On the ratio of directed lengths on the plane with generalized absolute value metric and related properties

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Abstract

In this paper, we show that a point of division divides a related line segment in the same ratio on the plane with generalized absolute value metric and Euclidean plane. Then the coordinates of the division point can be determined by the same formula as in the Euclidean plane. In the latter parts of the work, we give Ceva's and Menelaus'es theorems and the theorem of directed lines on the plane with generalized absolute value metric.

1 Introduction

H. Minkowski [22] published a whole family of metrics providing new insight into the study of plane geometry at the beginning of last century. Later, taxicab plane geometry was introduced by Menger, and developed by Krause, using the metric $d_T(K, L) = |x_1 - x_2| + |y_1 - y_2|$ instead of the well-known Euclidean metric $d_E(K, L) = ((x_1 - x_2)^2 + (y_1 - y_2)^2)^{1/2}$ for the distance between any two points $K = (x_1, y_1)$ and $L = (x_2, y_2)$ in the Cartesian coordinate plane (see [20], [17]). Recently, the plane geometry with the generalized absolute value metric d_G was introduced by Kaya et al. [13] and Akca et al. [3]. In $\mathbb{R}^2$, the d_G-distances between K and L are defined by

$$
\begin{aligned}
d_G(K, L) &= d_G((x_1, y_1), (x_2, y_2)) \\
&= k_1 \max\{|x_1 - x_2|, |y_1 - y_2|\} + k_2 \min\{|x_1 - x_2|, |y_1 - y_2|\}
\end{aligned} \tag{1}
$$

for all $k_1, k_2 \in \mathbb{R}$, $k_1 \geq k_2 \geq 0$, $k_1 \neq 0$. According to the definition of d_G-distances, the shortest way between two points K and L is the union of a vertical or a horizontal line segment and a line segment with the slope $\pm 2k_1 k_2/(k_1^2 - k_2^2)$. The family

of distances, d_G, which includes Chinese checkers, taxicab distance and maximum distance as special cases.

A *metric geometry* consists of a set P whose elements are called *points*, together with a collection L of non-empty subsets of P, called *lines*, and a distance function d, such that

1) Every two distinct points in P lie on a unique line,

2) There exist three points in P, which do not lie all on one line,

3) There exists a bijective function $f : l \to \mathbb{R}$ for all lines in L such that $|f(P) - f(Q)| = d(P,Q)$ for each pair of points P and Q on l.

A metric geometry defined above is denoted by $\{P, L, d\}$. However, if a metric geometry satisfies the plane separation axiom below, and it has an angle measure function m, then it is called *protractor geometry* and denoted by $\{P, L, d, m\}$.

4) For every l in L, there are two subsets H_1 and H_2 of P (called *half planes* determined by l) such that

(i) $H_1 \cup H_2 = P - l$ (P with l removed),

(ii) H_1 and H_2 are disjoint and each is convex,

(iii) If $A \in H_1$ and $B \in H_2$, then $[AB] \cap l = \emptyset$.

If L_E is the set of all lines in the Cartesian coordinate plane, and m_E is the standard angle measure function in the Euclidean plane, then $\{\mathbb{R}^2, L_E, d_C, m_E\}$, called the plane with the generalized absolute value metric is a model of protractor geometry (This can be shown easily: the proof is similar to that of taxicab plane; refer to [21] or [8] to see that the taxicab plane is a model of protractor geometry). The plane with the generalized absolute value metric is also in the class of non-Euclidean geometries since it fails to satisfy the side-angle-side axiom. However, the plane with the generalized absolute value metric is almost the same as Euclidean plane $\{\mathbb{R}^2, L_E, d_C, m_E\}$ since the points are the same, the lines are the same and the angles are measured in the same way. Since the plane with the generalized absolute value metric has distance function different from that in the Euclidean plane, it is interesting to study on the plane with the generalized absolute value metric of topics that include the distance concept in the Euclidean plane.([1], [2], [4], [5], [6], [7], [9], [12], [10], [11], [13], [14], [16], [23], [24], [25], [19], [26], [27], [28], [29]) These topics are division point, directed lengths, ratio of directed lengths, Menelaus'es Theorem, Ceva's Theorem and the theorem of directed lines. In this paper, GAM is the abbreviation for the plane geometry with the generalized absolute value metric.

2 On the ratio of directed lengths on the plane with generalized absolute value metric

Let A and B be any two points on a directed straight line l. We define *directed* GAM length of the line segment AB as follows:

$$d_G[AB] = \begin{cases} d_G(A,B), & \text{if } AB \text{ and } l \text{ have the same direction} \\ -d_G(A,B), & \text{if } AB \text{ and } l \text{ have the opposite direction.} \end{cases}$$

Thus, $d_G[AB] = -d_G[BA]$. Clearly, *directed length* in the Euclidean plane can be defined in a similar way. That is

$$d_E[AB] = \begin{cases} d_E(A,B), & if \ AB \text{ and } l \text{ have the same direction} \\ -d_E(A,B), & if \ AB \text{ and } l \text{ have the opposite direction.} \end{cases}$$

If P, Q, R are points on a same directed line and R is between points P and Q, we denote this by PRQ. If PRQ, then R divides the line segment PQ *internally* and the ratio of the directed GAM lengths is a positive real number, that is

$$\frac{d_G[PR]}{d_G[RQ]} = \delta > 0.$$

If PQR or RPQ then R divides the line segments PQ *externally*, $d_G[PR]/d_G[RQ] = \delta < 0$, that is, the line segments PR and RQ have opposite directions. In both cases R called the *division point* which divides the line segment PQ in ratio δ. It's obvious, $R \neq Q$. $R = P \Leftrightarrow \delta = 0$ and (R is at infinity $\Leftrightarrow \delta = -1$).

Let R and R' be two points such that R divides a given line segment PQ internally and R' divides PQ externally in the same proportion though with opposite signs. Thus, the ratio of the directed lengths, $d_G[PR]/d_G[RQ] = -d_G[PR']/d_G[R'Q]$ is the same positive number δ.

Lemma 1. *For any two points K and L in the Cartesian plane that do not lie on a vertical line, if m is the slope of the line through K and L, then*

$$\delta(m) = \frac{\sqrt{1+m^2}}{k_1 \max\{1, |m|\} + k_2 \min\{1, |m|\}}. \tag{2}$$

If K and L lie on a vertical line, then by definition, $d_E(K,L) = \delta(m) d_G(K,L)$.

Proof. For any two points $K = (x_1, y_1)$ and $L = (x_2, y_2)$ with $x_1 \neq x_2$; then $m = (y_2 - y_1)/(x_2 - x_1)$. Equation (2) is derived by a straightforward calculation with m and the coordinate definitions of $d_E(K,L)$ and $d_G(K,L)$ given in section 1. $\square$

Theorem 1. *For any two distinct points $K_1 = (x_1, y_1)$ and $K_2 = (x_2, y_2)$ in the analytical plane, if $L = (x, y)$ is a point on the line passing through K_1 and K_2, then*

$$d_G[K_1 L]/d_G[LK_2] = d_E[K_1 L]/d_E[LK_2].$$

That is, the ratios of the Euclidean and GAM directed lengths are the same.

Proof. If $L = K_1$ then both ratios are equal to 0. If L is at infinity then both ratios are equal to -1. Therefore without loss of generality, let $K_1 \neq L \neq K_2$. Then it's enough to show $d_G(K_1, L)/d_G(L, K_2) = d_E(K_1, L)/d_E(L, K_2)$. It's clear by Lemma 1. This completes proof.   □

The following corollary shows how one can find the coordinates of the division point which divides the line segment joining two given points in a given ratio.

Corollary 1. *For any two distinct points $K_1 = (x_1, y_1)$ and $K_2 = (x_2, y_2)$ in the GAM plane, if $L = (x, y)$ divides the line segment $K_1 K_2$, in ratio δ then,*

$$x = \frac{x_1 + \delta x_2}{1 + \delta} \quad , \quad y = \frac{y_1 + \delta y_2}{1 + \delta}; \qquad \delta \in \mathbb{R}, \quad \delta \neq -1$$

as in the Euclidean plane.

Proof. We would rather give a direct proof even though the Corollary follows from Theorem 1. The given formula is obvious when $\delta = 0$ or $\delta = -1$. If $\delta \neq 0, -1$ and L divides the line segment $K_1 K_2$ in ratio δ, we have $|d_G[K_1 L_1]/d_G[L_1 K_2]| = |\delta|$. That is

$$\frac{k_1 \max\{|x_1 - x|, |y_1 - y|\} + k_2 \max\{|x_1 - x|, |y_1 - y|\}}{k_1 \max\{|x - x_2|, |y - y_2|\} + k_2 \max\{|x - x_2|, |y - y_2|\}} = |\delta|. \qquad (3)$$

Since $K_1 \neq K_2$,

  Case (i) If $|x_1 - x| \geq |y_1 - y|$ then using the definition of d_G distance we get

$$|\delta| = |\delta| \left(\frac{k_1 |x_1 - x_2| + k_2 |y_1 - y_2|}{k_1 |x_1 - x_2| + k_2 |y_1 - y_2|} \right) = \frac{k_1 |\delta x_1 - \delta x_2| + k_2 |\delta y_1 - \delta y_2|}{k_1 |x_1 - x_2| + k_2 |y_1 - y_2|}.$$

Adding $x_1 - x_1$ and $y_1 - y_1$ to the first and second summands in the numerator and similarly $\delta x_2 - \delta x_2$ and $\delta y_2 - \delta y_2$ in the denominator respectively, one obtains

$$|\delta| = \frac{k_1 |\delta x_1 + x_1 - x_1 - \delta x_2| + k_2 |\delta y_1 + y_1 - y_1 - \delta y_2|}{k_1 |x_1 + \delta x_2 - \delta x_2 - x_2| + k_2 |y_1 + \delta y_2 - \delta y_2 - y_2|}.$$

Multiplying the numerator and the denominator of the right side of the last statement by $1/|1 + \delta|$, one gets

$$|\delta| \;=\; \frac{k_1 \left|\frac{\delta x_1 + x_1 - x_1 - \delta x_2}{1+\delta}\right| + k_2 \left|\frac{\delta y_1 + y_1 - y_1 - \delta y_2}{1+\delta}\right|}{k_1 \left|\frac{x_1 + \delta x_2 - \delta x_2 - x_2}{1+\delta}\right| + k_2 \left|\frac{y_1 + \delta y_2 - \delta y_2 - y_2}{1+\delta}\right|}$$

$$=\; \frac{k_1 \left|\frac{(1+\delta)x_1}{1+\delta} - \frac{x_1 + \delta x_2}{1+\delta}\right| + k_2 \left|\frac{(1+\delta)y_1}{1+\delta} - \frac{y_1 + \delta y_2}{1+\delta}\right|}{k_1 \left|\frac{x_1 + \delta x_2}{1+\delta} - \frac{(1+\delta)x_2}{1+\delta}\right| + k_2 \left|\frac{y_1 + \delta y_2}{1+\delta} - \frac{(1+\delta)y_2}{1+\delta}\right|}$$

$$=\; \frac{k_1 \left|x_1 - \frac{x_1 + \delta x_2}{1+\delta}\right| + k_2 \left|y_1 - \frac{y_1 + \delta y_2}{1+\delta}\right|}{k_1 \left|\frac{x_1 + \delta x_2}{1+\delta} - x_2\right| + k_2 \left|\frac{y_1 + \delta y_2}{1+\delta} - y_2\right|} \;=\; \frac{k_1 |x_1 - x| + k_2 |y_1 - y|}{k_1 |x_2 - x| + k_2 |y_2 - y|}.$$

Comparing this result with equation (2) we obtain

$$x = \frac{x_1 + \delta x_2}{1 + \delta}, \quad y = \frac{y_1 + \delta y_2}{1 + \delta}.$$

Case (ii) If $|y_1 - y| < |x_1 - x|$ then it can be proved similarly case (i). $\qquad\square$

3   Theorems of Menelaus and Ceva in the GAM plane

In this section, we study the GAM version of the Theorems of Menelaus and Ceva. Indeed, the validity of these theorems is obvious from the Theorem 1, but we prefer to state and give partial proofs for them.

Theorem 2. (Menelaus'es Theorem.) *Let $\{K_1, K_2, K_3\}$ be a triangle and L_1, L_2, L_3 be on the lines that contain the sides $K_1 K_2, K_2 K_3, K_3 K_1$ respectively, in the GAM plane. If L_1, L_2, L_3 are collinear, then*

$$\frac{d_G[K_1 L_1]}{d_G[L_1 K_2]} \cdot \frac{d_G[K_2 L_2]}{d_G[L_2 K_3]} \cdot \frac{d_G[K_3 L_3]}{d_G[L_3 K_1]} = -1$$

where none of L_1, L_2, L_3 coincide with any of K_1, K_2, K_3.

Proof. According to the positions of points K_1, K_2, K_3 and L_1, L_2, L_3, there are several possible cases. We give a proof of the theorem only in the following special case.

Let $K_i = (x_i, y_i)$, $i = 1, 2, 3$ and $x_i \neq y_{i+1}$ and let L_1, L_2, L_3 be on a line l given by $y = mx + k$ such that $L_i = l \wedge K_i K_{i+1(mod3)}$ and l is not parallel to the line $K_i K_{i+1}$, for $i = 1, 2, 3$ (see Figure 1). Clearly $mx_i - y_i + k \neq 0$ since $K_i \neq L_j$ for $i, j = 1, 2, 3$ and $m \neq (y_{i+1} - y_i)(x_{i+1} - x_i)^{-1}$. The equation of the line $K_i K_{i+1}$ is given by

$$y = \frac{(y_{i+1} - y_i)}{(x_{i+1} - x_i)} x - \frac{(x_i y_{i+1} - x_{i+1} y_i)}{(x_{i+1} - x_i)}.$$

It follows from a simple calculation that

$$L_i = \left(\frac{x_i y_{i+1} - x_{i+1} y_i - kx_i + kx_{i+1}}{mx_i - mx_{i+1} - y_i + y_{i+1}}, \frac{mx_i y_{i+1} - mx_{i+1} y_i - ky_i + ky_{i+1}}{mx_i - mx_{i+1} - y_i + y_{i+1}} \right)$$

$$= (u, v).$$

Now let us find $\frac{d_G[K_i L_i]}{d_G[L_i K_{i+1}]}$. $\frac{d_G[K_1 L_1]}{d_G[L_1 K_2]} = -\frac{d_G(K_1, L_1)}{d_G(L_1, K_2)}$. Since K_1, K_2 and L_1 are collinear,
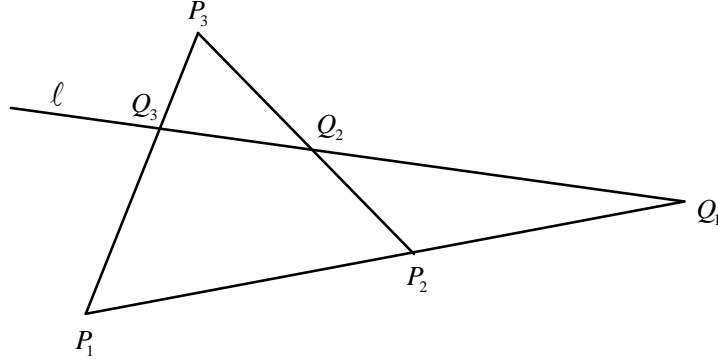

Figure 1:

Case (*i*) If $|x_1 - u| \geq |y_1 - v|$ then using the definition of d_G- distance one gets

$$= -\frac{k_1 \left| x_1 - \frac{x_1 y_2 - x_2 y_1 - kx_1 + kx_2}{mx_1 - mx_2 - y_1 + y_2} \right| + k_2 \left| y_1 - \frac{mx_1 y_2 - mx_2 y_1 - ky_1 + ky_2}{mx_1 - mx_2 - y_1 + y_2} \right|}{k_1 \left| \frac{x_1 y_2 - x_2 y_1 - kx_1 + kx_2}{mx_1 - mx_2 - y_1 + y_2} - x_2 \right| + k_2 \left| y_1 \frac{mx_1 y_2 - mx_2 y_1 - ky_1 + ky_2}{mx_1 - mx_2 - y_1 + y_2} - y_2 \right|}$$

$$= -\frac{k_1 \left| x_1 (mx_1 - y_1 + k) - x_2 (mx_1 - y_1 + k) \right| + k_2 \left| y_1 (mx_1 - y_1 + k) - y_2 (mx_1 - y_1 + k) \right|}{k_1 \left| x_2 (mx_2 - y_2 + k) - x_1 (mx_2 - y_2 + k) \right| + k_2 \left| y_2 (mx_2 - y_2 + k) - y_1 (mx_2 - y_2 + k) \right|}$$

$$= -\frac{|mx_1 - y_1 + k| (k_1 |x_1 - x_2| + k_2 |y_1 - y_2|)}{|mx_2 - y_2 + k| (k_1 |x_2 - x_1| + k_2 |y_2 - y_1|)}$$

$$= -\frac{|mx_1 - y_1 + k|}{|mx_2 - y_2 + k|}.$$

Similarly,

$$\frac{d_G[K_2 L_2]}{d_G[L_2 K_3]} = \frac{|mx_2 - y_2 + k|}{|mx_3 - y_3 + k|}$$

and

$$\frac{d_G[K_3 L_3]}{d_G[L_3 PK_1]} = \frac{|mx_3 - y_3 + k|}{|mx_1 - y_1 + k|}$$

and consequently,

$$\frac{d_G[K_i L_i]}{d_G[L_i K_{i+1}]} = s \frac{|mx_i - y_i + k|}{|mx_{i+1} - y_{i+1} + k|}, \qquad s = \begin{cases} -1, & \text{if } i = 1 \\ 1, & \text{if } i = 2, 3. \end{cases}$$

Now , it can be easily computed that

$$\prod_{i=1}^{3}(d_G[K_iL_i]/d_G[L_iK_{i+1}]) = -1.$$

Case (ii) If $|x_1 - u| < |y_1 - v|$ then it can be proved similarly case (i).   □

Theorem 3. (Converse of Menelaus'es Theorem.) *Let $\{K_1, K_2, K_3\}$ be a triangle and L_1, L_2, L_3 be three points on the lines that contain the sides K_1K_2, K_2K_3, K_3K_1 respectively, in the GAM plane. If*

$$\frac{d_G[K_1L_1]}{d_G[L_1K_2]} \cdot \frac{d_G[K_2L_2]}{d_G[L_2K_3]} \cdot \frac{d_G[K_3L_3]}{d_G[L_3K_1]} = -1,$$

then L_1, L_2, L_3 are collinear. Note that none of L_1, L_2, L_3 coincide with any of K_1, K_2, K_3.

Theorem 4. (Ceva's Theorem.) *Let $\{K_1, K_2, K_3\}$ be a triangle and lines l_1, l_2, l_3 pass through the vertices K_1, K_2, K_3, respectively and intersect lines containing the opposite sides at points L_1, L_2, L_3. The lines l_1, l_2, l_3 are concurrent (or parallel) if and only if*

$$\frac{d_G[K_1L_3]}{d_G[L_3K_2]} \cdot \frac{d_G[K_2L_1]}{d_G[L_1K_3]} \cdot \frac{d_G[K_3L_2]}{d_G[L_2K_1]} = 1.$$

Note that none of L_1, L_2, L_3 are K_1, K_2, K_3.

4   Theorems of directed lines (Strahlensätze)

In general, the axiom congruence and consequently properties of similarity for tringles are not valid in the GAM plane. But, it follows from Theorem 1 that the following *directed line theorem* [18] is valid in it.

Theorem 5. *Let a pencil of lines be intersected by a family of parallel lines in the GAM plane (see Figure 2)*
(i) The ratios of directed lengths of the corresponding segments on the lines belonging to the pencil are the same. For example

$$
\begin{aligned}
d_G[SP] : d_G[SQ] : d_G[SR] &= d_G[SP_1] : d_G[SQ_1] : d_G[SR_1] \\
&= d_G[SP_2] : d_G[SQ_2] : d_G[SR_2]
\end{aligned}
$$

or

$$d_G[SP_1] : d_G[SQ_1] = d_G[P_1P_2] : d_G[Q_1Q_2].$$

(*ii*) *The ratios of directed lengths of line segments on the parallel lines and corresponding segments on the lines belonging to the pencil, which are measured from the vertex, are the same. For example*

$$d_G[RQ] : d_G[R_1Q_1] : d_G[R_2Q_2] = d_G[SR] : d_G[SR_1] : d_G[SR_2]$$
$$= d_G[SQ] : d_G[SQ_1] : d_G[SQ_2]$$

or

$$d_g[PQ] : d_g[P_1Q_1] : d_g[P_2Q_2] = d_g[SP] : d_g[SP_1] : d_g[SP_2]$$
$$= d_g[SQ] : d_g[SQ_1] : d_g[SQ_2].$$

(*iii*) *The ratios of directed lengths of the corresponding segments on the parallel lines are the same. That is*

$$d_G[PQ] : d_G[QR] = d_G[P_1Q_1] : d_G[Q_1R_1] = d_G[P_2Q_2] : d_G[Q_2R_2].$$

Notice that here $p : q : r = p_1 : q_1 : r_1$ if and only if $p/p_1 = q/q_1 = r/r_1$.
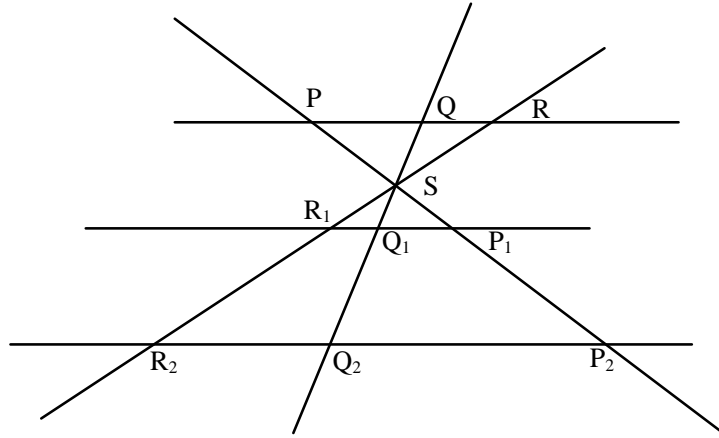

Figure 2:

Acknowledgements

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