(\bar{g_1}, ..., \bar{g_r})^{-1} &= v(\bar{h_1}\bar{m_1}, ..., \bar{h_i}\bar{m_i}\bar{h}\bar{m}, ..., \bar{h_r}\bar{m_r})v(\bar{h_1}\bar{m_1}, ..., \bar{h_r}\bar{m_r})^{-1} \\ &= v(\bar{h_1}, ..., \bar{h_i}\bar{h}, ..., \bar{h_r})v(\bar{h_1}, ..., \bar{h_r})^{-1} \\ &= \bar{1}. \end{aligned}$$

Hence $\dfrac{V(G)}{V(H)V(M,G)} = 1$. So $V(G) \subseteq V(H)V(M,G)$ and we have $V(H)V(M,G) = V(G)$. $\square$

The following lemma is needed in the next sections.

Lemma 2. *Let (G,M) be a pair of groups and H be a subgroup of G such that $G = HV^*(M,G)$. Then the following statements hold.*

(*a*) *$V^*(M \cap H, H) = V^*(M,G) \cap H$;*

(*b*) *$V(M \cap H, H) = V(M,G)$;*

(*c*) *$V^*(M \cap H, H) \cap V(M \cap H, H) = V^*(M,G) \cap V(M,G)$.*

Proof. (*a*) By definition $V^*(M,G) \subseteq V^*(M \cap H, H)$, thus $V^*(M,G) \cap H \subseteq V^*(M \cap H, H)$. Now let $m \in V^*(M \cap H, H)$. If $g_1, ..., g_r \in G$, then we can write $g_i = h_i x_i$ with $h_i \in H$ and $x_i \in V^*(M,G)$, $i = 1, ..., r$ and for every $v \in V$,

$$\begin{aligned} v(g_1, ..., g_i m, ..., g_r) &= v(h_1 x_1, ..., h_i x_i m, ..., h_r x_r) \\ &= v(h_1, ..., h_r) \\ &= v(h_1 x_1, ..., h_r x_r) \\ &= v(g_1, ..., g_r) \end{aligned}$$

Therefore $m \in V^*(M,G) \cap H$ and $V^*(M \cap H, H) \subseteq V^*(M,G) \cap H$.

(*b*) Since $V(M \cap H, H) \subseteq V(M,G)$, we show the reverse inclusion. If $m \in M$ and $g_i \in G$, then we can write $m = hx$ and $g_i = h_i x_i$ with $h, h_i \in H$ and $x, x_i \in$

$V^*(M,G)$, $i=1,...,r$. Now for every $v \in V$,

$$\begin{aligned}
v(g_1,...,g_im,...,g_r)v(g_1,...,g_r)^{-1} &= v(h_1x_1,...,h_ix_ihx,...,h_rx_r)v(h_1x_1,...,h_rx_r)^{-1}\\
&= v(h_1,...,h_ix_ih,...,h_r)v(h_1,...,h_r)^{-1}\\
&= h^{-1}v(hh_1h^{-1},...,hh_ix_i,...,hh_rh^{-1})hv(h_1,...,h_r)^{-1}\\
&= h^{-1}v(hh_1h^{-1},...,hh_i,...,hh_rh^{-1})hv(h_1,...,h_r)^{-1}\\
&= v(h_1,...,h_ih,...,h_r)v(h_1,...,h_r)^{-1}\\
&\in V(M\cap H,H).
\end{aligned}$$

Therefore $V(M,G) \subseteq V(M\cap H,H)$.

(c) This immediately follows from (a) and (b). □

3 Isologism between pairs of groups

In this section, we extend the notion of ν-isologism, as defined in [4], to the class of all pairs of groups and show that two pairs (G_1,M_1) and (G_2,M_2) of groups are ν-isologic if and only if there exists a pair (G,M) of groups such that (G_1,M_1) and (G_2,M_2) occur as factor pairs of (G,M), whereas (G,M), (G_1,M_1) and (G_2,M_2) are ν-isologic to each other.

Let (G,M) and (H,N) be pairs of groups. An homomorphism from (G,M) to (H,N) is a homomorphism $f: G \to H$ such that $f(M) \subseteq N$. We say that (G,M) and (H,N) are isomorphic and write $(G,M) \simeq (H,N)$, if f is an isomorphism and $f(M)=N$.

Definition 2. Let (G,M) and (H,N) be two pairs of groups and ν be a variety of groups defined by the set of laws V. An *ν-isologism* between (G,M) and (H,N) is a pair of isomorphism (α,β) with $\alpha : \dfrac{G}{V^*(M,G)} \to \dfrac{H}{V^*(N,H)}$ and $\beta : V(M,G) \to V(N,H)$, such that $\alpha(\dfrac{M}{V^*(M,G)}) = \dfrac{N}{V^*(N,H)}$ and for every $\nu \in V$, $m \in M$ and $g_1,...,g_r \in G$

$$\beta(\nu(g_1,...,g_im,...,g_r)\nu(g_1,...,g_r)^{-1}) = \nu(h_1,...,h_in,...,h_r)\nu(h_1,...,h_r)^{-1},$$

whenever $h_i \in \alpha(g_iV^*(M,G))$ and $n \in \alpha(mV^*(M,G))$. We say that (G,M) and (H,N) are ν-isologic, if there exists an ν-isologism between them. In this case we write $(G,M) \sim_\nu (H,N)$.

If ν is the variety of abelian groups or nilpotent groups of class at most n, then ν-isologism coincides with isoclinism and n-isoclinism between pairs of groups. So we have here a generalized notion of isoclinism and n-isoclinism between pairs of groups. In addition, If $M=G$ and $N=H$, then ν-isologism between two pairs of groups is an ν-isologism between G and H. The following Lemma gives an equivalent condition for two pairs of groups to be ν-isologic.

Lemma 3. *Let (G,M) and (H,N) be two pairs of groups.Then $(G,M) \sim_\nu (H,N)$ if and only if there exist $M_1 \leq V^*(M,G)$, $N_1 \leq V^*(N,H)$ and isomorphisms*

$\alpha : \dfrac{G}{M_1} \to \dfrac{H}{N_1}$ *and* $\beta : V(M,G) \to V(N,H)$ *such that* $\alpha(\dfrac{M}{M_1}) = \dfrac{N}{N_1}$ *and for every* $\nu \in V$, $m \in M$ *and* $g_1, ..., g_r \in G$, $\beta(\nu(g_1, ..., g_i m, ..., g_r)\nu(g_1, ..., g_r)^{-1}) = \nu(h_1, ..., h_i n, ..., h_r)\nu(h_1, ..., h_r)^{-1}$, *where* $n \in \alpha(mM_1)$ *and* $h_i \in \alpha(g_i M_1), i = 1, ..., r$.

Proof. The 'only' part is trivial. For the another part it is sufficient to show that $\alpha(\dfrac{V^*(M,G)}{M_1}) = \dfrac{V^*(N,H)}{N_1}$. Let $n \in \alpha(mM_1)$ where $m \in V^*(M,G)$, then $n \in N$ and for every $h_i \in H$ and $g_i \in \alpha^{-1}(h_i N_1)$, $i = 1, ..., r$, it holds that $v(h_1, ..., h_i n, ..., h_r)$
$v(h_1, ..., h_r)^{-1} = \beta(v(g_1, ..., g_i m, ..., g_r)v(g_1, ..., g_r)^{-1}) = 1$. So $n \in V^*(N,H)$ and thus $\alpha(\dfrac{V^*(M,G)}{M_1}) \subseteq \dfrac{V^*(N,H)}{N_1}$. A similar argument for α^{-1} yields the reverse inclusion which gives the assertion. □

Lemma 4. *Let* (α, β) *be an* ν*-isologism between* (G,M) *and* (H,N). *Then the followings hold.*

(a) *If* G_1 *is a subgroup of* G *with* $V^*(M,G) \subseteq G_1$ *and* $\alpha(\dfrac{G_1}{V^*(M,G)}) = \dfrac{H_1}{V^*(N,H)}$, *then*

$$(G_1, G_1 \cap M) \sim_\nu (H_1, H_1 \cap N);$$

(b) *If* M_1 *is a normal subgroup of* G *with* $M_1 \subseteq V(M,G)$, *then*

$$(\frac{G}{M_1}, \frac{M}{M_1}) \sim_\nu (\frac{H}{\beta(M_1)}, \frac{N}{\beta(M_1)}).$$

Proof. (a) Set $M_1 = V^*(G_1 \cap M, G)$ and $N_1 = V^*(H_1 \cap N, H)$. Then $M_1 \leq V^*(G_1 \cap M, G_1)$ and $N_1 \leq V^*(H_1 \cap N, H_1)$. Now define $\bar{\alpha} : \dfrac{G_1}{M_1} \to \dfrac{H_1}{N_1}$ and $\bar{\beta} : V(G_1 \cap M, G_1) \to V(H_1 \cap N, H_1)$ by $\bar{\alpha}(gM_1) = hN_1$, where $h \in \alpha(gZ_n(M,G)$ and $\bar{\beta}(x) = \beta(x)$. Then $\bar{\alpha}, \bar{\beta}$ are isomorphism and $\alpha(\dfrac{G_1 \cap M}{M_1}) = \dfrac{H_1 \cap N}{N_1}$. So by Lemma 3, $(G_1, G_1 \cap M) \sim_\nu (H_1, H_1 \cap N)$.

(b) Put $\bar{G} = \dfrac{G}{M_1}$, $\bar{M} = \dfrac{M}{M_1}$, $\bar{H} = \dfrac{H}{\beta(M_1)}$ and $\bar{N} = \dfrac{N}{\beta(M_1)}$. In addition we set $\bar{M_1} = \dfrac{V^*(M,G)M_1}{M_1}$, $\bar{N_1} = \dfrac{V^*(N,H)\beta(M_1)}{\beta(M_1)}$, then $\bar{M_1} \leq V^*(\bar{M}, \bar{G})$ and $\bar{N_1} \leq V^*(\bar{N}, \bar{H})$. Now we define $\bar{\alpha} : \dfrac{\bar{G}}{\bar{M_1}} \to \dfrac{\bar{H}}{\bar{N_1}}$ and $\bar{\beta} : V(\bar{M}, \bar{G}) \to V(\bar{N}, \bar{H})$ by $\bar{\alpha}(\bar{g}\bar{M_1}) = \bar{h}\bar{N_1}$, where $g \in G, h \in \alpha(gV^*(M,G)$ and $\bar{\beta}(\bar{x}) = \overline{\beta(x)}$. Then $\bar{\alpha}, \bar{\beta}$ are isomorphisms such that $\bar{\alpha}(\dfrac{\bar{M}}{\bar{M_1}}) = \dfrac{\bar{N}}{\bar{N_1}}$ and $\bar{\beta}(v(\bar{g_1}, ..., \bar{g_i}\bar{m}, ..., \bar{g_r})v(\bar{g_1}, ..., \bar{g_r})^{-1}) = v(\bar{h_1}, ..., \bar{h_i}\bar{n}, ..., \bar{h_r})v(\bar{h_1}, ..., \bar{h_r})^{-1})$, where $\bar{n} \in \bar{\alpha}(\bar{m}\bar{M_1})$, $\bar{h_i} \in \bar{\alpha}(\bar{g_i}\bar{M_1}), i = 1, ..., n$. Thus by Lemma 3, the result holds. □

Lemma 5. *Let* (G,M) *be a pair of groups. If* N *is a normal subgroup of* G *with* $N \leq M$ *and* H *is a subgroup of* G, *then* *(a)* $(H, H\cap M) \sim_\nu (HV^*(M,G), (H\cap M)V^*(M,G))$. *In particular if* $G = HV^*(M,G)$, *then* $(H, H\cap M) \sim_\nu (G, M)$. *Conversely, if* $\dfrac{H}{V^*(H\cap M, H)}$ *satisfies the descending chain condition on normal subgroups and* $(H, H\cap M) \sim_\nu (G, M)$, *then* $G = HV^*(M,G)$*;*

(b) $(\dfrac{G}{N}, \dfrac{M}{N}) \sim_\nu (\dfrac{G}{N\cap V(M,G)}, \dfrac{M}{N\cap V(M,G)})$. *In particular if* $N\cap V(M,G) = 1$, *then* $(G, M) \sim_\nu (\dfrac{G}{N}, \dfrac{M}{N})$. *Conversely, if* $V(M,G)$ *satisfies the ascending chain condition on normal subgroups and* $(G,M) \sim_\nu (\dfrac{G}{N}, \dfrac{M}{N})$, *then* $N\cap V(M,G) = 1$.

Proof. (a) It is clear that $V^*(H\cap M, G) \leq V^*(H\cap M, H)$ and the subgroup $V^*((H\cap M)V^*(M,G), HV^*(M,G))$ contains $V^*((H\cap M)V^*(M,G), G)$. We put $M_1 = V^*(H\cap M, G)$ and $N_1 = V^*((H\cap M)V^*(M,G), G)$ and define $\alpha : \dfrac{H}{M_1} \to \dfrac{HV^*(M,G)}{N_1}$ by $\alpha(hM_1) = hN_1$, $h\in H$. Then α is an isomorphism. In addition, $V(H\cap M, H) = V((H\cap M)V^*(M,G), HV^*(M,G))$ so by Lemma 3, $(H, H\cap M) \sim_\nu (HV^*(M,G)$,

$(H\cap M)V^*(M,G))$. Conversely, let $\dfrac{H}{V^*(H\cap M, H)}$ satisfies the descending chain condition on normal subgroups and (α, β) be an isologism between (G, M) and $(H, H\cap M)$. By the above we can assume that $V^*(M,G) \subseteq H$. Let H_1 be a subgroup of H such that $\alpha(\dfrac{H}{V^*(M,G)}) = \dfrac{H_1}{V^*(M\cap H, H)}$. Now by Lemma 4$(a)$, $(H_1, M\cap H_1) \sim_\nu (G, M)$ and $G = H$ if and only if $H = H_1$. Similarly, there exists a subgroup $H_2 \leq H_1$ contains $V^*(M\cap H, H)$ such that $H = H_1$ if and only if $H_1 = H_2$. Pursuing this process, we obtain a sequence $H_0 = H \geq H_1 \geq, ..., \geq V^*(M\cap H, H)$ such that $(G, M) \sim_\nu (H_i, M\cap H_i)$, $i = 1, 2, \ldots$. By assumption there exists an integer $r \geq 0$ such that $H_r = H_{r+1}$. So the result holds.

(b) It is easy to see that the maps $\alpha : \dfrac{\bar{G}}{V^*(\bar{M}, \bar{G})} \to \dfrac{\tilde{G}}{V^*(\tilde{M}, \tilde{G})}$ and $\beta : V(\bar{M}, \bar{G}) \to V(\tilde{M}, \tilde{G})$ given by $\alpha(\bar{g}V^*(\bar{M}, \bar{G})) = \tilde{g}V^*(\tilde{M}, \tilde{G})$ and $\beta(\bar{x}) = \tilde{x}$ respectively are both isomorphisms , where $\bar{G} = \dfrac{G}{N}$, $\bar{M} = \dfrac{M}{N}$, $\tilde{G} = \dfrac{G}{N\cap V(M,G)}$ and $\tilde{M} = \dfrac{M}{N\cap V(M,G)}$. So by Lemma 3, the result holds. Now assume that $V(M,G)$ satisfies the ascending chain condition on normal subgroups. Set $K = N\cap V(M,G)$. By the above there exists an isomorphism $\beta : V(M,G) \to V(\dfrac{M}{K}, \dfrac{G}{K})$. We consider a normal subgroup K_1 of $V(M,G)$ such that $\beta(K) = \dfrac{K_1}{K}$ and see $K = 1$ if and only if $K = K_1$. In a similar manner there exists a subgroup K_2 of $V(M,G)$ such that contains K_1 and $\beta(K_1) = \dfrac{K_2}{K_1}$. Again $K_2 = K_1$ if and only if $K = K_1$. Pursuing the above process and obtain a sequence of subgroups

$K_0 = K \trianglelefteq K_1 \trianglelefteq, ..., \trianglelefteq V(M,G)$. By assumption there exists an integer $s \geq 0$ such that $K_s = K_{s+1}$. Thus $N \cap V(M,G) = K = 1$. $\square$

Now, we are able to state and prove the main results of this section.

Theorem 2. *Let (G_1, M_1) and (G_2, M_2) be pairs of groups. Then $(G_1, M_1) \sim_\nu (G_2, M_2)$ if and only if there exists a pair (G, M) of groups and there exist normal subgroups N_1 and N_2 of G with $N_1 \subseteq M$, $N_2 \subseteq M$ such that $(G_1, M_1) \simeq (\frac{G}{N_1}, \frac{M}{N_1})$, $(G_2, M_2) \simeq (\frac{G}{N_2}, \frac{M}{N_2})$ and $(G_1, M_1) \sim_\nu (G, M) \sim_\nu (G_2, M_2)$.*

Proof. The 'if' part is trivial. Thus suppose that $(G_1, M_1) \sim_\nu (G_2, M_2)$ and (α, β) is a ν-isologism between them. Let G be a subgroup of $G_1 \times G_2$ given by

$$G = \{(g_1, g_2) \in G_1 \times G_2 | \alpha(g_1 V^*(M_1, G_1)) = g_2 V^*(M_2, G_2)\}$$

and $M = G \cap (M_1 \times M_2)$. Now set $N_1 = \{(1, n_2) | n_2 \in V^*(M_2, G_2)\}$ and $N_2 = \{(n_1, 1) | n_1 \in V^*(M_1, G_1)\}$. Then $N_i \trianglelefteq G$, $N_i \subseteq M$ and $(G_i, M_i) \simeq (\frac{G}{N_i}, \frac{M}{N_i})$, $(i = 1, 2)$. In addition $V(M, G) = \langle (g_1, \beta(g_1)) | g_1 \in V(M_1, G_1) \rangle$. So $N_i \cap V(M, G) = 1$, $(i = 1, 2)$ and by Lemma 5(b), $(G_1, M_1) \sim_\nu (G, M) \sim_\nu (G_2, M_2)$. $\square$

Theorem 3. *Let ν be a variety of groups and (α, β) is a ν-isologism between (G_1, M_1) and (G_2, M_2). Then there exists a pair (H, N) of groups with pairs of subgroups (H_1, N_1) and (H_2, N_2) such that $(\frac{G_1}{[G_1, V^*(M_1, G_1)]}, \frac{M_1}{[G_1, V^*(M_1, G_1)]}) \simeq (H_1, N_1)$, $(\frac{G_2}{\beta([G_1, V^*(M_1, G_1)] \cap V(M_1, G_1))}, \frac{M_2}{\beta([G_1, V^*(M_1, G_1)] \cap V(M_1, G_1))}) \simeq (H_2, N_2)$ and $(H_1, N_1) \sim_\nu (H, N) \sim_\nu (H_2, N_2)$.*

Proof. First put $X = \{(g_1, g_2) \in G_1 \times G_2 | \alpha(g_1 V^*(M_1, G_1)) = g_2 V^*(M_2, G_2)\}$ and $Y = X \cap (M_1 \times M_2)$, Then $V(Y, X) = \{(g_1, \beta(g_1)) | g_1 \in V(M_1, G_1)\}$. Now set $Z_1 = \{(1, n_2) | n_2 \in V^*(M_2, G_2)\}$ and $Z_2 = \{(n_1, 1) | n_1 \in V^*(M_1, G_1)\}$, therefore Z_i is a normal subgroup of X, $Z_i \leq Y$ and $Z_i \cap V(Y, X) = 1$ $(i = 1, 2)$. Set

$$G = \frac{X}{Z_1} \times \frac{X}{V(Y, X)} \text{ and } M = \frac{Y}{Z_1} \times \frac{Y}{V(Y, X)}.$$

Thus $(G, M) \sim_\nu (G_1, M_1) \sim_\nu (X, Y)$. As $Z_1 \cap V(X, Y) = 1$, X can be embedded in G by a monomorphism $\iota : X \to G$ define by $\iota(x) = (xZ_1, xV(Y, X))$. Let K be the normal closure of $\iota(Z_2)$ in G, that is

$$K = \iota(Z_2)[\iota(Z_2), G].$$

Let $g_1 \in G_1$ and choose $g_2 \in G_2$ such that $(g_1, g_2) \in X$ and define $f_1 : G_1 \to \frac{G}{K}$ by $f_1(g_1) = ((g_1, g_2)Z_1, (1, 1)V(Y, X))K$. Similarly, define $f_2 : G_2 \to \frac{G}{K}$ by $f_2(g_2) =$

$((g_1, g_2)Z_1, (g_1, g_2)V(Y,X))K$. Then f_1 and f_2 are well-defined homomorphism. We claim that

$$f_1(G_1)V^*(\frac{M}{K}, \frac{G}{K}) = \frac{G}{K} = f_2(G_2)V^*(\frac{M}{K}, \frac{G}{K}) \tag{1}$$

Lemma 1(g) shows that $V^*(M,G) = \dfrac{V^*(Y,X)}{Z_1} \times \dfrac{X}{V(Y,X)}$. So for every (g_1, g_2) and (h_1, h_2) in X,

$$\begin{aligned} ((g_1, g_2)Z_1, (h_1, h_2)V(Y,X))K = \\ ((g_1, g_2)Z_1, (1,1)V(Y,X))K \ \ ((1,1)Z_1, (h_1, h_2)Z(Y,X))K \\ \in f_1(G_1)\frac{V^*(M,G)K}{K} \subseteq f_1(G_1)V^*(\frac{M}{K}, \frac{G}{K}) \end{aligned}$$

and

$$\begin{aligned} ((g_1, g_2)Z_1, (h_1, h_2)V(Y,X))K = \\ ((g_1, g_2)Z_1, (g_1, g_2)V(Y,X))K \ \ ((1,1)Z_1, (g_1^{-1}h_1, g_2^{-1}h_2V(Y,X))K \\ \in f_2(G_2)\frac{V^*(M,G)K}{K} \subseteq f_2(G_2)V^*(\frac{M}{K}, \frac{G}{K}). \end{aligned}$$

Thus (1) is hold. Now we show that

(a) $ker(f_1) = [G_1, V^*(M_1, G_1)]$;

(b) $ker(f_2) = \beta(V(M_1, G_1) \cap [G_1, V^*(M_1, G_1)])$.

To prove (a), let $g_1 \in [G_1, V^*(M_1, G_1)]$, then $(g_1, 1) \in Z_2$. Hence

$$\begin{aligned} f_1(g_1) &= ((g_1, 1)Z_1, (1,1)V(Y,X))K \\ &= ((g_1, 1)Z_1, (g_1, 1)V(Y,X))K \ \ ((1,1)Z_1, (g_1^{-1}, 1)V(Y,X))K \\ &= K. \end{aligned}$$

So $g_1 \in ker(f_1)$.

Conversely, let $g_1 \in ker(f_1)$ and choose $g_2 \in G_2$ such that $(g_1, g_2) \in X$. So $((g_1, g_2)Z_1, (1,1)V(Y,X)) \in K$ and for some $n_1 \in V^*(M_1, G_1)$ and for some $c_1 \in [G_1, V^*(M_1, G_1)]$,

$$(g_1, g_2)Z_1, (1,1)V(Y,X)) = ((n_1, 1)Z_1, (n_1, 1)V(Y,X))((1,1)Z_1, (c_1, 1)V(Y,X)).$$

Hence $g_1 = n_1$ and $(n_1c_1, 1) \in V(Y,X)$. Thus $n_1 = c_1^{-1} \in [G_1, V^*(M_1, G_1)]$ and $g_1 \in [G_1, V^*(M_1, G_1)]$. This proves (a).

To prove (b), suppose $g_2 \in ker(f_2)$. Choose $g_1 \in G_1$ such that $(g_1, g_2) \in X$, thus $((g_1, g_2)Z_1, (g_1, g_2)V(Y,X)) \in K$ and for some $n_1 \in V^*(M_1, G_1)$ and $c_1 \in [G_1, V^*(M_1, G_1)]$,

$$((g_1, g_2)Z_1, (g_1, g_2)V(Y,X)) = ((n_1, 1)Z_1, (n_1c_1, 1)V(Y,X)).$$

Hence $g_1 = n_1$ and $(g_1c_1^{-1}n_1^{-1}, g_2) \in V(Y,X)$, so $g_2 = \beta(g_1c_1^{-1}n_1^{-1}) \in \beta(V(M_1, G_1))$. But as $c_1 \in [G_1, V^*(M_1, G_1)]$, it holds that

$$g_1c_1^{-1}n_1^{-1} = n_1c_1^{-1}n_1^{-1} \in [G_1, V^*(M_1, G_1)].$$

Thus $g_2 \in \beta(V(M_1, G_1) \cap [G_1, V^*(M_1, G_1)])$. Conversely, let $g_2 \in \beta(V(M_1, G_1) \cap [G_1, V^*(M_1, G_1)])$, then there exists $g_1 \in V(M_1, G_1) \cap [G_1, V^*(M_1, G_1)]$ such that $g_2 = \beta(g_1)$. Therefore $\alpha(g_1 V^*(M_1, G_1)) = \beta(g_1)(V^*(M_2, G_2) = g_2(V^*(M_2, G_2))$ and so $(g_1, g_2) \in X$. In particular $(g_1, g_2) \in V(Y, X)$ and we have

$$\begin{aligned}((g_1, g_2)Z_1, (g_1, g_2)V(Y, X)) &= ((g_1, 1)Z_1, (g_1, 1)V(Y, X))((1, g_2)Z_1, (1, g_2)V(Y, X))\\ &= ((g_1, 1)Z_1, (g_1, 1)V(Y, X))((1, 1)Z_1, (g_1^{-1}, 1)V(Y, X))\\ &\in K\end{aligned}$$

So $f_2(g_2) = \bar{1}$. This proves (b).

Now set $(H, N) = (\frac{G}{K}, \frac{M}{K})$ and $(H_i, N_i) = (f_i(G_i), f_i(M_i))$, $(i = 1, 2)$. Then

$$(\frac{G_1}{[G_1, V^*(M_1, G_1)]}, \frac{M_1}{[G_1, V^*(M_1, G_1)]}) \simeq (H_1, N_1),$$

$$(\frac{G_2}{\beta([G_1, V^*(M_1, G_1)] \cap V(M_1, G_1))}, \frac{M_2}{\beta([G_1, V^*(M_1, G_1)] \cap V(M_1, G_1))}) \simeq (H_2, N_2)$$

and $(H_1, N_1) \sim_\nu (H, N) \sim_\nu (H_2, N_2)$. □

4 Irreducible pairs of groups

In this section, we introduce the notion of subgroup irreducible and quotient irreducible pairs of groups. We use Zorn's lemma and show that every pair of groups is ν-isologic with a quotient irreducible pair of groups.

Definition 3. Let (G, M) be a pair of groups. If G contains no proper subgroup H satisfying $G = HV^*(M, G)$, then (G, M) is called *subgroup irreducible with respect to* ν-isologism. If the group G contains no normal subgroup N with $N \cap M \neq 1$ and $N \cap V(M, G) = 1$, then (G, M) is called *quotient irreducible with respect to* ν-isologism.

Theorem 4. *Let (G, M) be a pair of groups. If $V^*(M, G) \subseteq V(M, G)$, then (G, M) is subgroup irreducible and quotient irreducible with respect to ν-isologism.*

Proof. Let N be a normal subgroup of G such that $N \cap V(M, G) = 1$. We set $K = N \cap M$ and claim that $K = 1$. By Lemma 1(e), $V(K, G) \subseteq K \cap V(M, G) \subseteq N \cap V(M, G)$, thus $K = 1$ and (G, M) is quotient irreducible. Now, let H be a subgroup of G such that $G = HV^*(M, G)$. By Lemma 2(b), $V(M, G) = V(M \cap H, H)$ and hence

$$V^*(M, G) \subseteq V(M, G) = V(M \cap H, H) \subseteq M \cap H \subseteq H$$

Thus $G = H$ and (G, M) is subgroup irreducible. □

Let G be a group. The *Frattini* subgroup $\Phi(G)$ of G is defined as the intersection of all the maximal subgroups of G. If G has no maximal subgroups one defines

$\Phi(G) = G$. In any group G, the Frattini subgroup is equal to the set of all non-generator elements of G (see [8, Theorem 5.2.12]). If (G, M) is a pair of groups and N is a maximal subgroup of G, then either $V^*(M,G) \subseteq N$ or $V^*(M,G)N = G$. Lemma 2(b) shows that in each case $V(M,G) \cap V^*(M,G) \subseteq N$. So $V(M,G) \cap V^*(M,G) \subseteq \Phi(G)$.

Theorem 5. *Let (G, M) be a pair of groups. If (G, M) is subgroup irreducible with respect to ν-isologism, then $V^*(M,G) \leq \Phi(G)$. The converse holds if $\dfrac{V^*(M,G)}{V^*(M,G) \cap V(M,G)}$ is finitely generated.*

Proof. Let $g \in V^*(M,G)$. We claim that g is a non-generator of G. If $G = \langle g, X\rangle$, for some subset X of G and $H = \langle X\rangle$, then $G = HV^*(M,G)$. As (G, M) is subgroup irreducible, $G = H$. Thus g is a non-generator and so $g \in \Phi(G)$. Thus $V^*(M,G) \leq \Phi(G)$. Conversely, let $\dfrac{V^*(M,G)}{V^*(M,G) \cap V(M,G)}$ be finitely generated. Then there exist $x_1, ..., x_n \in V^*(M,G)$ such that $V^*(M,G) = \langle x_1, ...x_n\rangle(V^*(M,G) \cap V(M,G)$. If $G = HV^*(M,G)$ with $H \leq G$, then by Lemma 2(c), $V^*(M,G) \cap V(M,G) = V(M \cap H, H) \cap V^*(M \cap H, H)$. Since $x_1, ..., x_n$ are non-generator, we have

$$\begin{aligned} G = HV^*(M,G) &= H\langle x_1, ..., x_n\rangle(V^*(M,G) \cap V(M,G)) \\ &= H\langle x_1, ..., x_n\rangle(V^*(M \cap H, H) \cap V(M \cap H, H)) \\ &= H. \end{aligned}$$

So (G, M) is subgroup irreducible. □

Definition 4. Let (G, M) be a pair of groups, then subgroup generated by all the minimal normal subgroups N of G with $N \cap M \neq 1$ is called *socle* of (G, M).

Now we state the main results of this section.

Theorem 6. *If (G, M) quotient irreducible with respect to ν -isologism, then socle of (G, M) is contained in $V(M,G)$ and $\dfrac{Z(M,G)}{Z(M,G) \cap V(M,G)}$ is a torsion group. If ν is a nilpotent variety, then the converse holds.*

Proof. First let (G, M) be quotient irreducible with respect to ν-isologism. If N is a minimal normal subgroup of G with $N \cap M \neq 1$, then $N \cap V(M,G) \neq 1$ and as N is minimal subgroup, $N \subseteq V(M,G)$. Thus socle of (G, M) is contained in $V(M,G)$. If $\bar{x} \neq 1$ is an element of $\dfrac{Z(M,G)}{Z(M,G) \cap V(M,G)}$, set $T = \langle x\rangle$. As $T \subseteq Z(M,G)$, T is a normal subgroup of G, and $T \cap M \neq 1$. Therefore $T \cap V(M,G) \neq 1$. Hence there exists integer $k \geq 1$ such that $x^k \in V(M,G)$. So $\bar{x}^k = 1$ and $\bar{x}$ is a torsion element of $\dfrac{Z(M,G)}{Z(M,G) \cap V(M,G)}$. Conversely, let ν be nilpotent, $V(M,G)$ contains socle of (G, M) and $\dfrac{Z(M,G)}{Z(M,G) \cap V(M,G)}$ be a torsion group. Suppose that N is a normal subgroup of G such that $N \cap V(M,G) = 1$. Set $S = M \cap N$ and $T = S \cap Z(M,G)$. If $T \neq 1$, then

$$T \simeq \frac{T}{T \cap (Z(M,G) \cap V(M,G))} \simeq \frac{T(Z(M,G) \cap V(M,G))}{Z(M,G) \cap V(M,G)} \leq \frac{Z(M,G)}{Z(M,G) \cap V(M,G)}.$$

Thus T is a torsion group. Now choose $x \in T$, $x \neq 1$ with minimal order. Then $\langle x \rangle$ is a minimal normal subgroup of G. So by assumption $\langle x \rangle \subseteq V(M,G)$. This contradicts with $T \cap V(M,G) = 1$, therefore $T = 1$. So for any integer $n \geq 1$, $S \cap Z_n(M,G) = 1$ (see [2, Lemma 2.5]). In addition, Lemma 1 shows that $S \subseteq V^*(M,G)$. As ν is nilpotent, there exists an integer $k \geq 1$ such that $V^*(M,G) \subseteq Z_k(M,G)$, so $S \subseteq Z_k(M,G)$ and $M \cap N = S = 1$. Thus (G,M) is quotient irreducible with respect to ν-isologism. □

Theorem 7. *Let (G,M) be a pair of groups. Then there exists a normal subgroup N of G with $N \subseteq M$, such that $(G,M) \sim_\nu (\frac{G}{N}, \frac{M}{N})$ and $(\frac{G}{N}, \frac{M}{N})$ is quotient irreducible with respect to ν-isologism.*

Proof. Let $\mathcal{S} = \{N : N \trianglelefteq G, N \subseteq M \ \textit{and} \ N \cap V(M,G) = 1\}$. Then $\mathcal{S}$ is partially ordered by inclusion. If $\{N_\alpha\}_{\alpha \in I}$ is a chain in $\mathcal{S}$, then $\bigcup_{\alpha \in I} N_\alpha$ is an upper bound in $\mathcal{S}$. So by Zorn's lemma there exists a normal subgroup N in which maximal in $\mathcal{S}$. As $N \cap V(M,G) = 1$, it follows from Lemma 2(b) that $(G,M) \sim_\nu (\frac{G}{N}, \frac{M}{N})$. Let $\frac{T}{N}$ be a normal subgroup of $\frac{G}{N}$ such that $\frac{T}{N} \cap V(\frac{M}{N}, \frac{G}{N}) = 1$. We claim that $T \in \mathcal{S}$. It follows from Lemma 1(a) that $T \cap V(M,G) \trianglelefteq N$. As $N \cap V(M,G) = 1$, we have $T \cap V(M,G) = 1$ and so $T \in \mathcal{S}$. But N is maximal in $\mathcal{S}$ and therefore $T = N$. Hence $\frac{T}{N} = 1$ and $(\frac{G}{N}, \frac{M}{M})$ is quotient irreducible with respect to ν-isologism. □

5 An application to the Baer-invariant of a pair of groups

In this section, we use Lemma 5(b) and give some inequalities for the Baer-invariant of a pair of groups.

Let $1 \longrightarrow R \longrightarrow F \longrightarrow G \longrightarrow 1$ be a free presentation of the group G and let $N \simeq S/R$ for a suitable normal subgroup S of the free group F. Then the Baer-invariant of a pair (G,M) of groups with respect to the variety ν, denoted by ν $M(G,M)$, to be

$$\frac{R \cap V(S,F)}{V(R,F)}.$$

It is a routine exercise to check that $\nu\ M(G,M)$ is an abelian group and that it is independent of the choice of the free presentation of G. A variety ν is called a Schur-Baer variety if, for any group G for which the marginal factor group $\frac{G}{V^*(G)}$

is finite, then the verbal subgroup $V(G)$ is also finite. In [6], Moghaddam et al. proved that for finite groups G, $\nu\ M(G,M)$ is finite, with respect to the Schur-Baer variety ν. Therefore throughout this section we assume that ν is a variety of groups, which enjoys the Schur-Baer property .

Lemma 6. *Let N be a normal subgroup of G contained in M, then the following sequence is exact.*

$$1 \longrightarrow \nu M(G,N) \longrightarrow \nu M(G,M) \longrightarrow \nu M(\frac{G}{N},\frac{M}{N}) \longrightarrow \frac{N\cap V(M,G)}{V(N,G)} \longrightarrow 1.$$

Proof. See [5]. □

Corollary 1. *Let (G,M) be a pair of finite groups and N a normal subgroup of G contained in M. If $(G,M)\sim_\nu (\frac{G}{N},\frac{M}{N})$, then*

(a) $|\nu M(G,M)|=|\nu\ M(G,N)||\nu\ M(\frac{G}{N},\frac{M}{N})|$;

(b) $d(\nu M(\frac{G}{N},\frac{M}{N}))\le d(\nu M(G,M))$;

(c) $e(\nu M(\frac{G}{N},\frac{M}{N}))$ *divides* $e(\ \nu M(G,M))$,

where $e(X)$ and $d(X)$ are the exponent and the minimal number of generators of the group X, respectively.

Proof. Lemma 5(b) shows that $N\cap V(M,G)=1$, so by Lemma 6 the following sequence is exact

$$1 \longrightarrow \nu M(G,N) \longrightarrow \nu M(G,M) \longrightarrow \nu M(\frac{G}{N},\frac{M}{N}) \longrightarrow 1.$$

Now the results are easily deduced. □

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