

# Weakly linear systems of fuzzy relation inequalities and their applications: A brief survey

Jelena Ignjatović<sup>a</sup>, Miroslav Ćirić<sup>a</sup>

<sup>a</sup>University of Niš, Faculty of Sciences and Mathematics, Višegradska 33, 18000 Niš, Serbia

**Abstract.** Weakly linear systems of fuzzy relation inequalities and equations have recently emerged from research in the theory of fuzzy automata. From the general aspect of the theory of fuzzy relation equations and inequalities homogeneous and heterogeneous weakly linear systems have been discussed in two recent papers. Here we give a brief overview of the main results from these two papers, as well as from a series of papers on applications of weakly linear systems in the state reduction of fuzzy automata, the study of simulation, bisimulation and equivalence of fuzzy automata, and in the social network analysis. Especially, we present algorithms for computing the greatest solutions to weakly linear systems.

## 1. Introduction

Systems of fuzzy relation equations and inequalities were first studied by Sanchez, who used them in medical research (cf. [122–125]). Later they found a much wider field of application, and nowadays they are used in fuzzy control, discrete dynamic systems, knowledge engineering, identification of fuzzy systems, prediction of fuzzy systems, decision-making, fuzzy information retrieval, fuzzy pattern recognition, image compression and reconstruction, and in many other areas (cf., e.g., [46, 49, 55, 56, 88, 110, 112]).

Most frequently studied systems were the ones that consist of equations and inequalities with one side containing the composition of an unknown fuzzy relation and a given fuzzy relation or fuzzy set, while the other side contains only another or the same given fuzzy relation or fuzzy set. Such systems are called *linear systems*. Solvability and methods for computing the greatest solutions to linear systems of fuzzy relation equations and inequalities were first studied in the above mentioned papers by Sanchez, who discussed linear systems over the Gödel structure. Later, linear systems over more general structures of truth values were investigated, including those over complete residuated lattices (cf., e.g., [37, 46, 87, 113, 115–117]).

More complex non-linear systems of fuzzy relation inequalities and equations, called *weakly linear*, have been recently introduced and studied in [76, 78]. Basically, weakly linear systems discussed in [76] consist of inequalities and equations of the form  $A_i \circ X \bowtie X \circ A_i$  and  $X \leq M$ , where  $A_i$  ( $i \in I$ ) and  $M$  are given fuzzy relations on a set  $U$ ,  $X$  is an unknown fuzzy relation on  $U$ ,  $\circ$  denotes the composition of fuzzy relations, and  $\bowtie$  is one of  $\leq$ ,  $\geq$  and  $=$ . Besides, these systems can also include additional inequalities and equations of the form  $A_i \circ X^{-1} \bowtie X^{-1} \circ A_i$  and  $X^{-1} \leq M$ , where  $X^{-1}$  denotes the converse (inverse, transpose) relation of  $X$ .

---

2010 *Mathematics Subject Classification*. Primary 03E72; Secondary 03G25, 68Q45, 68Q85, 91D30

*Keywords*. Fuzzy relations; fuzzy relational systems; fuzzy relation inequalities; fuzzy relation equations; matrix inequalities; residuals of fuzzy relations; fuzzy equivalence relations; complete residuated lattices; post-fixed points

Received: 11 June 2011; Accepted: 26 July 2011

Communicated by Stojan Bogdanović

Research supported by Ministry Education and Science, Republic of Serbia, Grant No. 174013

*Email addresses*: [jelena.ignjatovic@pmf.edu.rs](mailto:jelena.ignjatovic@pmf.edu.rs) (Jelena Ignjatović), [miroslav.ciric@pmf.edu.rs](mailto:miroslav.ciric@pmf.edu.rs) (Miroslav Ćirić)

Such weakly linear systems, which include only fuzzy relations on a single set, are called *homogeneous*. On the other hand, *heterogeneous weakly linear systems*, studied in [78], include fuzzy relations on two possible different sets and an unknown is a fuzzy relation between these two sets. Two basic types of heterogeneous weakly linear systems are systems of the form  $X^{-1} \circ A_i \leq B_i \circ X^{-1}$  ( $i \in I$ ),  $X \leq N$ , or  $A_i \circ X \leq X \circ B_i$  ( $i \in I$ ),  $X \leq N$ , where  $A_i$  and  $B_i$  ( $i \in I$ ) are respectively given fuzzy relations on non-empty sets  $U$  and  $V$ ,  $N$  is a given fuzzy relation between  $U$  and  $V$ , and  $X$  is an unknown fuzzy relation between  $U$  and  $V$ . The remaining four types of heterogeneous weakly linear systems are obtained by combining the previous two types of systems (for  $X$  and  $X^{-1}$ ).

Both homogeneous and heterogeneous weakly linear systems have their origins in the theory of fuzzy automata. Homogeneous weakly linear systems emerged from the research aimed at the reduction of the number of states of fuzzy finite automata, carried out in [40, 41, 129], whereas heterogeneous weakly linear systems turned up from the study of simulation, bisimulation and equivalence of fuzzy automata, conducted in [38, 39]. In addition, weakly linear systems play an important role in the social network analysis, as will be shown in Section 7.

From a general viewpoint, weakly linear systems have been discussed in [76, 78]. There has been proved that every weakly linear system (either homogeneous or heterogeneous), with a complete residuated lattice as the underlying structure of truth values, has the greatest solution, and an algorithm has been provided for computing this greatest solution. Incidentally, the greatest solution may be the empty relation, and the algorithm detects this situation outputting the empty relation. The mentioned algorithm is based on the computing of the greatest post-fixed point, contained in a given fuzzy relation, of an isotone function on the lattice of fuzzy relations. The algorithm represents an iterative procedure whose each single step can be viewed as solving a particular linear system, and for this reason these systems were called weakly linear. This iterative procedure terminates in a finite number of steps whenever the underlying complete residuated lattice is locally finite, for example, when dealing with Boolean or Gödel structure. Otherwise, some sufficient conditions under which the procedure ends in a finite number of steps have been determined. If the underlying complete residuated lattice satisfies infinite distributive laws for the supremum and multiplication over infimum, for example, when dealing with a structure defined by a continuous t-norm on the real unit interval  $[0, 1]$  (an BL-algebra on  $[0, 1]$ ), the greatest solution can be obtained as the infimum of fuzzy relations outputted after each single step of the iterative procedure.

In cases when the mentioned procedure fails to terminate in a finite number of steps, it is reasonable to search for the greatest crisp solutions to the system. By modifying the original algorithm, one obtains an algorithm for computing the greatest crisp solution to the system, which terminates in a finite number of steps independently on the properties of the underlying complete residuated lattice, and works even for more general structures of truth values. However, there are examples of weakly linear systems having non-trivial fuzzy solutions, but having no non-trivial crisp solutions.

It is worth noting that there is a very nice relationship between heterogeneous and homogeneous weakly linear systems (cf. [78]). Namely, if a fuzzy relation  $R$  is a solution to a heterogeneous weakly linear system, then  $R \circ R^{-1}$  and  $R^{-1} \circ R$  are solutions to two related homogeneous weakly linear systems. In addition, a uniform fuzzy relation  $R$  is a solution to a heterogeneous weakly linear system if and only if its kernel and co-kernel are solutions to related homogeneous weakly linear systems and there is a special isomorphism (induced by  $R$ ) between the corresponding quotient fuzzy relational systems.

The structure of the paper is as follows. In Section 2 we introduce basic notions and notation concerning fuzzy sets, fuzzy relations, uniform fuzzy relations, residuated functions, and residuated semigroups and semimodules. In Section 3 we present the main results on linear systems of fuzzy relational inequalities and equations. Section 4 contains definitions of homogeneous and heterogeneous weakly linear systems and results describing their basic properties. Especially, theorems on the existence of the greatest solutions and equivalent forms of representation of these systems are shown. Section 5 presents algorithms for computing the greatest solutions to weakly linear systems. In Section 6 we deal with quotient fuzzy relational systems and establish relationships between solutions to heterogeneous and homogeneous weakly linear systems. Finally, in Section 7 we show applications of weakly linear systems in the theory of fuzzy automata, especially in the state reduction and the study of simulation, bisimulation and equivalence of fuzzy automata, as well as in the social network analysis.

## 2. Preliminaries

In this section we introduce notions and notation needed in the future work. We introduce basic concepts concerning fuzzy sets, fuzzy relations, uniform fuzzy relations, residuated functions, and residuated semigroups and semimodules.

### 2.1. Fuzzy sets

The terminology and basic notions in this section are according to [9, 10, 56, 88].

A *residuated lattice* is an algebra  $\mathcal{L} = (L, \wedge, \vee, \otimes, \rightarrow, 0, 1)$  such that

(L1)  $(L, \wedge, \vee, 0, 1)$  is a lattice with the least element 0 and the greatest element 1,

(L2)  $(L, \otimes, 1)$  is a commutative monoid with the unit 1,

(L3)  $\otimes$  and  $\rightarrow$  form an *adjoint pair*, i.e., they satisfy the *adjunction property*: for all  $x, y, z \in L$ ,

$$x \otimes y \leq z \Leftrightarrow x \leq y \rightarrow z. \tag{1}$$

Moreover,  $\mathcal{L}$  is called a *complete residuated lattice* if it satisfies (L1), (L3), and

(L2')  $(L, \wedge, \vee, 0, 1)$  is a complete lattice with the least element 0 and the greatest element 1.

The operations  $\otimes$  (called *multiplication*) and  $\rightarrow$  (called *residuum*) are intended for modeling the conjunction and implication of the corresponding logical calculus, and supremum ( $\vee$ ) and infimum ( $\wedge$ ) are intended for modeling of the existential and general quantifier, respectively. An operation  $\leftrightarrow$  defined by

$$x \leftrightarrow y = (x \rightarrow y) \wedge (y \rightarrow x), \tag{2}$$

called *biresiduum* (or *biimplication*), is used for modeling the equivalence of truth values. Emphasizing their monoidal structure, in some sources residuated lattices are called integral, commutative, residuated  $\ell$ -monoids [71].

If  $\mathcal{L}$  is a complete residuated lattice, then for all  $x, y, z \in L$  the following holds:

$$x \leq y \text{ implies } x \otimes z \leq y \otimes z, \tag{3}$$

For other properties of complete residuated lattices we refer to [9, 10].

The most studied and applied structures of truth values, defined on the real unit interval  $[0, 1]$  with  $x \wedge y = \min(x, y)$  and  $x \vee y = \max(x, y)$ , are the *Lukasiewicz structure* (where  $x \otimes y = \max(x + y - 1, 0)$ ,  $x \rightarrow y = \min(1 - x + y, 1)$ ), the *Goguen (product) structure* ( $x \otimes y = x \cdot y$ ,  $x \rightarrow y = 1$  if  $x \leq y$ , and  $= y/x$  otherwise), and the *Gödel structure* ( $x \otimes y = \min(x, y)$ ,  $x \rightarrow y = 1$  if  $x \leq y$ , and  $= y$  otherwise). More generally, an algebra  $([0, 1], \wedge, \vee, \otimes, \rightarrow, 0, 1)$  is a complete residuated lattice if and only if  $\otimes$  is a left-continuous t-norm and the residuum is defined by  $x \rightarrow y = \bigvee \{u \in [0, 1] \mid u \otimes x \leq y\}$  (cf. [10]). Another important set of truth values is the set  $\{a_0, a_1, \dots, a_n\}$ ,  $0 = a_0 < \dots < a_n = 1$ , with  $a_k \otimes a_l = a_{\max(k+l-n, 0)}$  and  $a_k \rightarrow a_l = a_{\min(n-k+l, n)}$ . A special case of the latter algebras is the two-element Boolean algebra of classical logic with the support  $\{0, 1\}$ . The only adjoint pair on the two-element Boolean algebra consists of the classical conjunction and implication operations. This structure of truth values we call the *Boolean structure*. A residuated lattice  $\mathcal{L}$  satisfying  $x \otimes y = x \wedge y$  is called a *Heyting algebra*, whereas a Heyting algebra satisfying the prelinearity axiom  $(x \rightarrow y) \vee (y \rightarrow x) = 1$  is called a *Gödel algebra*. If any finitely generated subalgebra of a residuated lattice  $\mathcal{L}$  is finite, then  $\mathcal{L}$  is called *locally finite*. For example, every Gödel algebra, and hence, the Gödel structure, is locally finite, whereas the product structure is not locally finite.

Let  $(L, \wedge, \vee, 0, 1)$  be a lattice with the least element 0 and the greatest element 1. An  $\mathcal{L}$ -fuzzy subset of a set  $U$  is any function from  $U$  into  $L$  [63]. If the structure  $\mathcal{L}$  of membership values is known from the context, we will say simply *fuzzy subset* instead of  $\mathcal{L}$ -fuzzy subset. The set of all  $\mathcal{L}$ -fuzzy subsets of  $U$  is denoted by  $\mathcal{F}(U)$ . Let  $f, g \in \mathcal{F}(U)$ . The *equality* of  $f$  and  $g$  is defined as the usual equality of functions, i.e.,  $f = g$  if and only if  $f(u) = g(u)$ , for every  $u \in U$ . The *inclusion*  $f \leq g$  is also defined pointwise:  $f \leq g$  if and only if

$f(u) \leq g(u)$ , for every  $u \in U$ . Endowed with this order  $\mathcal{F}(U)$  forms a lattice, in which the meet (intersection)  $\bigwedge_{i \in I} f_i$  and the join (union)  $\bigvee_{i \in I} f_i$  of a finite family  $\{f_i\}_{i \in I}$  of  $\mathcal{L}$ -fuzzy subsets of  $A$  are functions from  $U$  into  $L$  defined by

$$\left( \bigwedge_{i \in I} f_i \right)(u) = \bigwedge_{i \in I} f_i(u), \quad \left( \bigvee_{i \in I} f_i \right)(u) = \bigvee_{i \in I} f_i(u). \tag{4}$$

If  $\mathcal{L}$  is a complete lattice, then in (4) we can allow  $I$  to be an infinite set, and in this case  $\mathcal{F}(U)$  forms a complete lattice. If  $\mathcal{L}$  is a residuated lattice, then  $\mathcal{F}(U)$  also forms a residuated lattice in which the product  $f \otimes g$  is an  $\mathcal{L}$ -fuzzy subset defined by  $f \otimes g(u) = f(u) \otimes g(u)$ , for every  $u \in U$ .

A *crisp subset* of a set  $U$  is an  $\mathcal{L}$ -fuzzy subset which takes values only in the set  $\{0, 1\}$ . If  $f$  is a crisp subset of  $A$ , then expressions “ $f(u) = 1$ ” and “ $u \in f$ ” will have the same meaning, i.e.,  $f$  is considered as an ordinary subset of  $U$ . The *crisp part* of an  $\mathcal{L}$ -fuzzy subset  $f$  of  $U$  is a crisp subset  $f^c : U \rightarrow L$  defined by  $f^c(u) = 1$ , if  $f(u) = 1$ , and  $f^c(u) = 0$ , if  $f(u) < 1$ , i.e.,  $f^c = \{u \in U \mid f(u) = 1\}$ . An  $\mathcal{L}$ -fuzzy subset  $f$  of  $U$  is *normalized* (or *modal*, in some sources) if  $f(u) = 1$  for at least one  $u \in U$ , i.e., if its crisp part is non-empty.

### 2.2. Fuzzy relations

Let  $U$  and  $V$  be non-empty sets. A *fuzzy relation between sets*  $U$  and  $V$  is any function from  $U \times V$  into  $L$ , that is to say, any fuzzy subset of  $U \times V$ , and the equality, inclusion (ordering), joins and meets of fuzzy relations are defined as for fuzzy sets. In particular, a *fuzzy relation on a set*  $U$  is any function from  $U \times U$  into  $L$ , i.e., any fuzzy subset of  $U \times U$ . The set of all fuzzy relations from  $U$  to  $V$  will be denoted by  $\mathcal{R}(U, V)$ , and the set of all fuzzy relations on a set  $U$  will be denoted by  $\mathcal{R}(U)$ . If the structure  $\mathcal{L}$  of membership values is known from the context, then we say simply *fuzzy relation* instead of  $\mathcal{L}$ -fuzzy relation. The *converse* (in some sources called *inverse* or *transpose*) of a fuzzy relation  $R \in \mathcal{R}(U, V)$  is a fuzzy relation  $R^{-1} \in \mathcal{R}(V, U)$  defined by  $R^{-1}(v, u) = R(u, v)$ , for all  $u \in U$  and  $v \in V$ . A *crisp relation* is a fuzzy relation which takes values only in the set  $\{0, 1\}$ , and if  $R$  is a crisp relation of  $U$  to  $V$ , then expressions “ $R(u, v) = 1$ ” and “ $(u, v) \in R$ ” will have the same meaning. By  $\nabla_U$  we denote the *universal relation* on a set  $U$ , which is given by  $\nabla_U(u, v) = 1$ , for all  $u, v \in U$ , and by  $\Delta_U$  we denote the *equality relation* on  $U$ , which is given by  $\Delta_U(u, v) = 1$ , if  $u = v$ , and  $\Delta_U(u, v) = 0$ , if  $u \neq v$ , for all  $u, v \in U$ .

In this paper we will consider fuzzy subsets and fuzzy relations of/on arbitrary sets, not necessarily finite. For that reason the underlying structure  $\mathcal{L}$  of membership values is required to be a complete lattice. However, whenever we work with fuzzy subsets and relations of/on a finite set, the assumption that  $\mathcal{L}$  is complete become superfluous, and it can be omitted.

In accordance with this remark, in the rest of the section, if not noted otherwise, let  $U$  be a non-empty set, not necessarily finite, and let  $\mathcal{L}$  be a complete lattice.

For non-empty sets  $U, V$  and  $W$ , and fuzzy relations  $R \in \mathcal{R}(U, V)$  and  $S \in \mathcal{R}(V, W)$ , their *composition*  $R \circ S$  is a fuzzy relation from  $\mathcal{R}(U, W)$  defined by

$$(R \circ S)(u, w) = \bigvee_{v \in V} R(u, v) \otimes S(v, w), \tag{5}$$

for all  $u \in U$  and  $w \in W$ . If  $R$  and  $S$  are crisp relations, then  $R \circ S$  is an ordinary composition of relations, i.e.,

$$R \circ S = \{(u, w) \in U \times W \mid (\exists v \in V) (u, v) \in R \ \& \ (v, w) \in S\},$$

and if  $R$  and  $S$  are functions, then  $R \circ S$  is an ordinary composition of functions, i.e.,  $(R \circ S)(u) = S(R(u))$ , for every  $u \in U$ . Next, if  $f \in \mathcal{F}(U)$ ,  $R \in \mathcal{R}(U, V)$  and  $g \in \mathcal{F}(V)$ , the *compositions*  $f \circ R$  and  $R \circ g$  are fuzzy subsets of  $V$  and  $U$ , respectively, which are defined by

$$(f \circ R)(v) = \bigvee_{u \in U} f(u) \otimes R(u, v), \quad (R \circ g)(u) = \bigvee_{v \in V} R(u, v) \otimes g(v), \tag{6}$$

for every  $u \in U$  and  $v \in V$ .

In particular, for  $f, g \in \mathcal{F}(U)$  we write

$$f \circ g = \bigvee_{u \in U} f(u) \otimes g(u). \tag{7}$$

The value  $f \circ g$  can be interpreted as the "degree of overlapping" of  $f$  and  $g$ . In particular, if  $f$  and  $g$  are crisp sets and  $R$  is a crisp relation, then

$$f \circ R = \{v \in V \mid (\exists u \in f)(u, v) \in R\}, \quad R \circ g = \{u \in U \mid (\exists v \in g)(u, v) \in R\}.$$

Let  $U, V, W$  and  $Z$  be non-empty sets. Then for any  $R_1 \in \mathcal{R}(U, V), R_2 \in \mathcal{R}(V, W)$  and  $R_3 \in \mathcal{R}(W, Z)$  we have

$$(R_1 \circ R_2) \circ R_3 = R_1 \circ (R_2 \circ R_3), \tag{8}$$

and for  $R_0 \in \mathcal{R}(U, V), R_1, R_2 \in \mathcal{R}(V, W)$  and  $R_3 \in \mathcal{R}(W, Z)$  we have that

$$R_1 \leq R_2 \text{ implies } R_1^{-1} \leq R_2^{-1}, \quad R_0 \circ R_1 \leq R_0 \circ R_2, \quad \text{and } R_1 \circ R_3 \leq R_2 \circ R_3. \tag{9}$$

Further, for any  $R \in \mathcal{R}(U, V), S \in \mathcal{R}(V, W), f \in \mathcal{F}(U), g \in \mathcal{F}(V)$  and  $h \in \mathcal{F}(W)$  we can easily verify that

$$(f \circ R) \circ S = f \circ (R \circ S), \quad (f \circ R) \circ g = f \circ (R \circ g), \quad (R \circ S) \circ h = R \circ (S \circ h) \tag{10}$$

and consequently, the parentheses in (10) can be omitted, as well as the parentheses in (8).

Finally, for all  $R, R_i \in \mathcal{R}(U, V) (i \in I)$  and  $S, S_i \in \mathcal{R}(V, W) (i \in I)$  we have that

$$(R \circ S)^{-1} = S^{-1} \circ R^{-1}, \tag{11}$$

$$R \circ \left( \bigvee_{i \in I} S_i \right) = \bigvee_{i \in I} (R \circ S_i), \quad \left( \bigvee_{i \in I} R_i \right) \circ S = \bigvee_{i \in I} (R_i \circ S), \tag{12}$$

$$\left( \bigvee_{i \in I} R_i \right)^{-1} = \bigvee_{i \in I} R_i^{-1}. \tag{13}$$

We note that if  $U, V$  and  $W$  are finite sets of cardinality  $|U| = k, |V| = m$  and  $|W| = n$ , then  $R \in \mathcal{R}(U, V)$  and  $S \in \mathcal{R}(V, W)$  can be treated as  $k \times m$  and  $m \times n$  fuzzy matrices over  $\mathcal{L}$ , and  $R \circ S$  is the matrix product. Analogously, for  $f \in \mathcal{F}(U)$  and  $g \in \mathcal{F}(V)$  we can treat  $f \circ R$  as the product of a  $1 \times k$  matrix  $f$  and a  $k \times m$  matrix  $R$  (vector-matrix product),  $R \circ g$  as the product of an  $k \times m$  matrix  $R$  and an  $m \times 1$  matrix  $g^t$ , the transpose of  $g$  (matrix-vector product), and  $f \circ g$  as the scalar product of vectors  $f$  and  $g$ .

An  $\mathcal{L}$ -fuzzy relation  $R$  on a set  $U$  is said to be

- (R) *reflexive* (or *fuzzy reflexive*) if  $\Delta_U \leq R$ , i.e., if  $R(u, u) = 1$ , for every  $u \in U$ ;
- (S) *symmetric* (or *fuzzy symmetric*) if  $R^{-1} \leq R$ , i.e., if  $R(u, v) = R(v, u)$ , for all  $u, v \in U$ ;
- (T) *transitive* (or *fuzzy transitive*) if  $R \circ R \leq R$ , i.e., if for all  $u, v, w \in U$  we have

$$R(u, v) \otimes R(v, w) \leq R(u, w).$$

For an  $\mathcal{L}$ -fuzzy relation  $R$  on a set  $U$ , an  $\mathcal{L}$ -fuzzy relation  $R^\infty$  on  $U$  defined by  $R^\infty = \bigvee_{n \in \mathbb{N}} R^n$  is the least transitive  $\mathcal{L}$ -fuzzy relation on  $U$  containing  $R$ , and it is called the *transitive closure* of  $R$ .

A reflexive and transitive  $\mathcal{L}$ -fuzzy relation on  $U$  is called an  $\mathcal{L}$ -fuzzy quasi-order, or just a *fuzzy quasi-order*, if  $\mathcal{L}$  is known from the context, and a reflexive and transitive crisp relation on  $A$  is called a *quasi-order*. In some sources quasi-orders and fuzzy quasi-orders are called preorders and fuzzy preorders. Note that a reflexive fuzzy relation  $R$  is a fuzzy quasi-order if and only if  $R^2 = R$ . A reflexive, symmetric and transitive  $\mathcal{L}$ -fuzzy relation on  $U$  is called an  $\mathcal{L}$ -fuzzy equivalence (or just a *fuzzy equivalence*), and a reflexive, symmetric and transitive crisp relation on  $U$  is called an *equivalence*. An  $\mathcal{L}$ -fuzzy equivalence  $E$  on  $U$  is called an  $\mathcal{L}$ -fuzzy equality (or just a *fuzzy equality*) if for any  $u, v \in U, E(u, v) = 1$  implies  $u = v$ . If  $R$  is an  $\mathcal{L}$ -fuzzy quasi-order

on  $U$ , then  $R \wedge R^{-1}$  is the greatest  $\mathcal{L}$ -fuzzy equivalence contained in  $R$ , and it is called the *natural fuzzy equivalence* of  $R$ .

With respect to the ordering of  $\mathcal{L}$ -fuzzy relations, the set  $\mathcal{Q}(U)$  of all  $\mathcal{L}$ -fuzzy quasi-orders on a set  $U$ , and the set  $\mathcal{E}(U)$  of all  $\mathcal{L}$ -fuzzy equivalences on  $U$ , form complete lattices. The meet both in  $\mathcal{Q}(U)$  and  $\mathcal{E}(U)$  is the ordinary intersection of  $\mathcal{L}$ -fuzzy relations, but in the general case, the joins in  $\mathcal{Q}(U)$  and  $\mathcal{E}(U)$  do not coincide with the ordinary union of  $\mathcal{L}$ -fuzzy relations. Namely, if  $\{R_i\}_{i \in I}$  is a family of  $\mathcal{L}$ -fuzzy quasi-orders (resp.  $\mathcal{L}$ -fuzzy equivalences) on  $U$ , then its join in  $\mathcal{Q}(U)$  (resp. in  $\mathcal{E}(U)$ ) is  $(\bigvee_{i \in I} R_i)^\infty$ , the transitive closure of the union of this family.

Let  $Q$  be an  $\mathcal{L}$ -fuzzy quasi-order on a set  $U$ . For each  $u \in U$ , the  $Q$ -*afterset* of  $u$  is the  $\mathcal{L}$ -fuzzy subset  $Q_u$  of  $U$  defined by  $Q_u(x) = Q(u, x)$ , for any  $x \in U$ , and the  $Q$ -*foreset* of  $u$  is the  $\mathcal{L}$ -fuzzy subset  $Q^u$  of  $U$  defined by  $Q^u(x) = Q(x, u)$ , for any  $x \in U$  (cf. [2, 14, 46, 92, 129]). The set of all  $Q$ -aftersets will be denoted by  $U/Q$ , and the set of all  $Q$ -foresets by  $U \setminus Q$ . If  $E$  is an  $\mathcal{L}$ -fuzzy equivalence, then for every  $u \in U$  we have that  $E_u = E^u$ , and  $E_u$  is called the *equivalence class* of  $u$  with respect to  $E$  (cf. [36]). The set of all equivalence classes of  $E$  is denoted by  $U/E$  and called the *factor set* of  $U$  with respect to  $E$ . For any  $\mathcal{L}$ -fuzzy quasi-order  $Q$  on a set  $U$  and its natural  $\mathcal{L}$ -fuzzy equivalence  $E$  we have that the set  $U/Q$  of all  $Q$ -aftersets, the set  $U \setminus Q$  of all  $Q$ -foresets, and the factor set  $U/E$  have the same cardinality (cf. [129]). This cardinality will be called the *index* of  $Q$ , and it will be denoted by  $\text{ind}(Q)$ . If  $U$  is a finite set with  $n$  elements and an  $\mathcal{L}$ -fuzzy quasi-order  $Q$  on  $U$  is treated as an  $n \times n$  fuzzy matrix over  $\mathcal{L}$ , then  $Q$ -aftersets are row vectors, whereas  $Q$ -foresets are column vectors of this matrix. For any  $\mathcal{L}$ -fuzzy subset  $f$  of  $U$ , let  $\mathcal{L}$ -fuzzy relations  $Q_f$ ,  $Q^f$ , and  $E_f$  on  $U$  be defined by

$$Q_f(u, v) = f(u) \rightarrow f(v), \quad Q^f(u, v) = f(v) \rightarrow f(u), \quad E_f(u, v) = f(u) \leftrightarrow f(v), \quad (14)$$

for all  $u, v \in U$ . We have that  $Q_f$  and  $Q^f$  are  $\mathcal{L}$ -fuzzy quasi-orders, and  $E_f$  is an  $\mathcal{L}$ -fuzzy equivalence on  $U$ . In particular, if  $f$  is a normalized  $\mathcal{L}$ -fuzzy subset of  $U$ , then it is an afterset of  $Q_f$ , a foreset of  $Q^f$ , and an equivalence class of  $E_f$ .

For more information on lattices and related concepts we refer to books [11, 13, 121], as well as to books [9, 10, 56, 88, 110], for more information on fuzzy sets and fuzzy relations.

### 2.3. Uniform fuzzy relations

In this section we recall some notions, notation and results from [37, 38], concerning uniform fuzzy relations and related concepts.

The original intention of the authors in [37] was to introduce uniform fuzzy relations as a basis for defining such concept of a fuzzy function which would provide a correspondence between fuzzy functions and fuzzy equivalence relations, analogous to the correspondence between crisp functions and crisp equivalence relations. This was done, but also, it turned out that uniform fuzzy relations establish natural relationships between fuzzy partitions of two sets, some kind of “uniformity” between these fuzzy partitions. Roughly speaking, uniform fuzzy relations can be conceived as fuzzy equivalence relations which relate elements of two possibly different sets. In [37], uniform fuzzy relations were employed to solve some systems of fuzzy relation equations, systems that have important applications in approximate reasoning, especially in fuzzy control. Afterwards, in [75], they were used to define and study fuzzy homomorphisms and fuzzy relational morphisms of algebras, and to establish relationships between fuzzy homomorphisms, fuzzy relational morphisms, and fuzzy congruences, analogous to relationships between homomorphisms, relational morphisms, and congruences in classical algebra. In the same paper, fuzzy relational morphisms were also applied to deterministic fuzzy automata. It has been shown in [38] that fuzzy relational morphisms are the same as forward bisimulations (in the terminology used in this paper) when these two concepts are considered in the context of deterministic fuzzy automata. As we shall see later, in Section 7.3, uniform fuzzy relations have shown their full strength in the study of equivalence between fuzzy automata, carried out in [38] (see also [35]).

Let  $U$  and  $V$  be non-empty sets and let  $E$  and  $F$  be fuzzy equivalences on  $U$  and  $V$ , respectively. If a fuzzy relation  $R \in \mathcal{R}(U, V)$  satisfies

(EX1)  $R(u_1, v) \otimes E(u_1, u_2) \leq R(u_2, v)$ , for all  $u_1, u_2 \in U$  and  $v \in V$ ,

then it is called *extensional with respect to E*, and if it satisfies

(EX2)  $R(u, v_1) \otimes F(v_1, v_2) \leq R(u, v_2)$ , for all  $u \in U$  and  $v_1, v_2 \in V$ ,

then it is called *extensional with respect to F*. If  $R$  is extensional with respect to  $E$  and  $F$ , and it satisfies

(PFF)  $R(u, v_1) \otimes R(u, v_2) \leq F(v_1, v_2)$ , for all  $u \in U$  and  $v_1, v_2 \in V$ ,

then it is called a *partial fuzzy function* with respect to  $E$  and  $F$ .

Partial fuzzy functions were introduced by Klawonn [87], and studied also by Demirci [43, 44]. By the adjunction property and symmetry, conditions (EX1) and (EX2) can be also written as

(EX1')  $E(u_1, u_2) \leq (R(u_1, v) \leftrightarrow R(u_2, v))$ , for all  $u_1, u_2 \in U$  and  $v \in V$ ;

(EX2')  $F(v_1, v_2) \leq (R(u, v_1) \leftrightarrow R(u, v_2))$ , for all  $u \in U$  and  $v_1, v_2 \in V$ .

For any fuzzy relation  $R \in \mathcal{R}(U, V)$  we can define a fuzzy equivalence  $E_U^R$  on  $U$  by

$$E_U^R(u_1, u_2) = \bigwedge_{v \in V} (R(u_1, v) \leftrightarrow R(u_2, v)), \tag{15}$$

for all  $u_1, u_2 \in U$ , and a fuzzy equivalence  $E_V^R$  on  $V$  by

$$E_V^R(v_1, v_2) = \bigwedge_{u \in U} (R(u, v_1) \leftrightarrow R(u, v_2)), \tag{16}$$

for all  $v_1, v_2 \in V$ . They will be called *fuzzy equivalences* on  $U$  and  $V$  induced by  $R$ , and in particular,  $E_U^R$  will be called the *kernel* of  $R$ , and  $E_V^R$  the *co-kernel* of  $R$ . According to (EX1') and (EX2'),  $E_U^R$  and  $E_V^R$  are the greatest fuzzy equivalences on  $U$  and  $V$ , respectively, such that  $R$  is extensional with respect to them. Also, the fuzzy relation  $R \circ R^{-1} \in \mathcal{R}(U)$  will be called the *projection of R on U*, and  $R^{-1} \circ R \in \mathcal{R}(V)$  the *projection of R on V*.

A fuzzy relation  $R \in \mathcal{R}(U, V)$  is called just a *partial fuzzy function* if it is a partial fuzzy function with respect to  $E_U^R$  and  $E_V^R$  [37]. Partial fuzzy functions were characterized in [37, 38] as follows:

**Theorem 2.1.** *Let  $U$  and  $V$  be non-empty sets and let  $R \in \mathcal{R}(U, V)$  be a fuzzy relation. Then the following conditions are equivalent:*

- (i)  $R$  is a partial fuzzy function;
- (ii)  $R^{-1}$  is a partial fuzzy function;
- (iii)  $R^{-1} \circ R \leq E_V^R$ ;
- (iv)  $R \circ R^{-1} \leq E_U^R$ ;
- (v)  $R \circ R^{-1} \circ R \leq R$ .

The name partial fuzzy function was introduced in [87], but it should be noted that the notion of a partial fuzzy function can not be considered as a natural analog of a notion of a partial function, because for a partial function relation  $R$ , its reverse  $R^{-1}$  is not necessarily a partial function. For the crisp counterpart of partial fuzzy functions has been called in [35] a partial uniform relation.

A fuzzy relation  $R \in \mathcal{R}(U, V)$  is called an  $\mathcal{L}$ -function if for any  $u \in U$  there exists  $v \in V$  such that  $R(u, v) = 1$  [45], and it is called *surjective* if for any  $v \in V$  there exists  $u \in U$  such that  $R(u, v) = 1$ , i.e., if  $R$  is an  $\mathcal{L}$ -function. For a surjective fuzzy relation  $R \in \mathcal{R}(U, V)$  we also say that it is a fuzzy relation of  $U$  onto  $V$ . If  $R$  is both an  $\mathcal{L}$ -function and surjective, i.e., if both  $R$  and  $R^{-1}$  are  $\mathcal{L}$ -functions, then  $R$  is called a *surjective  $\mathcal{L}$ -function*. If for any  $u \in U$  there exists a unique  $v \in V$  such that  $R(u, v) = 1$ , then  $R$  is called an *F-function* [105].

Let us note that a fuzzy relation  $R \in \mathcal{R}(U, V)$  is an  $\mathcal{L}$ -function if and only if there is a function  $\psi : U \rightarrow V$  such that  $R(u, \psi(u)) = 1$ , for all  $u \in U$  (cf. [44, 45]). A function  $\psi$  with this property is called a *crisp description* of  $R$ , and we denote by  $CR(R)$  the set of all such functions.

An  $\mathcal{L}$ -function which is a partial fuzzy function with respect to  $E$  and  $F$  is called a *perfect fuzzy function* with respect to  $E$  and  $F$ . Perfect fuzzy functions were introduced and studied by Demirci [43, 44]. A fuzzy relation  $R \in \mathcal{R}(U, V)$  which is a perfect fuzzy function with respect to  $E_U^R$  and  $E_V^R$  will be called just a *perfect fuzzy function*.

Let  $U$  and  $V$  be non-empty sets and let  $E$  be a fuzzy equivalence on  $V$ . An ordinary function  $\psi : U \rightarrow V$  is called  *$E$ -surjective* if for any  $v \in V$  there exists  $u \in U$  such that  $E(\psi(u), v) = 1$ . In other words,  $\psi$  is  $E$ -surjective if and only if  $\psi \circ E^\#$  is an ordinary surjective function of  $U$  onto  $V/E$ , where  $E^\# : V \rightarrow V/E$  is a function given by  $E^\#(v) = E_v$ , for each  $v \in V$ . It is clear that  $\psi$  is an  $E$ -surjective function if and only if its image  $\text{Im } \psi$  has a non-empty intersection with every equivalence class of the crisp equivalence  $\ker(E)$ .

Let  $U$  and  $V$  be non-empty sets and let  $R \in \mathcal{R}(U, V)$  be a partial fuzzy function. If, in addition,  $R$  is a surjective  $\mathcal{L}$ -function, then it will be called a *uniform fuzzy relation* [37]. In other words, a uniform fuzzy relation is a perfect fuzzy function having the additional property that it is surjective. A uniform fuzzy relation that is also a crisp relation is called a *uniform relation*. The following characterizations of uniform fuzzy relations provided in [37, 38] will be used in the further text.

**Theorem 2.2.** *Let  $U$  and  $V$  be non-empty sets and let  $R \in \mathcal{R}(U, V)$  be a fuzzy relation. Then the following conditions are equivalent:*

- (i)  $R$  is a uniform fuzzy relation;
- (ii)  $R^{-1}$  is a uniform fuzzy relation;
- (iii)  $R$  is a surjective  $\mathcal{L}$ -function and

$$R \circ R^{-1} \circ R = R; \tag{17}$$

- (iv)  $R$  is a surjective  $\mathcal{L}$ -function and

$$E_U^R = R \circ R^{-1}; \tag{18}$$

- (v)  $R$  is a surjective  $\mathcal{L}$ -function and

$$E_V^R = R^{-1} \circ R; \tag{19}$$

- (vi)  $R$  is an  $\mathcal{L}$ -function, and for all  $\psi \in CR(R)$ ,  $u \in U$  and  $v \in V$  we have that  $\psi$  is  $E_V^R$ -surjective and

$$R(u, v) = E_V^R(\psi(u), v); \tag{20}$$

- (vii)  $R$  is an  $\mathcal{L}$ -function, and for all  $\psi \in CR(R)$  and  $u_1, u_2 \in U$  we have that  $\psi$  is  $E_V^R$ -surjective and

$$R(u_1, \psi(u_2)) = E_U^R(u_1, u_2). \tag{21}$$

**Corollary 2.3.** [37] *Let  $U$  and  $V$  be non-empty sets, and let  $\varphi \in \mathcal{F}(U \times V)$  be a uniform fuzzy relation. Then for all  $\psi \in CR(\varphi)$  and  $u_1, u_2 \in A$  we have that*

$$E_U^\varphi(u_1, u_2) = E_V^\varphi(\psi(u_1), \psi(u_2)). \tag{22}$$

A fuzzy relation  $R \in \mathcal{R}(U, V)$  is called an *uniform FL-function* if it is both a uniform fuzzy relation and an  $F$ -function, i.e., if it is a uniform fuzzy relation and  $E_V^R$  is a fuzzy equality (cf. [37]).

Let  $U$  and  $V$  be non-empty sets. According to Theorem 2.2, a fuzzy relation  $R \in \mathcal{R}(U, V)$  is a uniform fuzzy relation if and only if its inverse relation  $R^{-1}$  is a uniform fuzzy relation. Moreover, by (iv) and (v) of Theorem 2.2, we have that the kernel of  $R^{-1}$  is the co-kernel of  $R$ , and conversely, the co-kernel of  $R^{-1}$  is the kernel of  $R$ , that is

$$E_V^{R^{-1}} = E_V^R \quad \text{and} \quad E_U^{R^{-1}} = E_U^R.$$

The next theorem proved in [37, 38] will be very useful in our further work.



**Theorem 2.4.** Let  $U$  and  $V$  be non-empty sets, let  $R \in \mathcal{R}(U, V)$  be a uniform fuzzy relation, let  $E = E_U^R$  and  $F = E_V^R$ , and let a function  $\widetilde{R} : U/E \rightarrow V/F$  be defined by

$$\widetilde{R}(E_u) = F_{\psi(u)}, \text{ for any } u \in U \text{ and } \psi \in CR(R). \tag{23}$$

Then  $\widetilde{R}$  is a well-defined function (it does not depend on the choice of  $\psi \in CR(R)$  and  $u \in U$ ), it is a bijective function of  $U/E$  onto  $V/F$ , and  $(\widetilde{R})^{-1} = \widetilde{R}^{-1}$ .

The bijective function  $\widetilde{R}$  establishes some kind of “uniformity” between fuzzy partitions on  $U$  and  $V$  which correspond to fuzzy equivalences  $E_U^R$  and  $E_V^R$ , and for that reason these fuzzy relations are called uniform.

2.4. Residuated functions. Residuated semigroups and semimodules

In order to define the concept of a residuated function, we need the next fundamental theorem. First we note that the notations  $I_P$  and  $I_Q$  in the next theorem are used for the identity functions on  $P$  and  $Q$ .

**Theorem 2.5 (cf. [13]).** Let  $P$  and  $Q$  be ordered sets. The following conditions for a function  $f : P \rightarrow Q$  are equivalent:

- (i)  $f$  is isotone and there exists an isotone function  $g : Q \rightarrow P$  such that

$$I_P \leq f \circ g, \quad g \circ f \leq I_Q; \tag{24}$$

- (ii) there exists a function  $g : Q \rightarrow P$  such that

$$f(x) \leq y \Leftrightarrow x \leq g(y), \tag{25}$$

for all  $x \in P$  and  $y \in Q$ ;

- (iii) the inverse image under  $f$  of every principal down-set of  $Q$  is a principal down-set of  $P$ ;

- (iv)  $f$  is isotone and the set  $\{x \in P \mid f(x) \leq y\}$  has the greatest element, for every  $y \in Q$ .

Furthermore, if there is a function  $g$  which satisfies (24) or (25), then it is unique.

A function  $f$  that satisfies either of the equivalent conditions of Theorem 2.5 is called a *residuated function*, and the unique function  $g$  that satisfies (24) or (25) is called the *residual* of  $f$  (cf. [13]) and denoted by  $f^\sharp$ . For a residuated function  $f : P \rightarrow Q$  and  $y \in Q$  we have that

$$f^\sharp(y) = \top\{x \in P \mid f(x) \leq y\}, \tag{26}$$

where  $\top H$  denotes the greatest element of a subset  $H$  of an ordered set, if it exists. It should be noted that  $f \circ f^\sharp \circ f = f$  and  $f^\sharp \circ f \circ f^\sharp = f^\sharp$ , for any residuated function  $f$ , and for ordered sets  $P, Q$  and  $R$  and residuated functions  $f : P \rightarrow Q$  and  $g : Q \rightarrow R$  we have that  $f \circ g$  is also residuated and  $(f \circ g)^\sharp = g^\sharp \circ f^\sharp$ .

An *ordered semigroup* is a triple  $(S, \otimes, \leq)$  such that  $(S, \otimes)$  is a semigroup (not necessarily commutative),  $(S, \leq)$  is an ordered set, and the order  $\leq$  is compatible with respect to the multiplication  $\otimes$ , i.e., for all  $a, b, x, y \in S$ , by  $a \leq b$  it follows  $x \otimes a \leq x \otimes b$  and  $a \otimes y \leq b \otimes y$ . In addition, if  $(S, \otimes)$  is a monoid, then  $(S, \otimes, \leq)$  is called an *ordered monoid*.

Let  $(S, \otimes)$  be a semigroup. For any  $a \in S$ , we define functions  $\lambda_a$  and  $\rho_a$  of  $S$  into itself by  $\lambda_a(x) = a \otimes x$  and  $\rho_a(x) = x \otimes a$ , for each  $x \in S$ . The function  $\lambda_a$  is called the *left translation* on  $S$  determined by  $a$ , and  $\rho_a$  is called the *right translation* on  $S$  determined by  $a$ . An ordered semigroup  $(S, \otimes, \leq)$  is called *right residuated* if each left translation on  $S$  is a residuated function. In this case, for arbitrary  $a, b \in S$ , the element

$$a \setminus b = \lambda_a^\sharp(b) = \top\{x \in S \mid a \otimes x \leq b\} \tag{27}$$

is called the *right residual* of  $b$  by  $a$ , thinking of it as what remains of  $b$  on the right after “dividing”  $b$  on the left by  $a$ . Analogously,  $(S, \otimes, \leq)$  is called *left residuated* if each right translation on  $S$  is a residuated function, and in this case, for arbitrary  $a, b \in S$ , the element

$$b/a = \varrho_a^\#(b) = \top\{x \in S \mid x \otimes a \leq b\} \tag{28}$$

is called the *left residual* of  $b$  by  $a$ . An ordered semigroup that is both right and left residuated is called a *residuated semigroup*, or a *residuated monoid*, if it has an identity. Clearly, in a commutative semigroup the concepts of a right residual and a left residual coincide. It should be noted that for arbitrary elements  $a, b$  and  $c$  of a residuated semigroup the following is true:

$$a \otimes b \leq c \iff a \leq c/b \iff b \leq a \setminus c. \tag{29}$$

Further, let  $\mathcal{S} = (S, \oplus, \otimes, 0, 1)$  be a semiring with the zero  $0$  and the identity  $1$ . A *left  $\mathcal{S}$ -semimodule* is a commutative monoid  $(A, +, 0)$  for which an external multiplication  $S \times A \rightarrow A$ , denoted by  $(\lambda, x) \mapsto \lambda x$  and called the *left scalar multiplication*, is defined and which for all  $\lambda, \lambda_1, \lambda_2 \in S$  and  $x, x_1, x_2 \in A$  satisfies the following equalities:

$$(\lambda_1 \otimes \lambda_2)x = \lambda_1(\lambda_2 x), \tag{30}$$

$$\lambda(x_1 + x_2) = \lambda x_1 + \lambda x_2, \tag{31}$$

$$(\lambda_1 \oplus \lambda_2)x = \lambda_1 x + \lambda_2 x, \tag{32}$$

$$1x = x, \tag{33}$$

$$\lambda 0 = 0x = 0. \tag{34}$$

The definition of a *right  $\mathcal{S}$ -semimodule* is analogous, where the external multiplication is defined as a function  $A \times S \rightarrow A$ , denoted by  $(x, \lambda) \mapsto x\lambda$  and called the *right scalar multiplication*, and conditions dual to (30)–(34) are satisfied. For two semirings  $\mathcal{T} = (T, \oplus, \otimes, 0, 1)$  and  $\mathcal{S} = (S, \oplus, \otimes, 0, 1)$ , an  *$\mathcal{S}$ - $\mathcal{T}$ -bisemimodule* is a commutative monoid  $(A, +, 0)$  which is both a left  $\mathcal{S}$ -semimodule and a right  $\mathcal{T}$ -semimodule, and for all  $\lambda \in S, x \in A$  and  $\mu \in T$  the following is true:

$$(\lambda x)\mu = \lambda(x\mu). \tag{35}$$

If  $\mathcal{S} = \mathcal{T}$ , we say simply an  *$\mathcal{S}$ -bisemimodule*.

In addition, if  $(A, +, 0, \leq)$  is a commutative ordered monoid, then we can talk about residuated semimodules. A left  $\mathcal{S}$ -semimodule is *residuated* if for each scalar  $\lambda \in S$  the function  $x \mapsto \lambda x$  is residuated, and in this case, for arbitrary  $\lambda \in S$  and  $a \in A$  the element

$$\lambda \setminus a = \top\{x \in A \mid \lambda x \leq a\} \tag{36}$$

is called the *right residual* of  $a$  by  $\lambda$ . Analogously, a right  $\mathcal{S}$ -semimodule is called *residuated* if for each scalar  $\lambda \in S$  the function  $x \mapsto x\lambda$  is residuated, and in this case, for arbitrary  $\lambda \in S$  and  $a \in A$  the element

$$a/\lambda = \top\{x \in A \mid x\lambda \leq a\} \tag{37}$$

is called the *left residual* of  $a$  by  $\lambda$ . Finally, an  $\mathcal{S}$ - $\mathcal{T}$ -bisemimodule is called *residuated* if it is both a residuated left  $\mathcal{S}$ -semimodule and a residuated right  $\mathcal{T}$ -semimodule, and in this case we define the concepts of a right and a left residual as in (36) and (37).

### 3. Linear systems and residuals of fuzzy relations

In this section we recall some fundamental results concerning systems of linear fuzzy relation inequalities and equations, which will be briefly called *linear systems*. One can distinguish two basic types of linear systems. The first type, which we call *homogeneous linear systems*, are systems composed of fuzzy relations on

a single set, and the second type, which we call *heterogeneous linear systems*, are systems composed of fuzzy relations on two possible different sets, where an unknown is a fuzzy relation between these two sets. We will give more precise definitions.

Let  $U$  and  $V$  be non-empty sets (not necessarily finite), let  $\{A_i\}_{i \in I}$  be a given family of fuzzy relations on  $U$  and  $\{B_i\}_{i \in I}$  a given family of fuzzy relations on  $V$  (where  $I$  is also not necessarily finite), and let  $X$  be an unknown fuzzy relation between  $U$  and  $V$ . Consider the following six types of fuzzy relation equations:

$$A_i \circ X \leq B_i \quad (i \in I), \tag{11}$$

$$B_i \leq A_i \circ X \quad (i \in I), \tag{12}$$

$$A_i \circ X = B_i \quad (i \in I), \tag{13}$$

$$X \circ A_i \leq B_i \quad (i \in I), \tag{14}$$

$$B_i \leq X \circ A_i \quad (i \in I), \tag{15}$$

$$X \circ A_i = B_i \quad (i \in I), \tag{16}$$

Inequalities and equations that form these systems will be called *linear*, and the systems will be called *linear systems*. If  $U = V$ , then they are called *homogeneous linear systems*, and if  $U \neq V$ , then they are called *heterogeneous linear systems*.

Linear systems emerged from Sanchez’s research aimed at medical applications (cf. [122–125]). He proved that system (11), as well as the reverse system (14), are solvable and have the greatest solutions. We will describe these greatest solutions later. Other systems are not necessary solvable, but if (13) is solvable, then it has the same greatest solution as (11), and also, if (16) is solvable, then it has the same greatest solution as (14) (cf. [122–124]). In particular, if  $U = V$  and  $A_i = B_i$ , for each  $i \in I$ , then all systems are solvable. In fact, Sanchez’s results were proved for fuzzy relations over the Gödel structure, but we will see in the sequel that they also hold for fuzzy relations over an arbitrary complete residuated lattice.

We will interpret Sanchez’s results in terms of residuated semigroups and residuated semimodules. First we note that for an arbitrary non-empty set  $U$ , the quintuple  $(\mathcal{R}(U), \vee, \circ, \emptyset, \Delta)$  forms a semiring, where  $\Delta$  denotes the equality relation on  $U$ . Also, for non-empty sets  $U$  and  $V$ ,  $(\mathcal{R}(U, V), \vee, \emptyset, \leq)$  is an ordered monoid, and if we consider composition of a fuzzy relation from  $\mathcal{R}(U)$  and a fuzzy relation from  $\mathcal{R}(U, V)$  as a left scalar multiplication, and composition of a fuzzy relation from  $\mathcal{R}(U, V)$  and a fuzzy relation from  $\mathcal{R}(V)$  as a right scalar multiplication, then  $\mathcal{R}(U, V)$  can be also viewed as an  $\mathcal{R}(U)$ - $\mathcal{R}(V)$ -bisemimodule. In the sequel,  $\mathcal{R}(U)$ ,  $\mathcal{R}(V)$  and  $\mathcal{R}(U, V)$  will be treated exactly in this way.

**Theorem 3.1.** *Let  $U$  and  $V$  be arbitrary non-empty sets. Then  $\mathcal{R}(U, V)$  is a residuated  $\mathcal{R}(U)$ - $\mathcal{R}(V)$ -bisemimodule.*

*For arbitrary  $A \in \mathcal{R}(U)$ ,  $B \in \mathcal{R}(V)$  and  $R \in \mathcal{R}(U, V)$ , the right residual of  $R$  by  $A$  is a fuzzy relation  $A \setminus R \in \mathcal{R}(U, V)$  defined by*

$$(A \setminus R)(x, y) = \bigwedge_{x' \in U} (A(x', x) \rightarrow R(x', y)), \tag{38}$$

*for all  $x \in U$  and  $y \in V$ , and the left residual of  $R$  by  $B$  is a fuzzy relation  $R / B \in \mathcal{R}(U, V)$  defined by*

$$(R / B)(x, y) = \bigwedge_{y' \in V} (B(y, y') \rightarrow R(x, y')), \tag{39}$$

*for all  $x \in U$  and  $y \in V$ .*

*In particular, if  $U = V$ , then  $(\mathcal{R}(U), \circ, \leq)$  is a residuated semigroup with residuals defined as in (38) and (39).*

The previous theorem is valid if the underlying structure  $\mathcal{L}$  of membership values is a complete residuated lattice. Note that existence of an operation forming an adjoint pair with the multiplication is not only sufficient, but also a necessary condition for the existence of residuals of  $\mathcal{L}$ -fuzzy relations. Namely, the following is true.

**Theorem 3.2 ([76]).** Let  $\mathcal{L} = (L, \wedge, \vee, \otimes, 0, 1)$  be an algebra satisfying the conditions (L1') and (L2) of the definition of a complete residuated lattice and (3). Then the following conditions are equivalent:

- (i)  $\mathcal{L}$  is a residuated lattice;
- (ii) every left semimodule of  $\mathcal{L}$ -fuzzy relations is residuated;
- (iii) every right semimodule of  $\mathcal{L}$ -fuzzy relations is residuated;
- (iv) every bi-semimodule of  $\mathcal{L}$ -fuzzy relations is residuated.

Note that the previous theorem remains valid if the terms “left semimodule”, “right semimodule” and “bi-semimodule” in (ii), (iii) and (iv) are replaced by “semigroup”, and the term “residuated” in (ii) and (iii) is replaced by “right residuated” and “left residuated”.

**Corollary 3.3 ([76]).** Let  $\mathcal{L} = ([0, 1], \wedge, \vee, \otimes, 0, 1)$ , where  $\otimes$  is a  $t$ -norm. Then the following conditions are equivalent:

- (i)  $\otimes$  is a left-continuous  $t$ -norm;
- (ii) every left semimodule of  $\mathcal{L}$ -fuzzy relations is residuated;
- (iii) every right semimodule of  $\mathcal{L}$ -fuzzy relations is residuated;
- (iv) every bi-semimodule of  $\mathcal{L}$ -fuzzy relations is residuated.

According to Theorems 2.5 and 3.1, the set of all solutions to the fuzzy relation inequality  $A \circ X \leq R$  is the principal down-set of  $(\mathcal{R}(U, V), \leq)$  generated by the right residual  $A \setminus R$  of  $R$  by  $A$ , and hence,  $A \setminus R$  is the greatest solution to  $A \circ X \leq R$ . Consequently, the greatest solution to system (I1) is simply the intersection of the greatest solutions to individual inequalities in (I1), i.e., the greatest solution to (I1) is

$$\bigwedge_{i \in I} A_i \setminus B_i, \quad (40)$$

and the set of all solutions to (I1) is the principal down-set of  $(\mathcal{R}(U, V), \leq)$  generated by the fuzzy relation (40).

Analogously, the set of all solutions to  $X \circ B \leq R$  is the principal down-set of  $(\mathcal{R}(U, V), \leq)$  generated by the left residual  $R/B$  of  $R$  by  $B$ , so  $R/B$  is the greatest solution to this inequality. The greatest solution to (I4) is

$$\bigwedge_{i \in I} B_i / A_i, \quad (41)$$

i.e., the intersection of the greatest solutions to individual inequalities in (I4), and the set of all solutions to (I4) is the principal down-set of  $(\mathcal{R}(U, V), \leq)$  generated by the fuzzy relation (41).

If system (I2) has at least one solution, then the set of all its solutions is an up-set of  $(\mathcal{R}(U, V), \leq)$ , and hence, the set of all solutions to (I3) is the intersection of this up-set and the principal down-set generated by the fuzzy relation (40). Therefore, in this case we have that the fuzzy relation (40) is also the greatest solution to (I3). Similar conclusions can be drawn for systems (I5) and (I6).

If  $U = V$  and  $A_i = B_i$ , for each  $i \in I$ , then the equality relation is a solution to each of the systems (I1)–(I6), and the greatest solution both to (I1) and (I3) is given as in (40), and the greatest solution both to (I4) and (I6) is given as in (41).

#### 4. Weakly linear systems

Now we move to consideration of systems of weakly linear fuzzy relation inequalities and equations, that will be called simply *weakly linear systems*. As for linear systems, we can distinguish two basic types of weakly linear systems: *homogeneous* and *heterogeneous weakly linear systems*.

Let  $U$  be a non-empty set (not necessarily finite), let  $\{A_i\}_{i \in I}$  be a given family of fuzzy relations on  $U$  (where  $I$  is also not necessarily finite), let  $M$  be a given fuzzy relation on  $U$ , and let  $X$  be an unknown fuzzy relation on  $U$ . Two basic types of homogeneous weakly linear systems are systems of the form

$$X \circ A_i \leq A_i \circ X \quad (i \in I), \quad X \leq M, \tag{wl1.1}$$

$$A_i \circ X \leq X \circ A_i \quad (i \in I), \quad X \leq M, \tag{wl1.2}$$

and the third one is the conjunction of the first two:

$$X \circ A_i = A_i \circ X \quad (i \in I), \quad X \leq M. \tag{wl1.3}$$

Besides, in many situations we need that both a fuzzy relation  $R$  and its inverse  $R^{-1}$  are solutions to the above mentioned systems, and then we consider the following three systems:

$$X \circ A_i \leq A_i \circ X, \quad X^{-1} \circ A_i \leq A_i \circ X^{-1}, \quad (i \in I), \quad X \leq M \wedge M^{-1}, \tag{wl1.4}$$

$$A_i \circ X \leq X \circ A_i, \quad A_i \circ X^{-1} \leq X^{-1} \circ A_i, \quad (i \in I), \quad X \leq M \wedge M^{-1}, \tag{wl1.5}$$

$$X \circ A_i = A_i \circ X, \quad X^{-1} \circ A_i = A_i \circ X^{-1} \quad (i \in I), \quad X \leq M \wedge M^{-1}, \tag{wl1.6}$$

Clearly, a symmetric fuzzy relation is a solution to (wl1.4) (resp. (wl1.5), (wl1.6)) if and only if it is solution to (wl1.1) (resp. (wl1.2), (wl1.3)). If  $M(a_1, a_2) = 1$ , for all  $a_1, a_2 \in U$ , then the inequality  $X \leq M$  becomes trivial, and it can be omitted. Systems (wl1.1)–(wl1.6) are called *homogeneous weakly linear systems* [76, 78]. For the sake of convenience, for each  $t \in \{1, \dots, 6\}$ , system (wl1.t) will be denoted by  $WL^{1.t}(U, I, A_i, M)$ . If  $M(a_1, a_2) = 1$ , for all  $a_1, a_2 \in U$ , then system (wl1.t) is denoted simply by  $WL^{1.t}(U, I, A_i)$ . Note that the condition  $X \leq M \wedge M^{-1}$  appearing in (wl1.4)–(wl1.6) is equivalent to  $X \leq M$  and  $X^{-1} \leq M$ .

Another situation is when we work with two possibly different non-empty sets  $U$  and  $V$  (which are not necessarily finite). Let  $\{A_i\}_{i \in I}$  be a given family of fuzzy relations on  $U$  and  $\{B_i\}_{i \in I}$  a given family of fuzzy relations on  $V$  (where  $I$  is also not necessarily finite), let  $N$  be a given fuzzy relation between  $U$  and  $V$ , and let  $X$  be an unknown fuzzy relation between  $U$  and  $V$ . Two basic types of heterogeneous weakly linear systems are systems of the form

$$X^{-1} \circ A_i \leq B_i \circ X^{-1} \quad (i \in I), \quad X \leq N, \tag{wl2.1}$$

$$A_i \circ X \leq X \circ B_i \quad (i \in I), \quad X \leq N. \tag{wl2.2}$$

Besides, there are four systems obtained by combinations of the previous two systems (for  $X$  and  $X^{-1}$ )

$$X^{-1} \circ A_i \leq B_i \circ X^{-1} \quad X \circ B_i \leq A_i \circ X \quad (i \in I), \quad X \leq N, \tag{wl2.3}$$

$$A_i \circ X \leq X \circ B_i \quad B_i \circ X^{-1} \leq X^{-1} \circ A_i \quad (i \in I), \quad X \leq N, \tag{wl2.4}$$

$$A_i \circ X = X \circ B_i \quad (i \in I), \quad X \leq N, \tag{wl2.5}$$

$$X^{-1} \circ A_i = B_i \circ X^{-1} \quad (i \in I), \quad X \leq N. \tag{wl2.6}$$

Systems (wl2.1)–(wl2.6) are called *heterogeneous weakly linear systems*. For the sake of convenience, for each  $t \in \{1, \dots, 6\}$ , system (wl2.t) is denoted by  $WL^{2.t}(U, V, I, A_i, B_i, N)$ . If  $N(a, b) = 1$ , for all  $a \in U$  and  $b \in V$ , then system (wl2.t) is denoted simply by  $WL^{2.t}(U, V, I, A_i, B_i)$ .

Note first that some of the above introduced systems are mutually dual. For instance, (wl1.1) and (wl1.2) are dual, in the sense that a fuzzy relation  $R$  is a solution to  $WL^{1.1}(U, I, A_i, M)$  if and only if its inverse  $R^{-1}$  is a solution to  $WL^{1.2}(U, I, A_i^{-1}, M^{-1})$ . Similarly, dual pairs are systems (wl1.4) and (wl1.5), (wl2.1) and (wl2.2), (wl2.3) and (wl2.4), as well as (wl2.5) and (wl2.6). The duality means that for any universally valid statement concerning one of the systems in a dual pair there is the corresponding universally valid statement concerning another system in this dual pair, and vice versa. For this reason, we deal mainly with one of the systems which form a dual pair.

It is easy to see that every weakly linear system, either homogeneous or heterogeneous, has at least one solution, the empty relation on  $U$ , or the empty relation between  $U$  and  $V$ . These solutions will be called

trivial solutions. Moreover, if the relation  $M$  is reflexive, then the equality relation on  $U$  is also a solution to any homogeneous weakly linear system.

An important feature of weakly linear systems is that the sets of their solutions are closed under composition and arbitrary joins. More formally, the following is true.

**Proposition 4.1 ([76]).** For arbitrary fuzzy relations  $R_1, R_2, R_\alpha \in \mathcal{R}(U)$  ( $\alpha \in Y$ ) and  $t \in \{1, \dots, 6\}$  we have

- (a) If  $R_1$  and  $R_2$  are respectively solutions to systems  $WL^{1,t}(U, I, A_i, M_1)$  and  $WL^{1,t}(U, I, A_i, M_2)$ , then  $R_1 \circ R_2$  is a solution to the system  $WL^{1,t}(U, I, A_i, M_1 \circ M_2)$ .
- (b) If  $R_\alpha$  is a solution to the system  $WL^{1,t}(U, I, A_i, M)$ , for each  $\alpha \in Y$ , then  $\bigvee_{\alpha \in Y} R_\alpha$  is also a solution to the system  $WL^{1,t}(U, I, A_i, M)$ .

**Proposition 4.2 ([78]).** For arbitrary fuzzy relations  $R_1, R_\alpha \in \mathcal{R}(U, V)$  ( $\alpha \in Y$ ) and  $R_2 \in \mathcal{R}(V, W)$ , and an arbitrary  $t \in \{1, \dots, 6\}$ , we have

- (a) If  $R_1$  and  $R_2$  are respectively solutions to systems  $WL^{2,t}(U, V, I, A_i, B_i, N_1)$  and  $WL^{2,t}(V, W, I, B_i, C_i, N_2)$ , then  $R_1 \circ R_2$  is a solution to the system  $WL^{2,t}(U, W, I, A_i, C_i, N_1 \circ N_2)$ .
- (b) If  $R_\alpha$  is a solution to the system  $WL^{2,t}(U, V, I, A_i, B_i, M)$ , for each  $\alpha \in Y$ , then  $\bigvee_{\alpha \in Y} R_\alpha$  is also a solution to the system  $WL^{2,t}(U, V, I, A_i, B_i, M)$ .

According to the second statements in Propositions 4.1 and 4.2, every weakly linear system has the greatest solution, the join of all its solutions. Evidently, the greatest solution to a weakly linear system may be the empty relation, but if the given fuzzy relation  $M$  in a homogeneous weakly linear system is reflexive, then the greatest solution to this system must contain the equality relation, and hence, it must be reflexive. Besides, by the first statements in Propositions 4.1 and 4.2 we obtain the following.

**Theorem 4.3 ([76]).** All homogeneous weakly linear systems (wl1.1)–(wl1.6) have the greatest solutions. In addition, the following is true:

- (a) If  $M$  is a reflexive fuzzy relation, then the greatest solutions to (wl1.1)–(wl1.6) are also reflexive fuzzy relations.
- (b) If  $M$  is a fuzzy quasi-order, then the greatest solutions to (wl1.1)–(wl1.3) are also fuzzy quasi-orders.
- (c) If  $M$  is a fuzzy equivalence, then the greatest solutions to (wl1.4)–(wl1.6) are also fuzzy equivalences.

**Theorem 4.4 ([78]).** All heterogeneous weakly linear systems (wl2.1)–(wl2.6) have the greatest solutions.

In addition, if  $N$  is a partial fuzzy function, then the greatest solutions to systems (wl2.3) and (wl2.4) are also partial fuzzy functions.

As we have proved that weakly linear systems have the greatest solutions, we naturally come to the question how to compute these greatest solutions. For this purpose, it is convenient to represent these systems in another equivalent form, which will be done in the sequel.

We start with the homogeneous case. Let  $U$  be a non-empty set, let  $\{A_i\}_{i \in I}$  be a family of fuzzy relations on  $U$ , and let  $M$  be a fuzzy relation on  $U$ . Define functions  $\phi^{(1,t)} : \mathcal{R}(U) \rightarrow \mathcal{R}(U)$ , for  $1 \leq t \leq 6$ , as follows:

$$\phi^{(1.1)}(R) = \bigwedge_{i \in I} (A_i \circ R) / A_i \tag{42}$$

$$\phi^{(1.2)}(R) = \bigwedge_{i \in I} A_i \setminus (R \circ A_i) \tag{43}$$

$$\phi^{(1.3)}(R) = \bigwedge_{i \in I} [(A_i \circ R) / A_i] \wedge [A_i \setminus (R \circ A_i)] = \phi^{(1.1)}(R) \wedge \phi^{(1.2)}(R) \tag{44}$$

$$\phi^{(1.4)}(R) = \bigwedge_{i \in I} [(A_i \circ R) / A_i] \wedge [(A_i \circ R^{-1}) / A_i]^{-1} = \phi^{(1.1)}(R) \wedge [\phi^{(1.1)}(R^{-1})]^{-1} \tag{45}$$

$$\phi^{(1.5)}(R) = \bigwedge_{i \in I} [A_i \setminus (R \circ A_i)] \wedge [A_i \setminus (R^{-1} \circ A_i)]^{-1} = \phi^{(1.2)}(R) \wedge [\phi^{(1.2)}(R^{-1})]^{-1} \tag{46}$$

$$\begin{aligned} \phi^{(1.6)}(R) &= \bigwedge_{i \in I} [(A_i \circ R)/A_i] \wedge [(A_i \circ R^{-1})/A_i]^{-1} \wedge [A_i \setminus (R \circ A_i)] \wedge [A_i \setminus (R^{-1} \circ A_i)]^{-1} \\ &= \phi^{(1.4)}(R) \wedge [\phi^{(1.5)}(R)] = \phi^{(1.1)}(R) \wedge [\phi^{(1.1)}(R^{-1})]^{-1} \wedge \phi^{(1.2)}(R) \wedge [\phi^{(1.2)}(R^{-1})]^{-1} \end{aligned} \quad (47)$$

for each  $R \in \mathcal{R}(U)$ .

Using these functions we can represent systems (w1.1)–(w1.6) in the following way.

**Theorem 4.5 ([76]).** For every  $t \in \{1, \dots, 6\}$ , system (w1.t) is equivalent to system

$$X \leq \phi^{(1.t)}(X), \quad X \leq M^{(t)}, \quad (48)$$

where  $M^{(t)} = M$ , for  $t \in \{1, 2, 3\}$ , and  $M^{(t)} = M \wedge M^{-1}$ , for  $t \in \{4, 5, 6\}$ .

In the heterogeneous case we consider two possible different non-empty sets  $U$  and  $V$ , a family  $\{A_i\}_{i \in I}$  of fuzzy relations on  $U$ , a family  $\{B_i\}_{i \in I}$  of fuzzy relations on  $V$ , and a fuzzy relation  $N$  between  $U$  and  $V$ . Define functions  $\phi^{(2.t)} : \mathcal{R}(U, V) \rightarrow \mathcal{R}(U, V)$ , for  $1 \leq t \leq 6$ , as follows:

$$\phi^{(2.1)}(R) = \bigwedge_{i \in I} [(B_i \circ R^{-1})/A_i]^{-1} \quad (49)$$

$$\phi^{(2.2)}(R) = \bigwedge_{i \in I} A_i \setminus (R \circ B_i) \quad (50)$$

$$\phi^{(2.3)}(R) = \bigwedge_{i \in I} [(B_i \circ R^{-1})/A_i]^{-1} \wedge [(A_i \circ R)/B_i] = \phi^{(2.1)}(R) \wedge [\phi^{(2.1)}(R^{-1})]^{-1} \quad (51)$$

$$\phi^{(2.4)}(R) = \bigwedge_{i \in I} [A_i \setminus (R \circ B_i)] \wedge [B_i \setminus (R^{-1} \circ A_i)]^{-1} = \phi^{(2.2)}(R) \wedge [\phi^{(2.2)}(R^{-1})]^{-1} \quad (52)$$

$$\phi^{(2.5)}(R) = \bigwedge_{i \in I} [A_i \setminus (R \circ B_i)] \wedge [(A_i \circ R)/B_i] = \phi^{(2.2)}(R) \wedge [\phi^{(2.1)}(R^{-1})]^{-1} \quad (53)$$

$$\phi^{(2.6)}(R) = \bigwedge_{i \in I} [(B_i \circ R^{-1})/A_i]^{-1} \wedge [B_i \setminus (R^{-1} \circ A_i)]^{-1} = \phi^{(2.1)}(R) \wedge [\phi^{(2.2)}(R^{-1})]^{-1} \quad (54)$$

for each  $R \in \mathcal{R}(U, V)$ . Notice that in the expression “ $\phi^{(2.t)}(R^{-1})$ ” ( $t \in \{1, 2\}$ ) we denote by  $\phi^{(2.t)}$  a function from  $\mathcal{R}(V, U)$  into itself.

Now we represent systems (w2.1)–(w2.6) in the following equivalent form.

**Theorem 4.6 ([78]).** For every  $t \in \{1, \dots, 6\}$ , system (w.t) is equivalent to system

$$X \leq \phi^{(2.t)}(X), \quad X \leq N. \quad (55)$$

### 5. Computation of the greatest solutions

Equivalent forms of weakly linear systems provided in Theorems 4.5 and 4.6 suggest to reduce the problem of computing the greatest solutions to weakly linear systems to the problem of computing the greatest post-fixed points of the functions  $\phi^{(s.t)}$  contained in given fuzzy relations. Readers should be warned that “computation” means that we already know how to compute the composition of two relations and count this as a single step. However, this may be quite expensive, e.g., in continuous case.

Let  $\mathcal{R}$  denote either the lattice  $\mathcal{R}(U, V)$  of fuzzy relations between non-empty sets  $U$  and  $V$ , or the lattice  $\mathcal{R}(U)$  of fuzzy relations on a non-empty set  $U$ , and let  $\phi : \mathcal{R} \rightarrow \mathcal{R}$  be an isotone function. A fuzzy relation  $R \in \mathcal{R}$  is called a *post-fixed point* of  $\phi$  if  $R \leq \phi(R)$ . The well-known Knaster-Tarski fixed point theorem (stated and proved in a more general context, for complete lattices) asserts that the set of all post-fixed points of  $\phi$  form a complete lattice (cf. [121]). Moreover, for any fuzzy relation  $H \in \mathcal{R}$  we have that the set of all post-fixed points of  $\phi$  contained in  $H$  is non-empty, because it always contains the least element of  $\mathcal{R}$  (the empty relation), and it is also a complete lattice. According to Theorems 4.5 and 4.6, our main task is to find

an effective procedure for computing the greatest post-fixed point of the function  $\phi^{(s,t)}$  contained in the corresponding fuzzy relation, for all  $s \in \{1, 2\}$  and  $t \in \{1, \dots, 6\}$ .

Let  $\phi : \mathcal{R} \rightarrow \mathcal{R}$  be an isotone function and  $H \in \mathcal{R}$ . Define a sequence  $\{R_k\}_{k \in \mathbb{N}}$  of fuzzy relations from  $\mathcal{R}$  by:

$$R_1 = H, \quad R_{k+1} = R_k \wedge \phi(R_k), \text{ for each } k \in \mathbb{N}. \tag{56}$$

The sequence  $\{R_k\}_{k \in \mathbb{N}}$  is obviously descending. If we denote by  $\hat{R}$  the greatest post-fixed point of  $\phi$  contained in  $H$ , we can easily verify that

$$\hat{R} \leq \bigwedge_{k \in \mathbb{N}} R_k. \tag{57}$$

Now two very important questions arise. First, under what conditions the equality holds in (57)? Even more important question is: under what conditions the sequence  $\{R_k\}_{k \in \mathbb{N}}$  is finite? If this sequence is finite, then it is not hard to show that there exists  $k \in \mathbb{N}$  such that  $R_k = R_m$ , for every  $m \geq k$ , i.e., there exists  $k \in \mathbb{N}$  such that the sequence stabilizes on  $R_k$ . We can recognize that the sequence has stabilized when we find the smallest  $k \in \mathbb{N}$  such that  $R_k = R_{k+1}$ . In this case  $\hat{R} = R_k$ , and we have an algorithm which computes  $\hat{R}$  in a finite number of steps.

Some conditions under which equality holds in (57) or the sequence is finite were found in [76], in the case which considers fuzzy relations on a single set, and in [78] it was noted that the same results are also valid when fuzzy relations between two sets are considered.

A sequence  $\{R_k\}_{k \in \mathbb{N}}$  of fuzzy relations from  $\mathcal{R}$  is called *image-finite* if the set  $\bigcup_{k \in \mathbb{N}} \text{Im}(R_k)$  is finite, and it can be easily shown that this sequence is image-finite if and only if it is finite. Furthermore, the function  $\phi : \mathcal{R} \rightarrow \mathcal{R}$  is called *image-localized* if there exists a finite subset  $K \subseteq L$  such that for any fuzzy relation  $R \in \mathcal{R}$  we have

$$\text{Im}(\phi(R)) \subseteq \langle K \cup \text{Im}(R) \rangle, \tag{58}$$

where  $\langle K \cup \text{Im}(R) \rangle$  denotes the subalgebra of  $\mathcal{L}$  generated by the set  $K \cup \text{Im}(R)$ . Such  $K$  will be called a *localization set* of the function  $\phi$ .

**Theorem 5.1 ([76]).** *Let the function  $\phi$  be image-localized, let  $K$  be its localization set, let  $H \in \mathcal{R}$ , and let  $\{R_k\}_{k \in \mathbb{N}}$  be a sequence of fuzzy relations in  $\mathcal{R}$  defined by (56). Then*

$$\bigcup_{k \in \mathbb{N}} \text{Im}(R_k) \subseteq \langle K \cup \text{Im}(H) \rangle. \tag{59}$$

*If, moreover,  $\langle K \cup \text{Im}(H) \rangle$  is a finite subalgebra of  $\mathcal{L}$ , then the sequence  $\{R_k\}_{k \in \mathbb{N}}$  is finite.*

Further we consider  $\phi^{(s,t)}$ , for  $s \in \{1, 2\}$  and  $t \in \{1, \dots, 6\}$ , defined in (42)–(47) and (49)–(54). We have the following.

**Theorem 5.2 ([76, 78]).** *All functions  $\phi^{(s,t)}$ , for  $s \in \{1, 2\}$  and  $t \in \{1, \dots, 6\}$ , are isotone.*

*If  $U$  and  $I$ , resp.  $U, V$  and  $I$ , are finite sets, then all functions  $\phi^{(1,t)}$ , resp.  $\phi^{(2,t)}$ , are image-localized.*

**Theorem 5.3 ([76, 78]).** *Let  $U$  and  $I$ , resp.  $U, V$  and  $I$ , be finite sets, let  $\phi = \phi^{(1,t)}$ , resp.  $\phi = \phi^{(2,t)}$ , for some  $t \in \{1, \dots, 6\}$ , and let  $\{R_k\}_{k \in \mathbb{N}}$  be the sequence of fuzzy relations from  $\mathcal{R}$  defined by (56).*

*If  $\langle \text{Im}(H) \cup \bigcup_{i \in I} \text{Im}(A_i) \rangle$ , resp.  $\langle \text{Im}(H) \cup \bigcup_{i \in I} (\text{Im}(A_i) \cup \text{Im}(B_i)) \rangle$ , is a finite subalgebra of  $\mathcal{L}$ , then*

- (a) *the sequence  $\{R_k\}_{k \in \mathbb{N}}$  is finite and descending, and there is the least natural number  $k$  such that  $R_k = R_{k+1}$ ;*
- (b)  *$R_k$  is the greatest solution to system (wl1.t), resp. (wl2.t).*



Next, let  $\mathcal{L} = (L, \wedge, \vee, \otimes, \rightarrow, 0, 1)$  be a complete residuated lattice satisfying the following conditions:

$$x \vee \left( \bigwedge_{i \in I} y_i \right) = \bigwedge_{i \in I} (x \vee y_i), \tag{60}$$

$$x \otimes \left( \bigwedge_{i \in I} y_i \right) = \bigwedge_{i \in I} (x \otimes y_i), \tag{61}$$

for all  $x \in L$  and  $\{y_i\}_{i \in I} \subseteq L$ . Let us note that if  $\mathcal{L} = ([0, 1], \wedge, \vee, \otimes, \rightarrow, 0, 1)$ , where  $[0, 1]$  is the real unit interval and  $\otimes$  is a left-continuous t-norm on  $[0, 1]$ , then (60) follows immediately by linearity of  $\mathcal{L}$ , and  $\mathcal{L}$  satisfies (61) if and only if  $\otimes$  is a continuous t-norm, i.e., if and only if  $\mathcal{L}$  is a BL-algebra (cf. [9, 10]). Therefore, conditions (60) and (61) hold for every BL-algebra on the real unit interval. In particular, the Łukasiewicz, Goguen (product) and Gödel structures fulfill (60) and (61).

Under these conditions we have the following.

**Theorem 5.4 ([76, 78]).** *Let  $\phi = \phi^{(s,t)}$ , for some  $s \in \{1, 2\}$  and  $t \in \{1, \dots, 6\}$ , let  $\{R_k\}_{k \in \mathbb{N}}$  be the sequence of fuzzy relations from  $\mathcal{R}$  defined by (56), and let  $\mathcal{L}$  be a complete residuated lattice satisfying (60) and (61).*

*Then the fuzzy relation*

$$R = \bigwedge_{k \in \mathbb{N}} R_k,$$

*is the greatest solution to system (wls.t).*

It should be noted that the previous theorem gives a characterization of the greatest solution to system (wls.t), but does not provide an algorithm for computing this greatest solution because it is represented as an intersection of infinitely many fuzzy relations. Since the sequence  $\{R_k\}_{k \in \mathbb{N}}$  is descending, computing finitely many members of this sequence and their intersection we can get a fuzzy relation that is close enough to the greatest solution to system (wls.t), but this fuzzy relation is not a solution to (wls.t) if the sequence  $\{R_k\}_{k \in \mathbb{N}}$  is infinite.

In some situations we do not need solutions to systems of fuzzy relation equations and inequalities that are fuzzy relations, but those that are ordinary crisp relations. On the other hand, in cases when our algorithms for computing the greatest solutions to weakly linear systems fail to terminate in a finite number of steps, it is reasonable to search for the greatest crisp solutions to these systems. They can be understood as some kind of “approximations” of the greatest fuzzy solutions. It has been shown in [76, 78] that algorithms for computing the greatest fuzzy solutions to weakly linear systems can be modified to compute the greatest crisp solutions to these systems. This method will be presented in the sequel.

Let  $\mathcal{R}^c$  denote the set of all crisp relations from  $\mathcal{R}$ . It is easy to verify that  $\mathcal{R}^c$  is a complete sublattice of  $\mathcal{R}$ , i.e., the meet and the join in  $\mathcal{R}$  of an arbitrary family of crisp relations from  $\mathcal{R}^c$  are also crisp relations (in fact, they coincide with the ordinary intersection and union of crisp relations). Moreover, for any fuzzy relation  $R \in \mathcal{R}$  we have that  $R^c \in \mathcal{R}^c$ , where  $R^c$  denotes the *crisp part* of a fuzzy relation  $R$  (in some sources called the *kernel* of  $R$ ), i.e., a function  $R^c : U \times U \rightarrow \{0, 1\}$  (resp.  $R^c : U \times V \rightarrow \{0, 1\}$ ) defined by  $R^c(a, b) = 1$ , if  $R(a, b) = 1$ , and  $R^c(a, b) = 0$ , if  $R(a, b) < 1$ , for all  $a, b \in U$  (resp.  $a \in U$  and  $b \in V$ ). Equivalently,  $R^c$  is considered as an ordinary crisp relation given by  $R^c = \{(a, b) \mid R(a, b) = 1\}$ .

For each function  $\phi : \mathcal{R} \rightarrow \mathcal{R}$  we define a function  $\phi^c : \mathcal{R}^c \rightarrow \mathcal{R}^c$  by

$$\phi^c(R) = (\phi(R))^c, \text{ for any } R \in \mathcal{R}^c.$$

If  $\phi$  is isotone, then it can be easily shown that  $\phi^c$  is also an isotone function.

We have that the following is true.

**Theorem 5.5 ([76, 78]).** *Let  $\phi : \mathcal{R} \rightarrow \mathcal{R}$  be an isotone function and let  $H \in \mathcal{R}$  be a given fuzzy relation. A crisp relation  $\varrho \in \mathcal{R}^c$  is the greatest crisp solution in  $\mathcal{R}$  to the system*

$$X \leq \phi(X), \quad X \leq H, \tag{62}$$

if and only if it is the greatest solution in  $\mathcal{R}^c$  to the system

$$\xi \leq \phi^c(\xi), \quad \xi \leq H^c, \tag{63}$$

where  $X$  is an unknown fuzzy relation and  $\xi$  is an unknown crisp relation.

Furthermore, a sequence  $\{\varrho_k\}_{k \in \mathbb{N}} \subseteq \mathcal{R}$  defined by

$$\varrho_1 = H^c, \quad \varrho_{k+1} = \varrho_k \wedge \phi^c(\varrho_k), \text{ for every } k \in \mathbb{N}, \tag{64}$$

is a finite descending sequence of crisp relations, and the least member of this sequence is the greatest solution to the system (63) in  $\mathcal{R}^c$ .

Taking  $\phi$  to be any of the functions  $\phi^{(s,t)}$ , for  $s \in \{1, 2\}$  and  $t \in \{1, \dots, 6\}$ , Theorem 5.5 gives algorithms for computing the greatest crisp solutions to weakly linear systems. As we have seen in Theorem 5.5, these algorithms always terminate in a finite number of steps, independently on the properties of the underlying structure of truth values, and they could be used in cases when algorithms for computing the greatest fuzzy solutions do not terminate in a finite number of steps. However, examples provided in [76, 78]) show that there are cases when homogeneous weakly linear systems have non-trivial fuzzy solutions (different than the equality relation), but they do not have non-trivial crisp solutions, and there are cases when heterogeneous weakly linear systems have non-empty fuzzy solutions, but they do not have non-empty crisp solutions.

### 6. Quotient fuzzy relational systems

In this section we talk about fuzzy relational systems and present some results which are analogues of some well-known theorems of universal algebra (homomorphism, isomorphism, correspondence theorems, etc.). Then we establish relationships between solutions to heterogeneous and homogeneous weakly linear systems.

Loosely speaking, a *relational system* is a pair  $(U, \mathcal{R})$  consisting of a non-empty set  $A$  and a non-empty family  $\mathcal{R}$  of finitary relations on  $U$  which may have different arities. Two relational systems  $(U, \mathcal{R}_1)$  and  $(V, \mathcal{R}_2)$  are considered to be of the same type if a bijective function between  $\mathcal{R}_1$  and  $\mathcal{R}_2$  is given that preserves arity. When we deal only with binary relations, then relational systems  $(U, \mathcal{R}_1)$  and  $(V, \mathcal{R}_2)$  are of the same type if  $\mathcal{R}_1$  and  $\mathcal{R}_2$  can be written as  $\mathcal{R}_1 = \{A_i\}_{i \in I}$  and  $\mathcal{R}_2 = \{B_i\}_{i \in I}$ , for some non-empty index set  $I$ . In this case, the bijective function that we have mentioned above is just the function that maps  $A_i$  to  $B_i$ , for each  $i \in I$ .

Here we consider relational systems in the fuzzy context, and we work only with binary fuzzy relations. We define a *fuzzy relational system* as a pair  $\mathcal{U} = (U, \{A_i\}_{i \in I})$ , where  $U$  is a non-empty set and  $\{A_i\}_{i \in I}$  is a non-empty family of fuzzy relations on  $A$ , and by fuzzy relational systems of the same type we will mean systems of the form  $\mathcal{U} = (U, \{A_i\}_{i \in I})$  and  $\mathcal{V} = (V, \{B_i\}_{i \in I})$ . To avoid writing multiple indices, the fuzzy relational system  $\mathcal{U} = (U, \{A_i\}_{i \in I})$  will be sometimes denoted by  $\mathcal{U} = (U, I, A_i)$ . All fuzzy relational systems discussed in the sequel will be of the same type.

Let  $\mathcal{U} = (U, I, A_i)$  and  $\mathcal{V} = (V, I, B_i)$  be two fuzzy relational systems. A function  $\phi : U \rightarrow V$  is called an *isomorphism* if it is bijective and  $A_i(a_1, a_2) = B_i(\phi(a_1), \phi(a_2))$ , for all  $a_1, a_2 \in U$  and  $i \in I$ .

Let  $\mathcal{U} = (U, I, A_i)$  be a fuzzy relational system and let  $E$  be a fuzzy equivalence on  $U$ . For each  $i \in I$ , define a fuzzy relation  $A_i^{U/E}$  on the quotient (factor) set  $U/E$  as follows:

$$A_i^{U/E}(E_{a_1}, E_{a_2}) = (E \circ A_i \circ E)(a_1, a_2), \tag{65}$$

for all  $a_1, a_2 \in U$ . The right side of (65) can be equivalently written as

$$(E \circ A_i \circ E)(a_1, a_2) = \bigvee_{a'_1, a'_2 \in A} E(a_1, a'_1) \otimes A_i(a'_1, a'_2) \otimes E(a'_2, a_2) = E_{a_1} \circ A_i \circ E_{a_2},$$

and for all  $a_1, a_2, a'_1, a'_2 \in U$  such that  $E_{a_1} = E_{a'_1}$  and  $E_{a_2} = E_{a'_2}$  we have that  $(E \circ A_i \circ E)(a_1, a_2) = (E \circ A_i \circ E)(a'_1, a'_2)$ . Therefore, the fuzzy relation  $A_i^{U/E}$  is well-defined, and  $\mathcal{U}/E = (U/E, I, A_i^{U/E})$  is a fuzzy relational system

of the same type as  $\mathcal{U}$ , which is called the *quotient* (or *factor*) *fuzzy relational system* of  $\mathcal{U}$ , with respect to the fuzzy equivalence  $E$ .

Note that this concept of quotient fuzzy relational system emerges from the theory of fuzzy automata, namely, it originates from the concept of a factor (quotient) fuzzy automaton. Factor fuzzy automata were introduced in [40, 41], where they were used to reduce the number of states of fuzzy automata. We will see in Section 7 that quotient (fuzzy) relational systems can be also used to reduce the number of nodes of a (fuzzy) network, while keeping the basic structure of the network. It is also worth noting that quotient crisp relational systems have been recently defined in the same way in [27].

The following theorem can be conceived as an analogue of the well-known theorems of universal algebra which establish correspondences between functions and equivalence relations, as well as between homomorphisms and congruences (cf. [24, §2.6]).

**Theorem 6.1 ([78]).** *Let  $\mathcal{U} = (U, I, A_i)$  be a fuzzy relational system, let  $E$  be a fuzzy equivalence on  $U$ , and let  $\mathcal{U}/E = (U/E, I, A_i^{U/E})$  the quotient fuzzy relational system of  $\mathcal{U}$  with respect to  $E$ .*

*Then a fuzzy relation  $E^\natural \in \mathcal{R}(U, U/E)$  defined by*

$$E^\natural(a_1, E_{a_2}) = E(a_1, a_2), \quad \text{for all } a_1, a_2 \in U, \tag{66}$$

*is a uniform  $F$ -function whose kernel is  $E$ .*

*Moreover,  $E^\natural$  is a solution both to  $WL^{2.1}(U, U/E, I, A_i, A_i^{U/E})$  and  $WL^{2.2}(U, U/E, I, A_i, A_i^{U/E})$ .*

We also have the following.

**Theorem 6.2 ([78]).** *Let  $\mathcal{U} = (U, I, A_i)$  be a fuzzy relational system, let  $E$  be a fuzzy equivalence on  $U$ , and let  $\mathcal{U} = (U/E, I, A_i^{U/E})$  the quotient fuzzy relational system of  $\mathcal{U}$  with respect to  $E$ . Then the following conditions are equivalent:*

- (i)  $E$  is a solution to  $WL^{1.4}(U, I, A_i)$ ;
- (ii)  $E^\natural$  is a solution to  $WL^{2.3}(U, U/E, I, A_i, A_i^{U/E})$ ;
- (iii)  $E^\natural$  is a solution to  $WL^{2.5}(U, U/E, I, A_i, A_i^{U/E})$ .

The next theorem can be conceived as an analogue of the well-known Second Isomorphism Theorem from universal algebra (cf. [24, §2.6]).

**Theorem 6.3 ([78]).** *Let  $\mathcal{U} = (U, I, A_i)$  be a fuzzy relational system, let  $E$  and  $F$  be fuzzy equivalences on  $U$  such that  $E \leq F$ , and let  $\mathcal{U}/E = (U/E, I, A_i^{U/E})$  be the quotient fuzzy relational system of  $\mathcal{U}$  with respect to  $E$ . Then a fuzzy relation  $F/E$  on  $U/E$  defined by*

$$F/E(E_{a_1}, E_{a_2}) = F(a_1, a_2), \quad \text{for all } a_1, a_2 \in U, \tag{67}$$

*is a fuzzy equivalence on  $U/E$ , and the quotient fuzzy relational systems  $(\mathcal{U}/E)/(F/E)$  and  $\mathcal{U}/F$  are isomorphic.*

We also have an analogue of the Correspondence Theorem from universal algebra (cf. [24, §2.6]).

**Theorem 6.4 ([78]).** *Let  $\mathcal{U} = (U, I, A_i)$  be a fuzzy relational system and let  $E$  be fuzzy equivalence on  $U$ . The function  $\Phi : \mathcal{E}_E(U) \rightarrow \mathcal{E}(U/E)$ , where  $\mathcal{E}_E(U) = \{F \in \mathcal{E}(U) \mid E \subseteq F\}$ , defined by*

$$\Phi(F) = F/E, \quad \text{for all } F \in \mathcal{E}_E(U), \tag{68}$$

*is an order embedding of  $\mathcal{E}_E(U)$  into  $\mathcal{E}(U/E)$ , i.e.,*

$$F \leq G \Leftrightarrow \Phi(F) \leq \Phi(G), \quad \text{for all } F, G \in \mathcal{E}_E(U). \tag{69}$$

It is worth noting that in the case of Boolean (crisp) relational systems  $\Phi$  is also surjective, which means that it is an order isomorphism, and equivalently, a lattice isomorphism of  $\mathcal{E}_E(U)$  onto  $\mathcal{E}(U/E)$ . In the case of fuzzy relational systems we are not able to prove that fact, but this is not so important because in practice we usually use just the fact that  $\Phi$  is an order embedding.

**Theorem 6.5 ([78]).** Let  $\mathcal{U} = (U, I, A_i)$  be a fuzzy relational system, let  $E$  and  $F$  be fuzzy equivalences on  $U$  such that  $E \leq F$ , and let  $\mathcal{U}/E = (U/E, I, A_i^{U/E})$  be the quotient fuzzy relational system of  $\mathcal{U}$  with respect to  $E$ .

A fuzzy relation  $F_E \in \mathcal{R}(U, U/E)$  defined by

$$F_E(a_1, E_{a_2}) = F(a_1, a_2), \quad \text{for all } a_1, a_2 \in U, \tag{70}$$

is a uniform fuzzy relation with the kernel  $F$  and the co-kernel  $F/E$ .

In addition, if  $E$  is a solution to  $WL^{1.4}(U, I, A_i, M)$ , for some  $M \in \mathcal{R}(U)$ , then the following is true:

- (a)  $F$  is a solution to  $WL^{1.4}(U, I, A_i, M)$  if and only if  $F/E$  is a solution to  $WL^{1.4}(U/E, I, A_i^{U/E}, M/E)$ .
- (b)  $F$  is the greatest solution to system  $WL^{1.4}(U, I, A_i, M)$  if and only if  $F/E$  is the greatest solution to system  $WL^{1.4}(U/E, I, A_i^{U/E}, M/E)$ .
- (c)  $F$  is a solution to  $WL^{1.4}(U, I, A_i, M)$  if and only if  $F_E$  is a solution to  $WL^{2.3}(U, U/E, I, A_i, A_i^{U/E}, M_E)$ .

Using the above presented results on quotient fuzzy relational systems interesting relationships between solutions to heterogeneous and homogeneous weakly linear systems can be determined.

First, we have that the following is true.

**Theorem 6.6 ([78]).** Let a fuzzy relation  $R \in \mathcal{R}(U, V)$  be a solution to system  $WL^{2.3}(U, V, I, A_i, B_i, N)$ . Then

- (a)  $R \circ R^{-1}$  is a solution to system  $WL^{1.4}(U, I, A_i, N \circ N^{-1})$ ;
- (b)  $R^{-1} \circ R$  is a solution to system  $WL^{1.4}(V, I, B_i, N^{-1} \circ N)$ .

In the previous theorem we have considered the solution to system (w/2.3) which is an arbitrary fuzzy relation. In the next two theorems we deal with solutions to this system which are uniform fuzzy relations. We will see that the kernel and the co-kernel of a uniform solution to a heterogeneous weakly linear system are solutions to related homogeneous systems, and that such a connection also exists between the greatest solutions to a heterogeneous system and the related homogeneous systems.

**Theorem 6.7 ([78]).** Let  $R \in \mathcal{R}(U, V)$  be a uniform fuzzy relation and  $N \in \mathcal{R}(U, V)$  a fuzzy relation such that  $R \leq N$ . Then  $R$  is a solution to system  $WL^{2.3}(U, V, I, A_i, B_i, N)$  if and only if the following is true:

- (i)  $E_U^R$  is a solution to system  $WL^{1.4}(U, I, A_i, N \circ N^{-1})$ ;
- (ii)  $E_V^R$  is a solution to system  $WL^{1.4}(V, I, B_i, N^{-1} \circ N)$ ;
- (iii)  $\tilde{R}$  is an isomorphism of quotient fuzzy relational systems  $\mathcal{U}/E_U^R$  and  $\mathcal{V}/E_V^R$ ;

where  $\mathcal{U} = (U, I, A_i)$  and  $\mathcal{V} = (V, I, B_i)$ .

A natural question which arises here is the relationship between the greatest solution to a heterogeneous weakly linear system and the greatest solutions to the corresponding homogeneous weakly linear systems. The following theorem gives an answer to this question.

**Theorem 6.8 ([78]).** Let  $N \in \mathcal{R}(U, V)$  be a uniform fuzzy relation and let system  $WL^{2.3}(U, V, I, A_i, B_i, N)$  have a uniform solution.

Then the greatest solution  $R$  to  $WL^{2.3}(U, V, I, A_i, B_i, N)$  is a uniform fuzzy relation such that  $E_U^R$  is the greatest solution to  $WL^{1.4}(U, I, A_i, N \circ N^{-1})$  and  $E_V^R$  is the greatest solution to  $WL^{1.4}(V, I, B_i, N^{-1} \circ N)$ .

A result similar to Theorem 6.7 can be also obtained for system (w/2.5).

**Theorem 6.9 ([78]).** Let  $R \in \mathcal{R}(U, V)$  be a uniform fuzzy relation and  $N \in \mathcal{R}(U, V)$  a fuzzy relation such that  $R \leq N$ . Then  $R$  is a solution to system  $WL^{2.5}(U, V, I, A_i, B_i, N)$  if and only if the following is true:

- (i)  $E_U^R$  is a solution to system  $WL^{1.4}(U, I, A_i, N \circ N^{-1})$ ;
- (ii)  $E_V^R$  is a solution to system  $WL^{1.5}(V, I, B_i, N^{-1} \circ N)$ ;
- (iii)  $\tilde{R}$  is an isomorphism of quotient fuzzy relational systems  $\mathcal{U}/E_U^R$  and  $\mathcal{V}/E_V^R$ ;

where  $\mathcal{U} = (U, I, A_i)$  and  $\mathcal{V} = (V, I, B_i)$ .

It is an open question whether the analogue of Theorem 6.8 is valid for the system (wl2.5). The methodology used in Theorem 6.8 does not give results when it works with this system.

## 7. Some applications

### 7.1. Fuzzy automata

In this section, if not noted otherwise, let  $\mathcal{L}$  be a complete residuated lattice. A *fuzzy automaton* over  $\mathcal{L}$ , or simply a *fuzzy automaton*, is a quintuple  $\mathcal{A} = (A, X, \delta^A, \sigma^A, \tau^A)$ , where  $A$  and  $X$  are non-empty sets, called respectively the *set of states* and the *input alphabet*,  $\delta^A : A \times X \times A \rightarrow L$  is a fuzzy subset of  $A \times X \times A$ , called the *fuzzy transition function*, and  $\sigma^A : A \rightarrow L$  and  $\tau^A : A \rightarrow L$  are fuzzy subsets of  $A$ , called the *fuzzy set of initial states* and the *fuzzy set of terminal (final) states*, respectively. We can interpret  $\delta^A(a, x, b)$  as the degree to which an input letter  $x \in X$  causes a transition from a state  $a \in A$  into a state  $b \in A$ , whereas we can interpret  $\sigma^A(a)$  and  $\tau^A(a)$  as the degrees to which  $a$  is respectively an input state and a terminal state. Sometimes we disregard fuzzy sets of initial and terminal states and deal only with a triple  $\mathcal{T} = (A, X, \delta^A)$ , consisting of a set of states  $A$ , input alphabet  $X$ , and a transition function  $\delta^A$ , which is called a *fuzzy transition system*.

The set of states and the input alphabet of a fuzzy automaton or a fuzzy transition system are usually assumed to be finite sets, but for methodological reasons we sometimes allow the set of states  $A$  to be infinite. A fuzzy automaton, resp. fuzzy transition system, whose set of states is finite is called a *fuzzy finite automaton*, resp. *fuzzy finite transition system*. Here, there is no reason to consider fuzzy automata and fuzzy transition systems with infinite sets of states, and when we say fuzzy automaton or fuzzy transition system we will mean that its set of states is finite.

Let  $X^*$  denote the free monoid over the alphabet  $X$ , let  $\varepsilon \in X^*$  denote the empty word, and  $X^+ = X^* \setminus \{\varepsilon\}$ . The function  $\delta^A$  can be extended up to a function  $\delta_*^A : A \times X^* \times A \rightarrow L$  as follows: If  $a, b \in A$ , then

$$\delta_*^A(a, \varepsilon, b) = \begin{cases} 1, & \text{if } a = b, \\ 0, & \text{otherwise,} \end{cases} \tag{71}$$

and if  $a, b \in A, u \in X^*$  and  $x \in X$ , then

$$\delta_*^A(a, ux, b) = \bigvee_{c \in A} \delta_*^A(a, u, c) \otimes \delta^A(c, x, b). \tag{72}$$

Due to distributivity of the multiplication in  $\mathcal{L}$  over joins, we have that

$$\delta_*^A(a, uv, b) = \bigvee_{c \in A} \delta_*^A(a, u, c) \otimes \delta_*^A(c, v, b), \tag{73}$$

for all  $a, b \in A$  and  $u, v \in X^*$ , i.e., if  $w = x_1 \cdots x_n$ , for  $x_1, \dots, x_n \in X$ , then

$$\delta_*^A(a, w, b) = \bigvee_{(c_1, \dots, c_{n-1}) \in A^{n-1}} \delta^A(a, x_1, c_1) \otimes \delta^A(c_1, x_2, c_2) \otimes \cdots \otimes \delta^A(c_{n-1}, x_n, b). \tag{74}$$

Intuitively, the product  $\delta^A(a, x_1, c_1) \otimes \delta^A(c_1, x_2, c_2) \otimes \cdots \otimes \delta^A(c_{n-1}, x_n, b)$  represents the degree to which the input word  $w$  causes a transition from a state  $a$  into a state  $b$  through the sequence of intermediate states

$c_1, \dots, c_{n-1} \in A$ , and  $\delta_*^A(a, w, b)$  represents the supremum of degrees of all possible transitions from  $a$  into  $b$  caused by  $w$ . Also, we can visualize a fuzzy finite automaton  $\mathcal{A}$  representing it as a labelled directed graph whose nodes are states of  $\mathcal{A}$ , and an edge from a node  $a$  into a node  $b$  is labelled by pairs of the form  $x/\delta^A(a, x, b)$ , for any  $x \in X$ .

If  $\delta^A$  is a crisp subset of  $A \times X \times A$ , that is,  $\delta^A : A \times X \times A \rightarrow \{0, 1\}$ , and  $\sigma^A$  and  $\tau^A$  are crisp subsets of  $A$ , then  $\mathcal{A}$  is an ordinary *nondeterministic automaton*. In other words, nondeterministic automata are fuzzy automata over the Boolean structure. If  $\delta^A$  is a function of  $A \times X$  into  $A$ ,  $\sigma^A$  is a one-element crisp subset of  $A$ , that is,  $\sigma^A = \{a_0\}$ , for some  $a_0 \in A$ , and  $\tau^A$  is a fuzzy subset of  $A$ , then  $\mathcal{A}$  is called a *deterministic fuzzy automaton*, and it is denoted by  $\mathcal{A} = (A, X, \delta^A, a_0, \tau^A)$ . In [34, 54] the name *crisp-deterministic* was used. For more information on deterministic fuzzy automata we refer to [8, 73–75, 77, 85, 93]. Evidently, if  $\delta^A$  is a crisp subset of  $A \times X \times A$ , or a function of  $A \times X$  into  $A$ , then  $\delta_*^A$  is also a crisp subset of  $A \times X^* \times A$ , or a function of  $A \times X^*$  into  $A$ , respectively. A deterministic fuzzy automaton  $\mathcal{A} = (A, X, \delta^A, a_0, \tau^A)$ , where  $\tau^A$  is a crisp subset of  $A$ , is an ordinary deterministic automaton.

If for any  $u \in X^*$  we define a fuzzy relation  $\delta_u^A$  on  $A$  by

$$\delta_u^A(a, b) = \delta_*^A(a, u, b), \tag{75}$$

for all  $a, b \in A$ , called the *fuzzy transition relation* determined by  $u$ , then (73) can be written as

$$\delta_{uv}^A = \delta_u^A \circ \delta_v^A, \tag{76}$$

for all  $u, v \in X^*$ .

A *fuzzy language* in  $X^*$  over  $\mathcal{L}$ , or briefly a *fuzzy language*, is any fuzzy subset of  $X^*$ , i.e., any function from  $X^*$  to  $L$ . A *fuzzy language recognized by a fuzzy automaton*  $\mathcal{A} = (A, X, \delta^A, \sigma^A, \tau^A)$ , denoted as  $L(\mathcal{A})$ , is a fuzzy language in  $X^*$  defined by

$$L(\mathcal{A})(u) = \bigvee_{a, b \in A} \sigma^A(a) \otimes \delta_*^A(a, u, b) \otimes \tau^A(b), \tag{77}$$

or equivalently,

$$\begin{aligned} L(\mathcal{A})(\varepsilon) &= \sigma^A \circ \tau^A, \\ L(\mathcal{A})(u) &= \sigma^A \circ \delta_{x_1}^A \circ \delta_{x_2}^A \circ \dots \circ \delta_{x_n}^A \circ \tau^A, \end{aligned} \tag{78}$$

for any  $u = x_1x_2 \dots x_n \in X^+$ , where  $x_1, x_2, \dots, x_n \in X$ . In other words, the equality (77) means that the membership degree of the word  $u$  to the fuzzy language  $L(\mathcal{A})$  is equal to the degree to which  $\mathcal{A}$  recognizes or accepts the word  $u$ . Using notation from (6), and the second equality in (8), we can state (77) as

$$L(\mathcal{A})(u) = \sigma^A \circ \delta_u^A \circ \tau^A. \tag{79}$$

Fuzzy automata  $\mathcal{A}$  and  $\mathcal{B}$  are called *language-equivalent*, or just *equivalent*, if  $L(\mathcal{A}) = L(\mathcal{B})$ .

As we said before, the concept of quotient fuzzy relational system discussed in Section 6 comes from the concept of quotient fuzzy automaton. In fact, fuzzy relational systems are essentially the same as the fuzzy transition systems, so the concept of a quotient fuzzy relational system is identical to the concept of a quotient fuzzy transition system. This concept is the basis of the concept of a quotient fuzzy automaton, where additionally we have to define fuzzy sets of initial and terminal states on the factor set. In the sequel we give a complete definition of a quotient fuzzy automaton.

Let  $\mathcal{A} = (A, \delta^A, \sigma^A, \tau^A)$  be a fuzzy automaton and let  $E$  a fuzzy equivalence relation on  $A$ . Without any restriction on  $E$ , we can define a fuzzy transition function  $\delta^{A/E} : (A/E) \times X \times (A/E) \rightarrow L$  by

$$\delta^{A/E}(E_a, x, E_b) = \bigvee_{a', b' \in A} E(a, a') \otimes \delta^A(a', x, b') \otimes E(b', b) = (E \circ \delta_x^A \circ E)(a, b) = E_a \circ \delta_x^A \circ E_b, \tag{80}$$

for all  $a, b \in A, x \in X$ , and fuzzy sets  $\sigma^{A/E} : A/E \rightarrow L$  of initial states and  $\tau^{A/E} : A/E \rightarrow L$  of terminal states by

$$\sigma^{A/E}(E_a) = \bigvee_{a' \in A} \sigma^A(a') \otimes E(a', a) = (\sigma^A \circ E)(a) = \sigma^A \circ E_a, \tag{81}$$

$$\tau^{A/E}(E_a) = \bigvee_{a' \in A} E(a, a') \otimes \tau^A(a') = (E \circ \tau^A)(a) = E_a \circ \tau^A, \tag{82}$$

for any  $a \in A$ . Evidently,  $\delta^{A/E}, \sigma^{A/E}$  and  $\tau^{A/E}$  are well-defined and  $\mathcal{A}/E = (A/E, X, \delta^{A/E}, \sigma^{A/E}, \tau^{A/E})$  is a fuzzy automaton, called the *quotient* or *factor fuzzy automaton* of  $\mathcal{A}$  with respect to  $E$ . It was shown in [38, 41, 129] that theorems analogous to those presented in Section 6 are also valid for fuzzy automata.

The concept of a quotient fuzzy automaton can be generalized by taking a fuzzy quasi-order instead of a fuzzy equivalence. Namely, if  $\mathcal{A} = (A, X, \delta^A, \sigma^A, \tau^A)$  is a fuzzy automaton and  $Q$  is a fuzzy quasi-order on  $A$ , then we define the fuzzy transition function  $\delta^{A/Q} : A/Q \times X \times A/Q \rightarrow L$  by

$$\delta^{A/Q}(aQ, x, bQ) = \bigvee_{a', b' \in A} Q(a, a') \otimes \delta^A(a', x, b') \otimes Q(b', b) = (Q \circ \delta_x^A \circ Q)(a, b) = aQ \circ \delta_x^A \circ bQ, \tag{83}$$

for all  $a, b \in A$  and  $x \in X$ , and we define fuzzy sets  $\sigma^{A/Q} : A/Q \rightarrow L$  of initial states and  $\tau^{A/Q} : A/Q \rightarrow L$  of terminal states by

$$\sigma^{A/Q}(aQ) = \bigvee_{a' \in A} \sigma^A(a') \otimes Q(a', a) = (\sigma^A \circ Q)(a) = \sigma^A \circ Qa, \tag{84}$$

$$\tau^{A/Q}(aQ) = \bigvee_{a' \in A} Q(a, a') \otimes \tau^A(a') = (Q \circ \tau^A)(a) = aQ \circ \tau^A, \tag{85}$$

for any  $a \in A$ . Remind that  $aQ$  and  $Qa$  denote respectively the  $Q$ -afterset and  $Q$ -foreset determined by  $a$ , and  $A/Q = \{aQ \mid a \in A\}$  is the set of all  $Q$ -aftersets. It is easy to verify that  $\delta^{A/Q}, \sigma^{A/Q}$  and  $\tau^{A/Q}$  are well-defined and  $\mathcal{A}/Q = (A/Q, X, \delta^{A/Q}, \sigma^{A/Q}, \tau^{A/Q})$  is a fuzzy automaton, called the *afterset fuzzy automaton* of  $\mathcal{A}$  with respect to  $Q$ .

### 7.2. State reduction

From the very beginning of the theory of fuzzy sets, fuzzy automata and languages are studied as a means for bridging the gap between the precision of computer languages and vagueness and imprecision, which are frequently encountered in the study of natural languages. During the decades, fuzzy automata and languages have gained wide field of application, and nowadays they are used in lexical analysis, description of natural and programming languages, learning systems, control systems, neural networks, clinical monitoring, pattern recognition, error correction, databases, discrete event systems, and many other areas.

In real-life applications we typically start from an ordinary or fuzzy regular expression, which is then converted to a nondeterministic or fuzzy finite automaton (cf. [93, 128]). However, the practical implementation usually requires a deterministic finite automaton or a deterministic fuzzy finite automaton, and the obtained nondeterministic or fuzzy automaton has to be determinized. On the other hand, determinization can cause an exponential blow up in the number of states, and in the case of fuzzy finite automata over certain structures of membership values (such as the product structure), determinization can even result in an infinite automaton (cf. [34, 74, 77, 85, 93]). That is why the number of states of a fuzzy finite automaton has to be reduced prior to determinization.

Another important example that illustrates the significance of the state reduction is modeling of discrete event systems. A discrete event system (DES) is a dynamical system whose state space is described by a discrete set, and states evolve as a result of asynchronously occurring discrete events over time [26, 72]. Such systems have significant applications in many fields of computer science and engineering, such as concurrent and distributed software systems, computer and communication networks, manufacturing, transportation and traffic control systems, etc. Recently, fuzzy discrete event systems have been successfully applied to biomedical control for HIV/AIDS treatment planning, robotic control, intelligent vehicle control,

waste-water treatment, examination of chemical reactions, and in other fields. Usually, a discrete event system is modeled by a deterministic or nondeterministic finite automaton, and recently by a fuzzy finite automaton, with events modeled by input letters, and the behavior of a discrete event system is described by the language or fuzzy language generated by the automaton. Discrete event models of complex dynamical systems are built rarely in a monolithic manner. Instead, a modular approach is used where models of individual components are built first, followed by the composition of these models to obtain the model of the overall system. In the automaton modeling formalism the composition of individual automata (that model interacting system components) is typically formalized by the parallel composition of automata. Once a complete system model has been obtained by parallel composition of a set of automata, the resulting monolithic model can be used to analyze the properties of the system, such as safety properties, blocking properties, observability, diagnosability, controllability, etc. (cf. [26, 72]). The main problem that may arise here is that the size of the state set of the parallel composition may in the worst case grow exponentially in the number of automata that are composed. This process is known as the *curse of dimensionality* in the study of complex systems composed of many interacting components. The mentioned problem may be mitigated if we adopt modular reasoning, which can make it possible to replace components in the parallel composition by smaller equivalent automata that are obtained by the state reduction of the components, and then to analyze a simpler system.

In contrast to deterministic finite automata, for which there are many fast minimization algorithms, the state minimization problem for nondeterministic and fuzzy finite automata is computationally hard. For these automata a more practical problem is the *state reduction problem*, where a nondeterministic or fuzzy finite automaton should be replaced with an equivalent automaton with as small as possible number of states, which need not be minimal but must be effectively computable.

From different aspects, state reduction of fuzzy automata was studied in [3, 33, 90, 96, 111, 114, 118, 134, 135], as well as in the books [102, 112]. The basic idea exploited in these sources was to reduce the number of states of a fuzzy automaton by computing and merging indistinguishable states, modeled on the well-known method widely used in the minimization of deterministic finite automata. However, in the deterministic case we can effectively detect and merge indistinguishable states, but in the nondeterministic and fuzzy case we work with sets and fuzzy sets of states, and it is seemingly very difficult to decide whether two states are distinguishable or not. What we shall do is to find a superset such that one is certain not to merge state that should not be merged. There can always be states which could be merged but detecting those is too computationally expensive. Besides, detecting and merging indistinguishable states gives a crisp equivalence relation between states, but we will see later that the use of fuzzy relations (fuzzy equivalences and fuzzy quasi-orders) in general gives better reductions.

Two new ideas have been launched in [40, 41]. The first one is the use of fuzzy equivalences, rather than ordinary crisp equivalences, in reducing the number of states of fuzzy automata, and the other is the construction of such fuzzy equivalences by solving certain systems of fuzzy relation equations and inequalities. In other words, the idea is to form as small as possible quotient fuzzy automaton which would be equivalent to the original fuzzy finite automaton. However, although the quotient fuzzy automaton can be constructed without any restriction on a fuzzy equivalence, this fuzzy equivalence must fulfill certain requirements to ensure that the corresponding quotient fuzzy automaton is equivalent to the original one.

Let  $\mathcal{A} = (A, X, \delta^A, \sigma^A, \tau^A)$  be a fuzzy automaton and let  $E$  be a fuzzy equivalence on  $A$ . The fuzzy language  $L(\mathcal{A}/E)$  recognized by the quotient fuzzy automaton  $\mathcal{A}/E$  is given by

$$L(\mathcal{A}/E)(\varepsilon) = \sigma^A \circ E \circ \tau^A, \tag{86}$$

$$L(\mathcal{A}/E)(u) = \sigma^A \circ E \circ \delta_{x_1}^A \circ E \circ \delta_{x_2}^A \circ E \circ \dots \circ E \circ \delta_{x_n}^A \circ E \circ \tau^A, \tag{87}$$

whereas the fuzzy language  $L(\mathcal{A})$  recognized by the fuzzy automaton  $\mathcal{A}$  is given by

$$L(\mathcal{A})(\varepsilon) = \sigma^A \circ \tau^A, \tag{88}$$

$$L(\mathcal{A})(u) = \sigma^A \circ \delta_{x_1}^A \circ \delta_{x_2}^A \circ \dots \circ \delta_{x_n}^A \circ \tau^A, \tag{89}$$

for any  $u = x_1x_2 \dots x_n \in X^+$ , with  $x_1, x_2, \dots, x_n \in X$ . Therefore, fuzzy automata  $\mathcal{A}$  and  $\mathcal{A}/E$  recognize the same



fuzzy language if and only if the fuzzy equivalence  $E$  is a solution to a system of fuzzy relation equations

$$\begin{aligned} \sigma^A \circ \tau^A &= \sigma^A \circ R \circ \tau^A, \\ \sigma^A \circ \delta_{x_1}^A \circ \delta_{x_2}^A \circ \dots \circ \delta_{x_n}^A \circ \tau^A &= \sigma^A \circ R \circ \delta_{x_1}^A \circ R \circ \delta_{x_2}^A \circ R \circ \dots \circ R \circ \delta_{x_n}^A \circ R \circ \tau^A, \end{aligned} \tag{90}$$

for all  $n \in \mathbb{N}$  and  $x_1, x_2, \dots, x_n \in X$ , where  $R$  is an unknown fuzzy relation on  $A$ . We call (90) the *general system*. In this case, solutions are sought in the set  $\mathcal{E}(A)$  of all fuzzy equivalences on  $A$ , but it has been shown in [129] that solutions can also be sought in the set  $\mathcal{Q}(A)$  of all fuzzy quasi-orders on  $A$ . Namely, for a fuzzy quasi-order  $Q$  on  $A$ , the afterset fuzzy automaton  $\mathcal{A}/Q$  is equivalent to  $\mathcal{A}$  if and only if  $Q$  is a solution to the general system.

The general system has at least one solution, the equality relation on  $A$ . It has been proved in [129] that solutions to the general system in  $\mathcal{Q}(A)$  and  $\mathcal{E}(A)$  form ideals of these lattices, but the general system does not have the greatest solution neither in  $\mathcal{Q}(A)$  nor in  $\mathcal{E}(A)$ . Besides, the general system may consist of infinitely many equations, and finding its nontrivial solutions may be a very difficult task. For that reason it is more convenient to consider some instances of the general system, by which we mean systems, built from the same fuzzy relations, whose sets of solutions are contained in the set of all solutions to the general system. These instances have to be as general as possible, to obtain as large as possible solution, but they have to consist of finitely many equations and to be “easier” to solve.

The first equation in (90) has an important instance, the system consisting of the equations  $\sigma^A \circ R = \sigma^A$  and  $R \circ \tau^A = \tau^A$ . Both equations are linear and their greatest solutions can be easily computed. Namely, the greatest solution to the first equation is the fuzzy quasi-order  $Q_\sigma$ , the greatest solution to the second one is the fuzzy quasi-order  $Q^\tau$ , and the greatest solution to the system is their intersection  $Q_\sigma \wedge Q^\tau$ . If solutions are sought in the lattice  $\mathcal{E}(A)$  of fuzzy equivalences, then the greatest solution to the first equation is  $E_\sigma$ , the greatest solution to the second one is  $E_\tau$ , and the greatest solution to the system is  $E_\sigma \wedge E_\tau$ . Remind that  $Q_\sigma$ ,  $Q^\tau$ ,  $E_\sigma$  and  $E_\tau$  are defined in Section 2.

The system in the second row of (90) has two exceptionally important instances. The first one is the system

$$R \circ \delta_x^A \leq \delta_x^A \circ R \quad (x \in X), \tag{91}$$

which is actually the weakly linear system  $WL^{1.1}(A, X, \delta_x^A)$ , and as an instance of the whole system (90) we obtain

$$R \circ \delta_x^A \leq \delta_x^A \circ R \quad (x \in X), \quad R \leq Q_\sigma \wedge Q^\tau, \tag{92}$$

which is nothing else than the weakly linear system  $WL^{1.1}(A, X, \delta_x^A, Q_\sigma \wedge Q^\tau)$ . Another instance is the system

$$\delta_x^A \circ R \leq R \circ \delta_x^A \quad (x \in X), \tag{93}$$

i.e., the weakly linear system  $WL^{1.2}(A, X, \delta_x^A)$ , and as an instance of the whole system (90) we obtain

$$\delta_x^A \circ R \leq R \circ \delta_x^A \quad (x \in X), \quad R \leq Q_\sigma \wedge Q^\tau, \tag{94}$$

which is actually the weakly linear system  $WL^{1.2}(A, X, \delta_x^A, Q_\sigma \wedge Q^\tau)$ . Solutions to system (92) in  $\mathcal{Q}(A)$  are called *right invariant fuzzy quasi-orders*, and solutions to (94) in  $\mathcal{Q}(A)$  are called *left invariant fuzzy quasi-orders*. In the case when solutions are sought in  $\mathcal{E}(A)$ , we talk about *right* and *left invariant fuzzy equivalences*. It is worth noting that right and left invariant equivalences are also solutions to systems  $WL^{1.4}(A, X, \delta_x^A, E_\sigma \wedge E_\tau)$  and  $WL^{1.5}(A, X, \delta_x^A, E_\sigma \wedge E_\tau)$ , respectively. Right and left fuzzy equivalences have been introduced in [40, 41], and right and left invariant fuzzy quasi-orders in [129]. Their crisp analogues are right and left invariant equivalences and quasi-orders, used in [28, 29, 80–84] in the state reduction of nondeterministic finite automata. We will also see that right and left invariant fuzzy equivalences are actually forward and backward bisimulation fuzzy equivalences, which will be discussed in the next subsection.

Right and left invariant fuzzy equivalences are completely equal in the state reduction of fuzzy automata, which means that generally none of them can be considered better than the other. There are cases where one

of them better reduces the number of states, as well as other cases where the another one gives a better reduction. Besides, it has been proved that even better results in the state reduction are obtained by applying alternately reductions by means of the greatest right and left invariant fuzzy equivalences (cf. [41, 129]). There are also cases where each of them individually causes a polynomial reduction of the number of states, but alternately applying both types of fuzzy equivalences the number of states can be reduced exponentially (cf. [81, Section 11]). Everything that has been noted above for right and left invariant fuzzy equivalences is also valid for right and left invariant fuzzy quasi-orders. In addition, it has been proved in [129] that right and left invariant fuzzy quasi-orders generally give better reductions than right and left invariant fuzzy equivalences. It should be noted that there are some applications where only one of the right and left invariant fuzzy quasi-orders and fuzzy equivalences can be used. For example, left invariant fuzzy quasi-orders and fuzzy equivalences can be successfully applied in the conflict analysis of fuzzy discrete event systems (as shown in [129]), where the right invariant ones do not play any role. On the other hand, right invariant equivalences play an important role in the state reduction of nondeterministic automata constructed from regular expressions. In particular, it has been proved in [30–32, 79–82] that the well-known partial derivative automaton and the follow automaton can be obtained as factor automata of the position automaton with respect to certain right invariant equivalences. Related results for fuzzy automata have been recently given in [128].

The greatest right and left invariant fuzzy equivalences and fuzzy quasi-orders can be computed using the iterative procedure presented in Section 5. The question that naturally arises is what to do in cases where this iterative procedure fails to terminate in a finite number of steps. One answer to this question has already been proposed in Section 5. By means of the procedure provided in Section 5, we can compute the greatest right or left invariant crisp equivalence or quasi-order, and using it, we can reduce the given fuzzy finite automaton. The advantage of this approach is that the greatest right and left invariant crisp equivalences and quasi-orders can always be effectively computed, independently on the underlying structure of membership values, but the disadvantage is that in general we obtain worse reductions. Another answer to the above question is to consider some less general instances of system (90) whose greatest solution can be more easily computed. In particular, linear systems

$$R \circ \delta_x^A \leq \delta_x^A \quad (x \in X), \quad R \leq Q_\sigma \wedge Q^\tau, \tag{95}$$

and

$$\delta_x^A \circ R \leq \delta_x^A \quad (x \in X), \quad R \leq Q_\sigma \wedge Q^\tau, \tag{96}$$

have been investigated in [41, 129], and their solutions in  $Q(A)$  and  $\mathcal{E}(A)$  have been called *strongly right* and *left invariant fuzzy quasi-orders* and *fuzzy equivalences*. The greatest solutions to these systems can be easily computed as shown in Section 3, without any iterative procedure and for every complete residuated lattice as the underlying structure of membership values. However, it has been proved in [76] that they give even worse reductions than right and left invariant crisp equivalences and quasi-orders.

Different instances of the general system have been studied in [129]. These are linear systems

$$R \circ \tau_u^A \leq \tau_u^A, \quad (u \in X^*), \tag{97}$$

and

$$\sigma_u^A \circ R \leq \sigma_u^A, \quad (u \in X^*), \tag{98}$$

where  $\tau_u^A$  and  $\sigma_u^A$  are fuzzy subsets of  $A$  defined by  $\tau_u^A = \delta_u^A \circ \tau^A$  and  $\sigma_u^A = \sigma^A \circ \delta_u^A$ , for each  $u \in X^*$ . Solutions to (97) in  $Q(A)$  and  $\mathcal{E}(A)$  are called *weakly right invariant fuzzy quasi-orders* and *fuzzy equivalences*, and solutions to (98) in  $Q(A)$  and  $\mathcal{E}(A)$  are called *weakly left invariant fuzzy quasi-orders* and *fuzzy equivalences*. It has been shown in [129] that the greatest weakly right and left invariant fuzzy quasi-orders and fuzzy equivalences generally give better reductions than the greatest right and left invariant fuzzy quasi-orders and fuzzy equivalences. However, although it may seem that the greatest solutions to (97) and (98) can be easily computed, because the systems are linear, it is not so simple. The problem may be to compute the fuzzy sets  $\tau_u^A$  in (97)

or  $\sigma_u^A$  in (98), for each  $u \in X^*$ . Fuzzy sets  $\sigma_u^A$ , for  $u \in X^*$ , are states of the Nerode automaton of  $\mathcal{A}$ , which is obtained by determinization of the fuzzy automaton  $\mathcal{A}$  using a method developed in [74] (see also [34, 77, 85]), whereas fuzzy sets  $\tau_u^A$ , for  $u \in X^*$ , are states of the Nerode automaton of the reverse fuzzy automaton of  $\mathcal{A}$ . Therefore, before solving the system (97) or (98), we first have to compute all fuzzy sets  $\tau_u^A$  or  $\sigma_u^A$ , for  $u \in X^*$ , and the number of these fuzzy sets may be infinite, and even if it is finite, this number can be exponential in the number of states of  $\mathcal{A}$ . In other words, the number of inequalities in (97) or (98) may be infinite, or exponential in the number of states of  $\mathcal{A}$ . However, although theoretically determinization algorithms are computationally hard, they have surprisingly good performance in practice, so formation of systems (97) and (98) usually should not be a problem.

### 7.3. Simulation and bisimulation

Bisimulations have been introduced by Milner [99] and Park [107] in computer science, more precisely in concurrency theory, where they have been used to model equivalence between various systems, as well as to reduce the number of states of these systems. Roughly at the same time they have been also discovered in some areas of mathematics, e.g., in modal logic and set theory. They are employed today in a many areas of computer science, such as functional languages, object-oriented languages, types, data types, domains, databases, compiler optimizations, program analysis, verification tools, etc. For more information about bisimulations we refer to [1, 26, 53, 62, 95, 100, 101, 120, 126].

The most common structures on which bisimulations have been studied are labelled transition systems, but they have also been investigated in the context of deterministic, nondeterministic, weighted, probabilistic, timed and hybrid automata. Recently, bisimulations have been discussed in the context of fuzzy automata in [25, 38, 40, 41, 75, 118, 129, 131]. One can distinguish two general approaches to the concept of bisimulation for fuzzy automata. The first approach, which we encounter in [25, 118, 131], uses ordinary crisp relations and functions. Another approach, proposed in [38, 40, 41, 75, 129], is based on the use of fuzzy relations, which have been shown to provide better results both in the state reduction (as shown in the previous subsection) and the modeling of equivalence of fuzzy automata.

Two types of simulations and four types of bisimulations between fuzzy automata have recently been introduced in [38]. Let  $\mathcal{A} = (A, \delta^A, \sigma^A, \tau^A)$  and  $\mathcal{B} = (B, \delta^B, \sigma^B, \tau^B)$  be fuzzy automata, and let  $\varphi \in \mathcal{R}(A, B)$  be a non-empty fuzzy relation. The fuzzy relation  $\varphi$  is called a *forward simulation* if it satisfies

$$\sigma^A \leq \sigma^B \circ \varphi^{-1}, \tag{fs.1}$$

$$\varphi^{-1} \circ \delta_x^A \leq \delta_x^B \circ \varphi^{-1}, \quad \text{for every } x \in X, \tag{fs.2}$$

$$\varphi^{-1} \circ \tau^A \leq \tau^B, \tag{fs.3}$$

and a *backward simulation* if

$$\tau^A \leq \varphi \circ \tau^B, \tag{bs.1}$$

$$\delta_x^A \circ \varphi \leq \varphi \circ \delta_x^B, \quad \text{for every } x \in X, \tag{bs.2}$$

$$\sigma^A \circ \varphi \leq \sigma^B. \tag{bs.3}$$

Furthermore,  $\varphi$  is called a *forward bisimulation* if both  $\varphi$  and  $\varphi^{-1}$  are forward simulations, i.e., if  $\varphi$  satisfies

$$\sigma^A \leq \sigma^B \circ \varphi^{-1}, \quad \sigma^B \leq \sigma^A \circ \varphi, \tag{fb.1}$$

$$\varphi^{-1} \circ \delta_x^A \leq \delta_x^B \circ \varphi^{-1}, \quad \varphi \circ \delta_x^B \leq \delta_x^A \circ \varphi, \quad \text{for every } x \in X, \tag{fb.2}$$

$$\varphi^{-1} \circ \tau^A \leq \tau^B, \quad \varphi \circ \tau^B \leq \tau^A, \tag{fb.3}$$

and a *backward bisimulation*, if both  $\varphi$  and  $\varphi^{-1}$  are backward simulations, i.e., if  $\varphi$  satisfies

$$\tau^A \leq \varphi \circ \tau^B, \quad \tau^B \leq \varphi^{-1} \circ \tau^A, \tag{bb.1}$$

$$\delta_x^A \circ \varphi \leq \varphi \circ \delta_x^B, \quad \delta_x^B \circ \varphi^{-1} \leq \varphi^{-1} \circ \delta_x^A, \quad \text{for every } x \in X, \tag{bb.2}$$

$$\sigma^A \circ \varphi \leq \sigma^B, \quad \sigma^B \circ \varphi^{-1} \leq \sigma^A. \tag{bb.3}$$

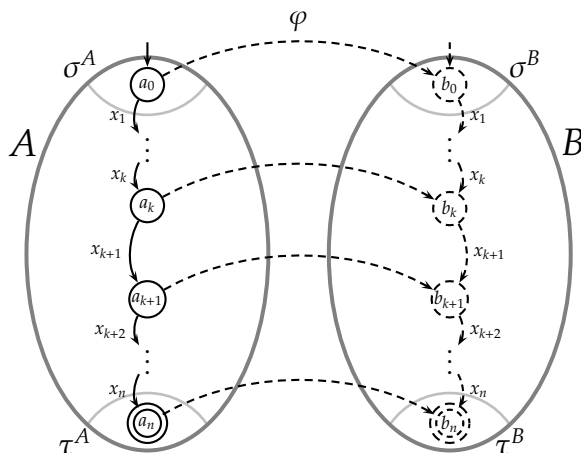


Figure 1: Forward and backward simulation

Also, if  $\varphi$  is a forward simulation and  $\varphi^{-1}$  is a backward simulation, i.e., if  $\varphi$  satisfies

$$\sigma^A \leq \sigma^B \circ \varphi^{-1}, \quad \tau^B \leq \varphi^{-1} \circ \tau^A, \tag{fbb.1}$$

$$\varphi^{-1} \circ \delta_x^A = \delta_x^B \circ \varphi^{-1}, \quad \text{for every } x \in X, \tag{fbb.2}$$

$$\sigma^B \circ \varphi^{-1} \leq \sigma^A, \quad \varphi^{-1} \circ \tau^A \leq \tau^B, \tag{fbb.3}$$

then  $\varphi$  is called a *forward-backward bisimulation*, and if  $\varphi$  is a backward and  $\varphi^{-1}$  a forward simulation, i.e.,

$$\sigma^B \leq \sigma^A \circ \varphi, \quad \tau^A \leq \varphi \circ \tau^B, \tag{bfb.1}$$

$$\delta_x^A \circ \varphi = \varphi \circ \delta_x^B, \quad \text{for every } x \in X, \tag{bfb.2}$$

$$\sigma^A \circ \varphi \leq \sigma^B \quad \varphi \circ \tau^B \leq \tau^A. \tag{bfb.3}$$

then  $\varphi$  is called a *backward-forward bisimulation*. For the sake of simplicity, we will call  $\varphi$  just a *simulation* if  $\varphi$  is either a forward or a backward simulation, and just a *bisimulation* if  $\varphi$  is any of the four types of bisimulations defined above. Moreover, forward and backward bisimulations are called *homotypic*, whereas backward-forward and forward-backward bisimulations are called *heterotypic*.

The meaning of forward and backward simulations can be best explained in the case when  $\mathcal{A}$  and  $\mathcal{B}$  are nondeterministic (Boolean) automata. For this purpose we will use the diagram shown in Figure 1. Let  $\varphi$  be a forward simulation between  $\mathcal{A}$  and  $\mathcal{B}$  and let  $a_0, a_1, \dots, a_n$  be an arbitrary successful run of the automaton  $\mathcal{A}$  on a word  $u = x_1x_2 \cdots x_n$  ( $x_1, x_2, \dots, x_n \in X$ ), i.e., a sequence of states of  $\mathcal{A}$  such that  $a_0 \in \sigma^A$ ,  $(a_k, a_{k+1}) \in \delta_{x_{k+1}}^A$ , for  $0 \leq k \leq n-1$ , and  $a_n \in \tau^A$ . According to (fs.1), there is an initial state  $b_0 \in \sigma^B$  such that  $(a_0, b_0) \in \varphi$ . Suppose that for some  $k$ ,  $0 \leq k \leq n-1$ , we have built a sequence of states  $b_0, b_1, \dots, b_k$  such that  $(b_{i-1}, b_i) \in \delta_{x_i}^B$  and  $(a_i, b_i) \in \varphi$ , for each  $i$ ,  $1 \leq i \leq k$ . Then  $(b_k, a_{k+1}) \in \varphi^{-1} \circ \delta_{x_{k+1}}^A$ , and by (fs.2) we obtain that  $(b_k, a_{k+1}) \in \delta_{x_{k+1}}^B \circ \varphi^{-1}$ , so there exists  $b_{k+1} \in B$  such that  $(b_k, b_{k+1}) \in \delta_{x_{k+1}}^B$  and  $(a_{k+1}, b_{k+1}) \in \varphi$ . Therefore, we have successively built a sequence  $b_0, b_1, \dots, b_n$  of states of  $\mathcal{B}$  such that  $b_0 \in \sigma^B$ ,  $(b_k, b_{k+1}) \in \delta_{x_{k+1}}^B$ , for every  $k$ ,  $0 \leq k \leq n-1$ , and  $(a_k, b_k) \in \varphi$ , for each  $k$ ,  $0 \leq k \leq n$ . Moreover, by (fs.3) we obtain that  $b_n \in \tau^B$ . Thus, the sequence  $b_0, b_1, \dots, b_n$  is a successful run of the automaton  $\mathcal{B}$  on the word  $u$  which simulates the original run  $a_0, a_1, \dots, a_n$  of  $\mathcal{A}$  on  $u$ . In contrast to forward simulations, where we build the sequence  $b_0, b_1, \dots, b_n$  moving forward, starting with  $b_0$  and ending with  $b_n$ , in the case of backward simulations we build this sequence moving backward, starting with  $b_n$  and ending with  $b_0$ . Similarly we can understand forward and backward simulations between arbitrary fuzzy automata, taking into account degrees of possibility of transitions and degrees of relationship.

In numerous papers dealing with simulations and bisimulations mostly forward simulations and forward bisimulations have been studied. They have been usually called just simulations and bisimulations, or *strong simulations* and *strong bisimulations* (cf. [100, 101, 120]), and the greatest bisimulation equivalence has

been usually called a *bisimilarity*. Distinction between forward and backward simulations, and forward and backward bisimulations, has been made, for instance, in [23, 68, 95] (for various kinds of automata), but less or more these concepts differ from the concepts having the same name which are considered here. More similar to our concepts of forward and backward simulations and bisimulations are those studied in [22], and in [69, 70] (for tree automata). Moreover, backward-forward bisimulations have been discussed in the context of weighted automata in [5–7, 12, 23, 57, 58].

It is easy to verify that condition (fs.3) can be written as  $\varphi \leq \tau^A \rightarrow \tau^B$ , where  $\tau^A \rightarrow \tau^B$  is a fuzzy relation between  $A$  and  $B$  defined by  $(\tau^A \rightarrow \tau^B)(a, b) = \tau^A(a) \rightarrow \tau^B(b)$ , for all  $a \in A$  and  $b \in B$ , and correspondingly for conditions (bs.3), (fb.3), (bb.3), (fbb.3) and (bfb.3). Therefore, the second and third conditions in the definitions of simulations and bisimulations form six heterogeneous weakly linear systems, precisely all those six weakly linear systems that were discussed in Section 4. The first conditions in these definitions determine whether there is any simulation or bisimulation of a given type between two fuzzy automata. On this basis, effective algorithms provided in [39] decide whether there is a simulation or bisimulation of a given type between two fuzzy finite automata, and whenever it exists, the same algorithm computes the greatest one. Namely, each of these algorithms first computes the greatest solution to the corresponding weakly linear system, and then check whether this solution satisfies the first condition in the definition of the considered type of simulations or bisimulations. If this condition is satisfied, then the computed solution is the greatest bisimulation or simulation of the considered type, and if the condition is not satisfied, then one concludes that there is no simulation or bisimulation of this type. Note that many algorithms have been proposed to compute the greatest (forward) bisimulation equivalence on a given labelled graph or a labeled transition system, and the faster ones are based on the crucial equivalence between the greatest bisimulation equivalence and the relational coarsest partition problem (cf. [53, 62, 86, 106, 119]). An algorithm that computes the greatest forward bisimulation between two nondeterministic finite automata is given in [89].

One of the most important problems of automata theory is to determine whether two given automata are equivalent, what usually means to determine whether their behaviour is identical. In the context of deterministic, nondeterministic and fuzzy automata the behaviour of an automaton is understood to be the language that is recognized by it, and two automata are considered *equivalent*, or more precisely *language-equivalent*, if they recognize the same language. Like the minimization problem, the equivalence problem is solvable in polynomial time for deterministic finite automata, but it is computationally hard for nondeterministic and fuzzy finite automata. Another important issue is to express the language-equivalence of two automata as a relation between their states, if such relationship exists, or find some kind of relations between states which would imply the language-equivalence. The language-equivalence of two deterministic automata can be expressed in terms of relationships between their states, but in the case of nondeterministic and fuzzy automata the problem is more complicated.

The most widely-used notion of “equivalence” between states of different nondeterministic automata is exactly the concept of bisimulation. However, bisimulations provide compatibility with the transitions, initial and terminal states of automata, but they are not sufficient to model equivalence, because in general they do not behave like equivalence relations. As we have already said in Section 2, a kind of relations that can be conceived as equivalences which relate elements of two possibly different sets, in the fuzzy setting, are uniform fuzzy relations. By merging these two concepts into one, which was done in [38] (see also [35]), a very powerful tool in the study of equivalence between fuzzy automata has been provided. In particular, analogues of Theorems 6.7, 6.8 and 6.9 have been proved in [38], which implies that two fuzzy automata  $\mathcal{A}$  and  $\mathcal{B}$  are FB-equivalent, i.e., there exists a uniform forward bisimulation between them, if and only if there is an isomorphism with certain special properties between the factor fuzzy automata  $\mathcal{A}/E$  and  $\mathcal{B}/F$ , where  $E$  and  $F$  are respectively the greatest right invariant fuzzy equivalences (or forward bisimulation fuzzy equivalences) on  $\mathcal{A}$  and  $\mathcal{B}$ . It is interesting to note that this result relates the problem of determining whether two fuzzy automata are FB-equivalent with the famous *graph isomorphism problem*, the computational problem of determining whether two finite graphs are isomorphic. Besides its practical importance, the graph isomorphism problem is a curiosity in computational complexity theory, as it is one of a very small number of problems belonging to NP that is neither known to be computable in polynomial time nor NP-complete. Along with integer factorization, it is one of the few important algorithmic problems whose rough computational complexity is still not known, and it is generally accepted that graph isomorphism is a

problem that lies between P and NP-complete if  $P \neq NP$  (cf. [127]). Although no worst-case polynomial-time algorithm is known, testing graph isomorphism is usually not very hard in practice. The basic algorithm examines all  $n!$  possible bijections between the nodes of two graphs (with  $n$  nodes), and tests whether they preserve adjacency of the nodes. Clearly, the major problem is the rapid growth in the number of bijections when the number of nodes is growing, which is also the crucial problem in testing isomorphism between fuzzy automata, but the algorithm can be made more efficient by suitable partitioning of the sets of nodes as described in [127]. However, a more effective way to decide whether two fuzzy automata are FB-equivalent is the above discussed algorithm from [39], which determines whether there is a forward bisimulation between these automata, and if it exists, the algorithm computes the greatest one. After that, it simply checks if the greatest forward bisimulation is a surjective  $\mathcal{L}$ -function (according to Theorem 4.4, the greatest forward bisimulation is always a partial fuzzy function).

Finally, note that the last section of the paper [38] gives a comprehensive overview of various concepts on deterministic, nondeterministic, fuzzy, and weighted automata, which are related to the algebraic concepts of a homomorphism, congruence and relational morphism. In particular, relationships between these concepts and the concepts of bisimulations have been discussed.

#### 7.4. Social network analysis

Social network analysis has originated as a branch of sociology and mathematics which provides formal models and methods for the systematic study of social structures. Social networks share many common properties with other types of networks, and methods of social network analysis are nowadays applied to the analysis of networks in general, including many kinds of networks that arise in computer science, physics, biology, etc., such as the hyperlink structure on the Web, the electric grid, computer networks, information networks or various large-scale networks appearing in nature.

The key difference between network analysis and other approaches is the focus on relationships among actors rather than the attributes of individual actors. Network analysis takes a global view on network structures, based on the belief that types and patterns of relationships emerge from individual connectivity and that the presence (or absence) of such types and patterns have substantial effects on the network and its constituents. In particular, the network structure provides opportunities and imposes constraints on the individual actors by determining the transfer or flow of resources (material or immaterial) across the network. Such an approach requires a set of methods and analytic concepts that are distinct from the methods of traditional statistics and data analysis. The natural means to model networks mathematically is provided by the notions of graphs, relations and matrices, and methods of network analysis primarily originate from graph theory, semigroup theory, linear algebra, and recently, of automata theory. This formality served network analysis to reduce the vagueness in formulating and testing its theories, and contributed to more coherence in the field by allowing researchers to carry out more precise discussions in the literature and to compare results across studies. More information on various aspects of the network analysis and its applications can be found in [19, 21, 47, 52, 67, 98, 109, 132].

However, the above mentioned vagueness in social networks (as well as in many other networks) can not be completely avoided, since relations between individuals are in essence vague. This vagueness can be overcome only applying the fuzzy approach to the network analysis, but still, just few authors dealt with this topic (cf. [42, 60, 61, 103, 104]). The study of social networks from the aspect of fuzzy set theory will be one of the main challenges in our future research.

A *fuzzy social network* (or just a *fuzzy network*) one can define as a fuzzy relational system  $\mathcal{U} = (U, I, A_i)$ , where  $A$  is a non-empty (usually finite) set of *individuals* (called also *nodes* or *actors*), and  $\{A_i\}_{i \in I}$  is a family of fuzzy relations on  $U$ . Ordinary social networks are typically visualized as directed multigraphs or directed labelled graphs (with labels taken from the index set  $I$ ), so fuzzy social networks can be treated as directed fuzzy multigraphs, directed labelled fuzzy graphs, or as fuzzy automata (with  $I$  as its input alphabet).

In large and complex systems it is impossible to understand the relationship between each pair of individuals, but to a certain extent, it may be possible to understand the system, by classifying individuals and describing relationships on the class level. In networks, for instance, nodes in the same class can be considered to occupy the same *position*, or play the same *role* in the network. The main aim of the positional

analysis of networks is to find similarities between nodes which have to reflect their position in a network. These similarities have been formalized first by Lorrain and White [94] by the concept of a *structural equivalence*. Informally speaking, two nodes are considered to be structurally equivalent if they have identical neighborhoods. However, in many situations this concept has shown oneself to be too strong, and weakening it sufficiently to make it more appropriate for modeling social positions, White and Reitz [133] have introduced the concept of a *regular equivalence*. Here, two nodes are considered to be regularly equivalent if they are equally related to equivalent others. Afterwards, regular equivalences have been studied in numerous papers, e.g., in [16–20, 59, 108, 109]. The main technique used in positional analysis of networks is the blockmodeling, where large and complex social networks are mapped into simpler structures, called blockmodel images. Blockmodel images are viewed as structural summaries of these large and complex networks. The structure depicted in a blockmodel image can be understood as the fundamental structure of the network and the considered network is an instantiation of the fundamental structure. The key role in blockmodeling play various types of equivalences, like structural and regular equivalences, which in a certain sense simultaneously partition nodes and edges of the network (cf. [4, 15, 50–52]).

The notion of a regular equivalence has been extended to the fuzzy framework by Fan et al. [60, 61]. They have defined a *regular fuzzy equivalence* on a fuzzy network  $\mathcal{U}$  as any fuzzy equivalence  $E$  on  $U$  which satisfies  $E \circ A_i = A_i \circ E$ , for each  $i \in I$ . In our terminology, these are just fuzzy equivalences which are solutions to the weakly linear system  $WL^{1.3}(U, I, A_i)$ , or equivalently, to the system  $WL^{1.6}(U, I, A_i)$ . On the other hand, *structural fuzzy equivalences*, defined in the same manner as in the crisp case (cf., e.g., [91]), are just fuzzy equivalences that are solutions to the linear system  $A_i \circ X = A_i$ ,  $X \circ A_i = A_i$  ( $i \in I$ ). In terms used in Subsection 7.2 in the theory of fuzzy automata, regular fuzzy equivalences are the ones that are both right and left invariant, while structural fuzzy equivalences are the ones that are both strongly right and left invariant. Fan et al. [60, 61] have also provided procedures for computing the greatest regular fuzzy equivalence and the greatest regular crisp equivalence contained in a given fuzzy (resp. crisp) equivalence, but they have considered only fuzzy relations over the Gödel structure which is locally finite, so the problems appearing in Section 5 do not appear in this case. In contrast to that, the methods presented in Section 5 can be applied to fuzzy networks over more general structures of truth values. It is worth noting that similarity between regular and bisimulation equivalences has been pointed out in [97], and the results presented here make this similarity even more clear.

A different procedure for computing the greatest regular fuzzy equivalence on a fuzzy social network  $\mathcal{U} = (U, I, A_i)$ , contained in a given fuzzy equivalence  $M$ , has been provided in [76]. As we have already pointed out, this is precisely the computing of the greatest solution to weakly linear system  $WL^{1.6}(U, I, A_i, M)$ . The paper [76] also considers more general types of fuzzy equivalences, called *right regular* and *left regular fuzzy equivalences*, and provides procedures for computing the greatest right and left regular fuzzy equivalences on  $\mathcal{U}$  contained in a given fuzzy equivalence  $M$ , i.e., the greatest solutions to weakly linear systems  $WL^{1.4}(U, I, A_i, M)$  and  $WL^{1.5}(U, I, A_i, M)$ , respectively. In addition, the paper [76] provides procedures for computing the greatest solutions to systems  $WL^{1.1}(U, I, A_i, M)$ ,  $WL^{1.2}(U, I, A_i, M)$  and  $WL^{1.3}(U, I, A_i, M)$ , where  $M$  is a given fuzzy quasi-order, i.e., the greatest *right regular*, *left regular* and *regular fuzzy quasi orders* contained in  $M$  (cf. Section 5). However, in [76] we have not discussed blockmodeling by means of right regular, left regular and regular fuzzy equivalences and fuzzy quasi-orders, and this issue is one of the subjects of our current research. Besides, we are currently dealing with the issue of recognizing structural similarities between two different fuzzy social networks  $\mathcal{U} = (U, I, A_i)$  and  $\mathcal{V} = (V, I, B_i)$ , for which purpose we use solutions to heterogeneous weakly linear systems  $WL^{2.t}(U, V, I, A_i, B_i, N)$ , where  $t \in \{1, \dots, 6\}$  and  $N$  is a particular fuzzy relation between  $U$  and  $V$ . In other words, we are studying simulation, bisimulation and equivalence between two fuzzy social networks. Currently we are also discussing related issues concerning two-mode (bipartite) fuzzy social networks (cf. [130]).

## References

- [1] L. Aceto, A. Ingólfssdóttir, K. G. Larsen, J. Srba, *Reactive Systems: Modelling, Specification and Verification*, Cambridge University Press, Cambridge, 2007.

- [2] W. Bandler, L.J. Kohout, Fuzzy relational products as a tool for analysis and synthesis of the behaviour of complex natural and artificial systems, in: S.K. Wang, P.P. Chang (Eds.), *Fuzzy Sets: Theory and Application to Policy Analysis and Information Systems*, Plenum Press, New York, 1980, pp. 341–367.
- [3] N. C. Basak, A. Gupta, On quotient machines of a fuzzy automaton and the minimal machine, *Fuzzy Sets and Systems* 125 (2002) 223–229.
- [4] V. Batagelj, Large scale social network analysis, in: R. A. Meyers (Ed.), *Encyclopedia of Complexity and Systems Science*, Springer, New York, 2009, pp. 8245–8265.
- [5] M. P. Béal, S. Lombardy, J. Sakarovitch, On the equivalence of  $\mathbb{Z}$ -automata, In: L. Caires et al. (eds.), *ICALP 2005*, Springer, Heidelberg, *Lecture Notes in Computer Science* 3580 (2005) 397–409.
- [6] M. P. Béal, S. Lombardy, J. Sakarovitch, Conjugacy and equivalence of weighted automata and functional transducers. In: D. Grigoriev, J. Harrison, and E. A. Hirsch (eds.), *CSR 2006*, Springer, Heidelberg, *Lecture Notes in Computer Science* 3967 (2006) 58–69.
- [7] M. P. Béal, D. Perrin, On the generating sequences of regular languages on  $k$  symbols, *Journal of the ACM* 50 (2003) 955–980.
- [8] R. Bělohávek, Determinism and fuzzy automata, *Information Sciences* 143 (2002) 205–209.
- [9] R. Bělohávek, *Fuzzy Relational Systems: Foundations and Principles*, Kluwer, New York, 2002.
- [10] R. Bělohávek, V. Vychodil, *Fuzzy Equational Logic*, Springer, Berlin/Heidelberg, 2005.
- [11] G. Birkhoff, *Lattice Theory*, third ed., American Mathematical Society, Providence, RI, 1973.
- [12] S. L. Bloom, Z. Ésik, *Iteration Theories: The Equational Logic of Iterative Processes*, *EATCS Monographs on Theoretical Computer Science*, Springer, Berlin-Heidelberg, 1993.
- [13] T. S. Blyth, *Lattices and Ordered Algebraic Structures*, Springer, London, 2005.
- [14] U. Bodenhofer, B. De Baets, J. Fodor, A compendium of fuzzy weak orders: Representations and constructions, *Fuzzy Sets and Systems* 158 (2007) 811–829.
- [15] S. P. Borgatti, Two-mode concepts in social network analysis, in: R. A. Meyers (Ed.), *Encyclopedia of Complexity and Systems Science*, Springer, New York, 2009, pp. 8279–8291.
- [16] S. P. Borgatti, M. G. Everett, The class of all regular equivalences: algebraic structure and computation, *Social Networks* 11 (1989) 65–88.
- [17] S. P. Borgatti, M. G. Everett, Regular blockmodels of multiway, multimode matrices, *Social Networks* 14 (1992) 91–120.
- [18] S. P. Borgatti, M. G. Everett, Two algorithms for computing regular equivalences, *Social Networks* 15 (1993) 361–376.
- [19] J. Boyd, *Social Semigroups*, George Mason University Press, Fairfax, VA, 1991.
- [20] J. P. Boyd, M. G. Everett, Relations, residuals, regular interiors, and relative regular equivalence, *Social Networks* 21 (1999) 147–165.
- [21] U. Brandes, T. Erlebach (eds.), *Network Analysis: Methodological Foundations*, (Lecture Notes in Computer Science, vol. 3418), Springer, 2005.
- [22] T. Brihaye, Words and bisimulations of dynamical systems, *Discrete Mathematics and Theoretical Computer Science* 9 (2) (2007) 11–32.
- [23] P. Buchholz, Bisimulation relations for weighted automata, *Theoretical Computer Science* 393 (2008) 109–123.
- [24] S. Burris, H. P. Sankappanavar, *A Course in Universal Algebra*, Springer-Verlag, New York, 1981.
- [25] Y. Cao, G. Chen, E. Kerre, Bisimulations for fuzzy transition systems, *IEEE Transactions on Fuzzy Systems* 19 (2011) 540–552.
- [26] C. G. Cassandras, S. Lafortune, *Introduction to Discrete Event Systems*, Springer, 2008.
- [27] I. Chajda, H. Länger, Quotients and homomorphisms of relational systems, *Acta Univ. Palacki. Olomuc., Fac. rer. nat., Mathematica* 49 (2010) 37–47.
- [28] J.-M. Champarnaud, F. Coulon, NFA reduction algorithms by means of regular inequalities, in: Z. Ésik, and Z. Fülöp (eds.), *DLT 2003*, *Lecture Notes in Computer Science* 2710 (2003) 194–205.
- [29] J.-M. Champarnaud, F. Coulon, NFA reduction algorithms by means of regular inequalities, *Theoretical Computer Science* 327 (2004) 241–253 (erratum: *Theoretical Computer Science* 347 (2005) 437–40).
- [30] J.-M. Champarnaud, D. Ziadi, New finite automaton constructions based on canonical derivatives, In: S. Yu, A. Paun (eds.), *CIAA 2000*, Springer, Berlin, *Lecture Notes in Computer Science* 2088 (2001) 94–104.
- [31] J.-M. Champarnaud, D. Ziadi, Computing the equation automaton of a regular expression in  $O(s^2)$  space and time, In: A. Amir, G. Landau (eds.), *CPM 2001*, Springer, Berlin, *Lecture Notes in Computer Science* 2089 (2001) 157–168.
- [32] J.-M. Champarnaud, D. Ziadi, Canonical derivatives, partial derivatives and finite automaton constructions, *Theoretical Computer Science* 289 (2002) 137–163.
- [33] W. Cheng, Z. Mo, Minimization algorithm of fuzzy finite automata, *Fuzzy Sets and Systems* 141 (2004) 439–448.
- [34] M. Ćirić, M. Droste, J. Ignjatović, H. Vogler, Determinization of weighted finite automata over strong bimonoids, *Information Sciences* 180 (2010) 3497–3520.
- [35] M. Ćirić, J. Ignjatović, M. Bašić, I. Jančić, Nondeterministic automata: Simulation, bisimulation and structural equivalence, submitted to *Computers & Mathematics with Applications*.
- [36] M. Ćirić, J. Ignjatović, S. Bogdanović, Fuzzy equivalence relations and their equivalence classes, *Fuzzy Sets and Systems* 158 (2007) 1295–1313.
- [37] M. Ćirić, J. Ignjatović, S. Bogdanović, Uniform fuzzy relations and fuzzy functions, *Fuzzy Sets and Systems* 160 (2009) 1054–1081.
- [38] M. Ćirić, J. Ignjatović, N. Damljanović, M. Bašić, Bisimulations for fuzzy automata, *Fuzzy Sets and Systems* (2011), doi: 10.1016/j.fss.2011.07.003.
- [39] M. Ćirić, J. Ignjatović, I. Jančić, N. Damljanović, Algorithms for computing the greatest simulations and bisimulations between fuzzy automata, submitted to *Fuzzy Sets and Systems*.
- [40] M. Ćirić, A. Stamenković, J. Ignjatović, T. Petković, Factorization of fuzzy automata, In: Csuhaj-Varju, E., Ésik, Z. (eds.), *FCT*



- 2007, Springer, Heidelberg, Lecture Notes in Computer Science 4639 (2007) 213–225.
- [41] M. Ćirić, A. Stamenković, J. Ignjatović, T. Petković, Fuzzy relation equations and reduction of fuzzy automata, *Journal of Computer and System Sciences* 76 (2010) 609–633.
- [42] G. B. Davis, K. M. Carley, Clearing the FOG: Fuzzy, overlapping groups for social networks, *Social Networks* 30 (2008) 201–212.
- [43] M. Demirci, Fuzzy functions and their applications, *Journal of Mathematical Analysis and Applications* 252 (2000) 495–517.
- [44] M. Demirci, Foundations of fuzzy functions and vague algebra based on many-valued equivalence relations, Part I: Fuzzy functions and their applications, *International Journal of general Systems* 32 (2) (2003) 123–155.
- [45] M. Demirci, A theory of vague lattices based on many-valued equivalence relations – I: general representation results, *Fuzzy Sets and Systems* 151 (2005) 437–472.
- [46] B. De Baets, Analytical solution methods for fuzzy relational equations, in: D. Dubois, H. Prade (eds.), *Fundamentals of Fuzzy Sets, The Handbooks of Fuzzy Sets Series, Vol. 1*, Kluwer Academic Publishers, 2000, pp. 291–340.
- [47] W. De Nooy, A. Mrvar, V. Batagelj, *Exploratory Network Analysis with Pajek*, Cambridge University Press, 2005.
- [48] A. Di Nola, A. Lettieri, I. Perfilieva, V. Novak, Algebraic analysis of fuzzy systems, *Fuzzy Sets and Systems* 158 (2007) 1–22.
- [49] A. Di Nola, E. Sanchez, W. Pedrycz, S. Sessa, *Fuzzy Relation Equations and Their Application to Knowledge Engineering*, Kluwer Academic Press, Dordrecht, 1989.
- [50] P. Doreian, Positional analysis and blockmodeling, in: R. A. Meyers (Ed.), *Encyclopedia of Complexity and Systems Science*, Springer, New York, 2009, pp. 6913–6927.
- [51] P. Doreian, V. Batagelj, A. Ferligoj, Generalized blockmodeling of two-mode network data, *Social Networks* 26 (2004) 29–53.
- [52] P. Doreian, V. Batagelj, A. Ferligoj, *Generalized Blockmodeling*, Cambridge University Press, 2005.
- [53] A. Dovier, C. Piazza, A. Policriti, An efficient algorithm for computing bisimulation equivalence, *Theoretical Computer Science* 311 (2004) 221–256.
- [54] M. Droste, T. Stüber, H. Vogler, Weighted finite automata over strong bimonoids, *Information Sciences* 180 (2010) 156–166.
- [55] D. Dubois, H. Prade, *Fuzzy Sets and Systems: Theory and Applications*, Academic Press, New York, 1980.
- [56] D. Dubois, H. Prade (eds.), *Fundamentals of Fuzzy Sets, The Handbooks of Fuzzy Sets Series, Vol. 1*, Kluwer Academic Publishers, 2000.
- [57] Z. Ésik, W. Kuich, A generalization of Kozen’s axiomatization of the equational theory of the regular sets, in: *Words, semigroups, and transductions*, World Scientific, River Edge, NJ, 2001, pp. 99–114.
- [58] Z. Ésik, A. Maletti, Simulation vs. Equivalence, *CoRR abs/1004.2426* (2010).
- [59] M. G. Everett, S. P. Borgatti, Regular equivalences: general theory, *Journal of Mathematical Sociology* 18 (1994) 29–52.
- [60] T. F. Fan, C. J. Liau, T. Y. Lin, Positional analysis in fuzzy social networks, in: *Proceedings of the Third IEEE International Conference on Granular Computing*, 2007, pp. 423–428.
- [61] T. F. Fan, C. J. Liau, T. S. Lin, A theoretical investigation of regular equivalences for fuzzy graphs, *International Journal of Approximate Reasoning* 49 (2008) 678–688.
- [62] R. Gentilini, C. Piazza, A. Policriti, From bisimulation to simulation: coarsest partition problems, *Journal of Automated Reasoning* 31 (2003) 73–103.
- [63] J. A. Goguen, L-fuzzy sets, *Journal of Mathematical Analysis and Applications* 18 (1967) 145–174.
- [64] J. S. Golan, *Semirings and their Applications*, Kluwer Academic Publishers, 1999.
- [65] M. Gondran, M. Minoux, *Graphs, Dioids and Semirings – New Models and Algorithms*, Springer, Berlin, 2008.
- [66] P. Hájek, *Mathematics of fuzzy logic*, Kluwer, Dordrecht, 1998.
- [67] R. A. Hanneman, M. Riddle, *Introduction to Social Network Methods*, University of California, Riverside, 2005.
- [68] T. A. Henzinger, P. W. Kopke, A. Puri, P. Varaiya, What’s decidable about hybrid automata? *Journal of Computer and System Sciences* 57 (1998) 94–124.
- [69] J. Högberg, A. Maletti, J. May, Backward and forward bisimulation minimisation of tree automata, in: J. Holub, J. Ždárek (eds.), *IAA07*, Springer, Heidelberg, Lecture Notes in Computer Science 4783 (2007) 109–121.
- [70] J. Högberg, A. Maletti, J. May, Backward and forward bisimulation minimisation of tree automata, *Theoretical Computer Science* 410 (2009) 3539–3552.
- [71] U. Höhle, Commutative, residuated  $\ell$ -monoids, in: U. Höhle and E. P. Klement (Eds.), *Non-Classical Logics and Their Applications to Fuzzy Subsets*, Kluwer Academic Publishers, Boston, Dordrecht, 1995, pp. 53–106.
- [72] B. Hruz, M. C. Zhou, *Modeling and control of discrete-event dynamical systems: with Petri nets and other tools*, Springer, 2007.
- [73] J. Ignjatović, M. Ćirić, Formal power series and regular operations on fuzzy languages, *Information Sciences* 180 (2010) 1104–1120.
- [74] J. Ignjatović, M. Ćirić, S. Bogdanović, Determinization of fuzzy automata with membership values in complete residuated lattices, *Information Sciences* 178 (2008) 164–180.
- [75] J. Ignjatović, M. Ćirić, S. Bogdanović, Fuzzy homomorphisms of algebras, *Fuzzy Sets and Systems* 160 (2009), 2345–2365.
- [76] J. Ignjatović, M. Ćirić, S. Bogdanović, On the greatest solutions to weakly linear systems of fuzzy relation inequalities and equations, *Fuzzy Sets and Systems* 161 (2010) 3081–3113.
- [77] J. Ignjatović, M. Ćirić, S. Bogdanović, T. Petković, Myhill-Nerode type theory for fuzzy languages and automata, *Fuzzy Sets and Systems* 161 (2010) 1288–1324.
- [78] J. Ignjatović, M. Ćirić, N. Damljanović, I. Jančić, Weakly linear systems of fuzzy relation inequalities: The heterogeneous case, submitted to *Fuzzy Sets and Systems*.
- [79] L. Ilie, S. Yu, Constructing NFAs by optimal use of positions in regular expressions, In: A. Apostolico, M. Takeda (eds.), *CPM 2002*, Springer, Berlin, Lecture Notes in Computer Science 2373 (2002) 279–288.
- [80] L. Ilie, S. Yu, Algorithms for computing small NFAs, in: K. Diks et al. (eds): *MFCS 2002*, Lecture Notes in Computer Science 2420 (2002) 328–340.

- [81] L. Ilie, S. Yu, Reducing NFAs by invariant equivalences, *Theoretical Computer Science* 306 (2003) 373–390.
- [82] L. Ilie, S. Yu, Follow automata, *Information and Computation* 186 (2003) 140–162
- [83] L. Ilie, G. Navarro, S. Yu, On NFA reductions, in: J. Karhumäki et al. (eds): *Theory is Forever, Lecture Notes in Computer Science* 3113 (2004) 112–124.
- [84] L. Ilie, R. Solis-Oba, S. Yu, Reducing the size of NFAs by using equivalences and preorders, in: A. Apostolico, M. Crochemore, and K. Park (eds): *CPM 2005, Lecture Notes in Computer Science* 3537 (2005) 310–321.
- [85] Z. Jančić, J. Ignjatović, M. Ćirić, An improved algorithm for determinization of weighted and fuzzy automata, *Information Sciences* 181 (2011) 1358–1368.
- [86] P. C. Kannelakis, S. A. Smolka, CCS expressions, finite state processes, and three problems of equivalence, *Information and Computation* 86 (1990) 43–68.
- [87] F. Klawonn, Fuzzy points, fuzzy relations and fuzzy functions, in: V. Novák and I. Perfilieva (Eds.), *Discovering World with Fuzzy Logic*, Physica-Verlag, Heidelberg, 2000, pp. 431–453.
- [88] G. J. Klir, B. Yuan, *Fuzzy Sets and Fuzzy Logic, Theory and Application*, Prentice-Hall, Englewood Cliffs, NJ, 1995.
- [89] D. C. Kozen, *Automata and Computability*, Springer, 1997.
- [90] H. Lei, Y. M. Li, Minimization of states in automata theory based on finite lattice-ordered monoids, *Information Sciences* 177 (2007) 1413–1421.
- [91] J. Lerner, Role assignments, in: U. Brandes, T. Erlebach (eds.), *Network Analysis: Methodological Foundations, Lecture Notes in Computer Science*, vol. 3418, Springer, 2005, pp. 216–252.
- [92] T.-J. Li, Y. Leung, W.-X. Zhang, Generalized fuzzy rough approximation operators based on fuzzy coverings, *International Journal of Approximate Reasoning* 48 (2008) 836–856.
- [93] Y. M. Li, W. Pedrycz, Fuzzy finite automata and fuzzy regular expressions with membership values in lattice ordered monoids, *Fuzzy Sets and Systems* 156 (2005) 68–92.
- [94] F. Lorrain, H. C. White, Structural equivalence of individuals in social networks, *Journal of Mathematical Sociology* 1 (1971) 49–80.
- [95] N. Lynch, F. Vaandrager, Forward and backward simulations: Part I. Untimed systems, *Information and Computation* 121 (1995), 214–233.
- [96] D. S. Malik, J. N. Mordeson, M. K. Sen, Minimization of fuzzy finite automata, *Information Sciences* 113 (1999) 323–330.
- [97] M. Marx, M. Masuch, Regular equivalence and dynamic logic, *Social Networks* 25 (2003) 51–65.
- [98] P. Mika, *Social Networks and the Semantic Web*, Springer, 2007.
- [99] R. Milner, A calculus of communicating systems, in G. Goos and J. Hartmanis (eds.), *Lecture Notes in Computer Science*, vol. 92, Springer, 1980.
- [100] R. Milner, *Communication and Concurrency*, Prentice-Hall International, 1989.
- [101] R. Milner, *Communicating and Mobile Systems: the  $\pi$ -Calculus*, Cambridge University Press, Cambridge, 1999.
- [102] J. N. Mordeson, D. S. Malik, *Fuzzy Automata and Languages: Theory and Applications*, Chapman & Hall/CRC, Boca Raton, London, 2002.
- [103] P. S. Nair, S. Sarasamma, Data mining through fuzzy social network analysis, in: *Proceedings of the 26th Annual Meeting of the North American Fuzzy Information Processing Society*, 2007, pp. 251–255.
- [104] M. E. J. Newman, Analysis of weighted networks, *Physical Review E* 70 (2004), 056131:1–9.
- [105] V. Novák, *Fuzzy Sets and Their Applications*, Adam Hilger, Bristol, 1989.
- [106] R. Paige, R. E. Tarjan, Three partition refinement algorithms, *SIAM Journal of Computing* 16 (1987) 973–989.
- [107] D. Park, Concurrency and automata on infinite sequences, in: P. Deussen (ed.), *Proc. 5th GI Conf.*, Karlsruhe, Germany, *Lecture Notes in Computer Science* 104 (1981), Springer-Verlag, pp. 167–183.
- [108] P. Pattison, The analysis of semigroups of multirelational systems, *Journal of Mathematical Psychology* 25 (1982) 87–117.
- [109] P. E. Pattison, *Algebraic Models for Social Networks*, Cambridge University Press, Cambridge, 1993.
- [110] W. Pedrycz, F. Gomide, *Fuzzy Systems Engineering: Toward Human-Centric Computing*, Wiley-IEEE Press, 2007.
- [111] K. Peeva, Finite L-fuzzy machines, *Fuzzy Sets and Systems* 141 (2004) 415–437.
- [112] K. Peeva, Y. Kyosev, *Fuzzy Relational Calculus: Theory, Applications, and Software (with CD-ROM)*, in Series “Advances in Fuzzy Systems – Applications and Theory”, Vol 22, World Scientific, 2004.
- [113] K. Peeva, Y. Kyosev, Algorithm for solving max-product fuzzy relational equations, *Soft Computing* 11 (2007) 593–605.
- [114] K. Peeva, Z. Zahariev, Computing behavior of finite fuzzy machines – Algorithm and its application to reduction and minimization, *Information Sciences* 178 (2008) 4152–4165.
- [115] I. Perfilieva, Fuzzy function as an approximate solution to a system of fuzzy relation equations, *Fuzzy Sets and Systems* 147 (2004) 363–383.
- [116] I. Perfilieva, S. Gottwald, Fuzzy function as a solution to a system of fuzzy relation equations, *International Journal of General Systems* 32 (2003) 361–372.
- [117] I. Perfilieva, V. Novák, System of fuzzy relation equations as a continuous model of IF-THEN rules, *Information Sciences* 177 (2007) 3218–3227.
- [118] T. Petković, Congruences and homomorphisms of fuzzy automata, *Fuzzy Sets and Systems* 157 (2006) 444–458.
- [119] F. Ranzato, F. Tapparo, Generalizing the Paige-Tarjan algorithm by abstract interpretation, *Information and Computation* 206 (2008) 620–651.
- [120] M. Roggenbach, M. Majster-Cederbaum, Towards a unified view of bisimulation: a comparative study, *Theoretical Computer Science* 238 (2000) 81–130.
- [121] S. Roman, *Lattices and Ordered Sets*, Springer, New York, 2008.
- [122] E. Sanchez, *Equations de relations floues*, Thèse de Doctorat, Faculté de Médecine de Marseille, 1974.
- [123] E. Sanchez, Resolution of composite fuzzy relation equations, *Information and Control* 30 (1976) 38–48.

- [124] E. Sanchez, Solutions in composite fuzzy relation equations: application to medical diagnosis in Brouwerian logic, in: M. M. Gupta, G. N. Saridis, B. R. Gaines (Eds.), *Fuzzy Automata and Decision Processes*, North-Holland, Amsterdam, 1977, pp. 221–234.
- [125] E. Sanchez, Resolution of eigen fuzzy sets equations, *Fuzzy Sets and Systems* 1 (1978) 69–74.
- [126] D. Sangiorgi, On the origins of bisimulation and coinduction, *ACM Transactions on Programming Languages and Systems* 31 (4) (2009) 111–151.
- [127] S. S. Skiena, *The Algorithm Design Manual*, Springer, London, 2008.
- [128] A. Stamenković, M. Ćirić, Construction of fuzzy automata from fuzzy regular expressions, submitted to *Fuzzy Sets and Systems*.
- [129] A. Stamenković, M. Ćirić, J. Ignjatović, Reduction of fuzzy automata by means of fuzzy quasi-orders, submitted to *Information Sciences*.
- [130] I. Stanković, J. Ignjatović, M. Ćirić, Boolean relation equations in data analysis, to appear in *Proceedings of the 9th IEEE International Symposium on Intelligent Systems and Informatics (SISY 2011)*, Subotica, Serbia, 2011.
- [131] D. D. Sun, Y. M. Li, W. W. Yang, Bisimulation relations for fuzzy finite automata, *Fuzzy Systems and Mathematics* 23 (2009) 92–100 (in Chinese).
- [132] S. Wasserman, K. Faust, *Social Network Analysis: Methods and Applications*, Cambridge University Press, Cambridge, 1994.
- [133] D. R. White, K. P. Reitz, Graph and semigroup homomorphisms on networks and relations, *Social Networks* 5 (1983) 143–234.
- [134] L. H. Wu, D. W. Qiu, Automata theory based on complete residuated lattice-valued logic: Reduction and minimization, *Fuzzy Sets and Systems* 161 (2010) 1635–1656.
- [135] H. Xing, D. W. Qiu, F. C. Liu, Z. J. Fan, Equivalence in automata theory based on complete residuated lattice-valued logic, *Fuzzy Sets and Systems* 158 (2007) 1407–1422.
- [136] A. Žiberna, Generalized blockmodeling of valued networks, *Social Networks* 29 (2007) 105–126.