Filomat 26:2 (2012), 363–370
DOI 10.2298/FIL1202363T

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Ricci solitons and gradient Ricci solitons in three-dimensional trans-Sasakian manifolds

Mine Turan[a], Uday Chand De[b], Ahmet Yildiz[a]

[a] *Art and Science Faculty, Department of Mathematics, Dumlupınar University, Kütahya, Turkey*
[b] *Department of Pure Mathematics, University of Calcutta, 35, B.C. Road, Kolkata 700019, West Bengal, India*

Abstract. The object of the present paper is to study 3-dimensional trans-Sasakian manifolds admitting Ricci solitons and gradient Ricci solitons. We prove that if (g, V, λ) is a Ricci soliton where V is collinear with the characteristic vector field ξ, then V is a constant multiple of ξ and the manifold is of constant scalar curvature provided α, β =constant. Next we prove that in a 3-dimensional trans-Sasakian manifold with constant scalar curvature if g is a gradient Ricci soliton, then the manifold is either a β-Kenmotsu manifold or an Einstein manifold. As a consequence of this result we obtain several corollaries.

1. Introduction

A Ricci soliton is a generalization of an Einstein metric. In a Riemannian manifold (M, g), g is called a Ricci soliton if [17]

$$\pounds_V g + 2S + 2\lambda g = 0, \tag{1}$$

where £ is the Lie derivative, S is the Ricci tensor, V is a complete vector field on M and λ is a constant. Metrics satisfying (1) are interesting and useful in physics and are often referred as quasi-Einstein (e.g. [10, 19, 20]). Compact Ricci solitons are the fixed points of the Ricci flow $\frac{\partial}{\partial t}g = -2S$ projected from the space of metrics onto its quotient modulo diffeomorphisms and scalings, and often arise as blow-up limits for the Ricci flow on compact manifolds. Theoretical physicists have also been looking into the equation of Ricci soliton in relation with string theory. The initial contribution in this direction is due to Friedan who discusses some aspects of it in [10].

The Ricci soliton is said to be shrinking, steady and expanding according as λ is negative, zero and positive respectively. If the vector field V is the gradient of a potential function $-f$, then g is called a gradient Ricci soliton and equation (1) assumes the form $\nabla\nabla f = S + \lambda g$.

A Ricci soliton on a compact manifold has constant curvature in dimension 2 [17] and also in dimension 3 [21]. For details we refer to Chow and Knopf [3] and Derdzinski [1].

On the other hand, the roots of contact geometry lie in differential equations as in 1872 Sophus Lie introduced the notion of contact transformation as a geometric tool to study systems of differential equations. This subject has manifold connections with the other fields of pure mathematics, and substantial applications

2010 *Mathematics Subject Classification*. Primary 53C15; Secondary 53C25, 53A30
Keywords. Trans-Sasakian manifold, Ricci soliton, gradient Ricci soliton
Received: 15 November 2009; Accepted: 27 June 2011
Communicated by Vladimir Dragović
This work was done when the second author visited Dumlupinar University in November, 2009

in applied areas such as mechanics, optics, phase space of dynamical system, thermodynamics and control theory. For more details see [12, 18, 24, 25].

Trans-Sasakian manifolds arose in a natural way from the classification of almost contact metric structures by Chinae and Gonzales [9], and they appear as a natural generalization of both Sasakian and Kenmotsu manifolds. Again in the Gray-Hervella classification of almost Hermitian manifolds [2], there appears a class, W_4, of Hermitian manifolds which are closely related to locally conformal Kaehler manifolds. An almost contact metric structure on a manifold M is called a trans-Sasakian structure [15] if the product manifold $M \times \mathbb{R}$ belongs to the class W_4. The class $C_6 \oplus C_5$ [14] coincides with the class of the trans-Sasakian structures of type (α, β). In [14], the local nature of the two subclasses C_5 and C_6 of trans-Sasakian structures is characterized completely. In [9], some curvature identities and sectional curvatures for C_5, C_6 and trans-Sasakian manifolds are obtained. It is known that [8] trans-Sasakian structures of type $(0, 0)$, $(0, \beta)$ and $(\alpha, 0)$ are cosymplectic, β-Kenmotsu [8] and α-Sasakian [8], respectively.

The local structure of trans-Sasakian manifolds of dimension $n \geq 5$ has been completely characterized by J. C. Marrero [13]. He proved that a trans-Sasakian manifold of $n \geq 5$ is either cosymplectic or α-Sasakian or β-Kenmotsu.

In [16], Sharma has started the study of Ricci solitons in K-contact manifolds. In a K-contact manifold the structure vector field ξ is Killing, that is, $\pounds_\xi g = 0$, which is not in general, in a trans-Sasakian manifold.

Motivated by these circumtances in this paper we study Ricci solitons and gradient Ricci solitons in three-dimensional trans-Sasakian manifolds. Throughout the paper we assume that α, β =constant.

The present paper is organized as follows: After Preliminaries in section 3, we give an example of a 3-dimensional trans-Sasakian manifold with α, β =constant. In section 4, we study Ricci solitons in a 3-dimensional trans-Sasakian manifold and prove that if the vector field V is collinear with the Reeb vector field ξ, then V is a constant multiple of ξ and the manifold is of constant scalar curvature. Also we prove that if g is a Ricci soliton and $V = \xi$, then the Ricci soliton is shrinking. Finally we prove that if a 3-dimensional trans-Sasakian manifold with constant scalar curvature admits gradient Ricci soliton, then the manifold is either a β-Kenmotsu manifold or an Einstein manifold. As a consequence of this result we obtain several corollaries.

2. Preliminaries

Let M be a connected almost contact metric manifold with an almost contact metric structure(ϕ, ξ, η, g), that is, ϕ is an $(1, 1)$-tensor field , ξ is a vector field , η is a 1-form and g is the compatible Riemannian metric such that

$$\phi^2(X) = -X + \eta(X)\xi, \quad \eta(\xi) = 1, \quad \phi\xi = 0, \quad \eta \circ \phi = 0,$$

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y),$$

$$g(X, \phi Y) = -g(\phi X, Y), \quad g(X, \xi) = \eta(X),$$

for all $X, Y \in TM$ ([4], [5]). The fundamental 2-form Φ of the manifold is defined by $\Phi(X, Y) = g(X, \phi Y)$, for $X, Y \in TM$.

An almost contact metric structure (ϕ, ξ, η, g) on a connected manifold M is called a trans-Sasakian structure [15] if $(M \times \mathbb{R}, J, G)$ belongs to the class W_4 [2], where J is the almost complex structure on $M \times \mathbb{R}$ defined by $J(X, f d/dt) = (\phi X - f\xi, \eta(X)d/dt)$, for all vector fields X on M and smooth functions f on $M \times \mathbb{R}$, and G is the product metric on $M \times \mathbb{R}$. This may be expressed by the condition [6]

$$(\nabla_X \phi)Y = \alpha(g(X, Y)\xi - \eta(Y)X) + \beta(g(\phi X, Y)\xi - \eta(Y)\phi X), \tag{2}$$

for smooth functions α and β on M. Here we say that the trans-Sasakian structure is of type (α, β). From the formula (2) it follows that

$$\nabla_X \xi = -\alpha\phi X + \beta(X - \eta(X)\xi), \tag{3}$$

$$(\nabla_X\eta)Y = -\alpha g(\phi X, Y) + \beta g(\phi X, \phi Y).$$

An explicit example of 3-dimensional proper trans-Sasakian manifold is constructed in [13]. In [23], the Ricci tensor and curvature tensor for 3-dimensional trans-Sasakian manifolds are studied and their explicit formulae are given.

From [23] we know that for a 3-dimensional trans-Sasakian manifold

$$2\alpha\beta + \xi\alpha = 0, \tag{4}$$

$$S(X,\xi) = (2(\alpha^2-\beta^2) - \xi\beta)\eta(X) - X\beta - (\phi X)\alpha,$$

$$S(X,Y) = (\frac{\tau}{2}+\xi\beta-(\alpha^2-\beta^2))g(X,Y)-(\frac{\tau}{2}+\xi\beta-3(\alpha^2-\beta^2))\eta(X)\eta(Y)-(Y\beta+(\phi Y)\alpha)\eta(X)-(X\beta+(\phi X)\alpha)\eta(Y),$$

and

$$\begin{aligned} R(X,Y)Z \;=\; & (\frac{\tau}{2}+2\xi\beta-2(\alpha^2-\beta^2))(g(Y,Z)X-g(X,Z)Y)-g(Y,Z)[(\frac{\tau}{2}+\xi\beta-3(\alpha^2-\beta^2))\eta(X)\xi \\ & -\eta(X)(\phi grad\alpha - grad\beta)+(X\beta+(\phi X)\alpha)\xi]+g(X,Z)[(\frac{\tau}{2}+\xi\beta-3(\alpha^2-\beta^2))\eta(Y)\xi \\ & -\eta(Y)(\phi grad\alpha - grad\beta)+(Y\beta+(\phi Y)\alpha)\xi]-[(Z\beta+(\phi Z)\alpha)\eta(Y)+(Y\beta+(\phi Y)\alpha)\eta(Z) \\ & +(\frac{\tau}{2}+\xi\beta-3(\alpha^2-\beta^2))\eta(Y)\eta(Z)]X+[(Z\beta+(\phi Z)\alpha)\eta(X)+(X\beta+(\phi X)\alpha)\eta(Z) \\ & +(\frac{\tau}{2}+\xi\beta-3(\alpha^2-\beta^2))\eta(X)\eta(Z)]Y, \end{aligned}$$

where S is the Ricci tensor of type $(0,2)$, R is the curvature tensor of type $(1,3)$ and τ is the scalar curvature of the manifold M.

For α, β =constant the above relations become

$$S(X,Y) = (\frac{\tau}{2}-(\alpha^2-\beta^2))g(X,Y)-(\frac{\tau}{2}-3(\alpha^2-\beta^2))\eta(X)\eta(Y), \tag{5}$$

$$S(X,\xi) = 2(\alpha^2-\beta^2)\eta(X), \tag{6}$$

$$R(X,Y)\xi = (\alpha^2-\beta^2)(\eta(Y)X-\eta(X)Y), \tag{7}$$

$$QX = (\frac{\tau}{2}-(\alpha^2-\beta^2))X-(\frac{\tau}{2}-3(\alpha^2-\beta^2))\eta(X)\xi. \tag{8}$$

From (4) it follows that if α, β =constant, then the manifold is either α-Sasakian or β-Kenmotsu or cosymplectic.

Proposition 2.1. *A 3-dimensional trans-Sasakian manifold with α, β =constant, is either α-Sasakian or β-Kenmotsu or cosymplectic.*

We note that α-Sasakian manifolds are quasi-Sasakian [7]. They provide examples of $C(\lambda)$-manifolds with $\lambda > 0$.

A β-Kenmotsu manifold is a $C(-\beta^2)$-manifold.

Cosymplectic manifolds provide a natural setting for time dependent mechanical systems as they are local products of a Kaehler manifold and real line or a circle [11].

3. Example of a 3-dimensional trans-Sasakian manifold

In this section we give an example of a 3-dimensional trans-Sasakian manifold with α, β =constant.

We consider the 3-dimensional manifold $M = \{(x,y,z) \in \mathbb{R}^3, z \neq 0\}$, where (x,y,z) are the standard coordinates in $\mathbb{R}^3$. The vector fields

$$e_1 = z\frac{\partial}{\partial x}, \quad e_2 = z\frac{\partial}{\partial y}, \quad e_3 = z\frac{\partial}{\partial z}$$

are linearly independent at each point of M. Lat g be the Riemannian metric defined by

$$\begin{aligned} g(e_1, e_3) &= g(e_2, e_3) = g(e_1, e_2) = 0, \\ g(e_1, e_1) &= g(e_2, e_2) = g(e_3, e_3) = 1. \end{aligned}$$

Let η be the 1-form defined by $\eta(Z) = g(Z, e_3)$ for any $Z \in \chi(M)$. Let ϕ be the $(1,1)$ tensor field defined by $\phi(e_1) = -e_2$, $\phi(e_2) = e_1$, $\phi(e_3) = 0$. Then using linearity of ϕ and g we have

$$\eta(e_3) = 1, \quad \phi^2 Z = -Z + \eta(Z)e_3,$$

$$g(\phi Z, \phi W) = g(Z, W) - \eta(Z)\eta(W),$$

for any $Z, W \in \chi(M)$. Thus for $e_3 = \xi$, (ϕ, ξ, η, g) defines an almost contact metric structure on M. Now, by direct computations we obtain $[e_1, e_2] = 0$ $[e_2, e_3] = -e_2$, $[e_1, e_3] = -e_1$.

The Riemannian connection ∇ of the metric tensor g is given by the Koszul's formula which is

$$2g(\nabla_X Y, Z) = Xg(Y, Z) + Yg(Z, X) - Zg(X, Y) - g(X, [Y, Z]) - g(Y, [X, Z]) + g(Z, [X, Y]). \tag{9}$$

Using (9) we have

$$\begin{aligned} 2g(\nabla_{e_1} e_3, e_1) &= 2g(-e_1, e_1), \\ 2g(\nabla_{e_1} e_3, e_2) &= 0 = 2g(-e_1, e_2), \\ 2g(\nabla_{e_1} e_3, e_3) &= 0 = 2g(-e_1, e_3). \end{aligned}$$

Hence $\nabla_{e_1} e_3 = -e_1$. Similarly, $\nabla_{e_2} e_3 = -e_2$ and $\nabla_{e_3} e_3 = 0$. Equation (9) further yields

$$\begin{aligned} \nabla_{e_1} e_2 &= 0, \qquad \nabla_{e_1} e_1 = e_3, \\ \nabla_{e_2} e_2 &= e_3, \qquad \nabla_{e_2} e_1 = 0, \\ \nabla_{e_3} e_2 &= 0, \qquad \nabla_{e_3} e_1 = 0. \end{aligned}$$

We see that

$$\begin{aligned} (\nabla_{e_1}\phi)e_1 &= \nabla_{e_1}\phi e_1 - \phi\nabla_{e_1} e_1 = -\nabla_{e_1} e_2 - \phi e_3 = -\nabla_{e_1} e_2 = 0 \\ &= 0(g(e_1, e_1)e_3 - \eta(e_1)e_1) - 1(g(\phi e_1, e_1)e_3 - \eta(e_1)\phi e_1). \end{aligned} \tag{10}$$

$$\begin{aligned} (\nabla_{e_1}\phi)e_2 &= \nabla_{e_1}\phi e_2 - \phi\nabla_{e_1} e_2 = -\nabla_{e_1} e_1 - 0 = e_3 \\ &= 0(g(e_1, e_2)e_3 - \eta(e_2)e_1) - 1(g(\phi e_1, e_2)e_3 - \eta(e_2)\phi e_1). \end{aligned} \tag{11}$$

$$\begin{aligned} (\nabla_{e_1}\phi)e_3 &= \nabla_{e_1}\phi e_3 - \phi\nabla_{e_1} e_3 = 0 + \phi e_1 = -e_2 \\ &= 0(g(e_1, e_3)e_3 - \eta(e_3)e_1) - 1(g(\phi e_1, e_3)e_3 - \eta(e_3)\phi e_1). \end{aligned} \tag{12}$$

By (10), (11) and (12) we see that the manifold satisfies (2) for $X = e_1$, $\alpha = 0$, $\beta = -1$, and $e_3 = \xi$. Similarly, it can be shown that for $X = e_2$ and $X = e_3$ the manifold also satisfies (2) for $\alpha = 0$, $\beta = -1$, and $e_3 = \xi$. Hence the manifold is a trans-Sasakian manifold of type $(0, -1)$.

4. Ricci soliton

Let M be a 3-dimensional trans-Sasakian manifold with metric g. A Ricci soliton is a generalization of an Einstein metric and defined on a Riemannian manifold (M, g) by

$$\pounds_V g + 2S + 2\lambda g = 0. \tag{13}$$

Let V be pointwise collinear with ξ i.e. $V = b\xi$, where b is a function on the 3-dimensional trans-Sasakian manifold. Then $(\pounds_V g + 2S + 2\lambda g)(X, Y) = 0$, implies

$$g(\nabla_X b\xi, Y) + g(\nabla_Y b\xi, X) + 2S(X, Y) + 2\lambda g(X, Y) = 0,$$

or,

$$bg((\nabla_X \xi, Y) + (Xb)\eta(Y) + bg(\nabla_Y \xi, X) + (Yb)\eta(X) + 2S(X, Y) + 2\lambda g(X, Y) = 0.$$

Using (3), we obtain

$$bg(-\alpha\phi X + \beta(X - \eta(X)\xi), Y) + (Xb)\eta(Y) + bg(-\alpha\phi Y + \beta(Y - \eta(Y)\xi), X) + (Yb)\eta(X) + 2S(X, Y) + 2\lambda g(X, Y) = 0,$$

which yields

$$2b\beta g(X, Y) - 2b\beta\eta(X)\eta(Y) + (Xb)\eta(Y) + (Yb)\eta(X) + 2S(X, Y) + 2\lambda g(X, Y) = 0. \tag{14}$$

In (14) replacing Y by ξ it follows that

$$Xb + (\xi b)\eta(X) + 2(2(\alpha^2 - \beta^2)\eta(X)) + 2\lambda\eta(X) = 0. \tag{15}$$

Again putting $X = \xi$ in (15) yields $\xi b = -2(\alpha^2 - \beta^2) - \lambda$.
Putting this value in (15), we get

$$Xb + (-2(\alpha^2 - \beta^2) - \lambda)\eta(X) + 2(2(\alpha^2 - \beta^2))\eta(X) + 2\lambda\eta(X) = 0,$$

or,

$$db = -\{\lambda + 2(\alpha^2 - \beta^2)\}\eta. \tag{16}$$

Applying d on (16), we get $\{\lambda + 2(\alpha^2 - \beta^2)\}d\eta = 0$.
Since $d\eta \neq 0$ we have

$$\lambda + 2(\alpha^2 - \beta^2) = 0. \tag{17}$$

Using (17) in (16) yields b is a constant. Therefore from (14) it follows

$$S(X, Y) = -(\lambda + b\beta)g(X, Y) + b\beta\eta(X)\eta(Y),$$

which implies that M is of constant scalar curvature provided β =constant. This leads to the following:

Theorem 4.1. *If in a 3-dimensional trans-Sasakian manifold the metric g is a Ricci soliton and V is pointwise collinear with ξ, then V is a constant multiple of ξ and g is of constant scalar curvature provided β = constant.*

Now let $V = \xi$. Then the equation (13) reduces to

$$\pounds_\xi g + 2S + 2\lambda g = 0. \tag{18}$$

Using (3), we get

$$(\pounds_\xi g)(X, Y) = 2\beta\{g(X, Y) - \eta(X)\eta(Y)\}. \tag{19}$$

Therefore

$$A(X, Y) = (\pounds_\xi g + 2S)(X, Y) = (\pounds_\xi g)(X, Y) + 2S(X, Y). \tag{20}$$

Now using (5) and (19) from (20) we obtain

$$A(X, Y) = 2\beta\{g(X, Y) - \eta(X)\eta(Y)\} + 2\{(\frac{\tau}{2} - (\alpha^2 - \beta^2))g(X, Y) - (\frac{\tau}{2} - 3(\alpha^2 - \beta^2))\eta(X)\eta(Y)\}. \tag{21}$$

Applying (21) in (18) we get

$$2\{\beta - (\alpha^2 - \beta^2) + \lambda\}g(X, Y) - 2\{\beta - \frac{\tau}{2} + 3(\alpha^2 - \beta^2)\}\eta(X)\eta(Y) = 0. \tag{22}$$

Now taking $X = Y = \xi$ in (22) we obtain $\lambda = 2(\beta^2 - \alpha^2)$.
Since $\alpha^2 \neq \beta^2$, Therefore $\lambda > 0$ and Ricci soliton is shrinking.

Theorem 4.2. *If a 3-dimensional trans-Sasakian manifold admits a Ricci soliton (g, ξ, λ) then the Ricci soliton is shrinking.*

5. Gradient Ricci Soliton

If the vector field V is the gradient of a potential function $-f$ then g is called a gradient Ricci soliton and (13) assume the form

$$\nabla\nabla f = S + \lambda g. \tag{23}$$

This reduces to

$$\nabla_Y Df = QY + \lambda Y, \tag{24}$$

where D denotes the gradient operator of g. From (24) it follows

$$R(X,Y)Df = (\nabla_X Q)Y - (\nabla_Y Q)X. \tag{25}$$

Differentiating (8) we have

$$(\nabla_W Q)(X) = \frac{d\tau(W)}{2}(X - \eta(X)\xi) - (\frac{\tau}{2} - 3(\alpha^2 - \beta^2))(-\alpha g(\phi W, X) + \beta(g(W,X) - \eta(X)\eta(W)) + \eta(X)\nabla_W \xi). \tag{26}$$

In (26) replacing W by ξ yields

$$(\nabla_\xi Q)(X) = \frac{d\tau(\xi)}{2}(X - \eta(X)\xi).$$

Then we have

$$g((\nabla_\xi Q)(X) - (\nabla_X Q)(\xi), \xi) = g(\frac{d\tau(\xi)}{2}(X - \eta(X)\xi), \xi) = \frac{d\tau(\xi)}{2}(g(X,\xi) - \eta(X)) = 0. \tag{27}$$

Using (27) from (25), we obtain

$$g(R(\xi, X)Df, \xi) = 0. \tag{28}$$

From (7) we get

$$g(R(\xi, Y)Df, \xi) = (\alpha^2 - \beta^2)(g(Y, Df) - \eta(Y)\eta(Df)).$$

Using (28) in the above equation yields

$$(\alpha^2 - \beta^2)(g(Y, Df) - \eta(Y)\eta(Df)) = 0,$$

or,

$$(\alpha^2 - \beta^2)(g(Y, Df) - \eta(Y)g(Df, \xi)) = 0,$$

which implies

$$Df = (\xi f)\xi, \quad \text{since } \alpha^2 \neq \beta^2. \tag{29}$$

Using (29) in (24)

$$\begin{aligned} S(X,Y) + \lambda g(X,Y) &= g(\nabla_Y Df, X) = g(\nabla_Y (\xi f)\xi, X) \\ &= (\xi f)g(\nabla_Y \xi, X) + Y(\xi f)\eta(X) \\ &= (\xi f)g((-\alpha\phi Y + \beta(Y - \eta(X)\xi), X) + Y(\xi f)\eta(X) \\ &= -\alpha(\xi f)g(\phi Y, X) + \beta(\xi f)g(Y,X) - \beta(\xi f)\eta(Y)\eta(X) + Y(\xi f)\eta(X). \end{aligned} \tag{30}$$

Putting $X = \xi$ in (30) and using (6) we get

$$S(Y,\xi) + \lambda\eta(Y) = Y(\xi f) = \{\lambda + 2(\alpha^2 - \beta^2)\}\eta(Y). \tag{31}$$

Interchanging X and Y in (30) we obtain

$$S(X,Y) + \lambda g(X,Y) = -\alpha(\xi f)g(Y,\phi X) + \beta(\xi f)g(X,Y) - \beta(\xi f)\eta(X)\eta(Y) + X(\xi f)\eta(Y). \tag{32}$$

Adding (30) and (32) we get

$$2S(X,Y) + 2\lambda g(X,Y) = 2\beta(\xi f)g(X,Y) - 2\beta(\xi f)\eta(X)\eta(Y) + Y(\xi f)\eta(X) + X(\xi f)\eta(Y). \tag{33}$$

Using (31) in (33) we have

$$S(X,Y) + \lambda g(X,Y) = \beta(\xi f)\{g(X,Y) - \eta(X)\eta(Y)\} + (\lambda + 2(\alpha^2 - \beta^2))\eta(X)\eta(Y).$$

Then using (24) we have

$$\nabla_Y Df = \beta(\xi f)(Y - \eta(Y)\xi) + (\lambda + 2(\alpha^2 - \beta^2))\eta(Y)\xi. \tag{34}$$

Using (34) we calculate

$$\begin{aligned} R(X,Y)Df &= \nabla_X\nabla_Y Df - \nabla_Y\nabla_X Df - \nabla_{[X,Y]}Df \\ &= \beta X(\xi f)Y - \beta Y(\xi f)X + \beta Y(\xi f)\eta(X)\xi - \beta X(\xi f)\eta(Y)\xi \\ &\quad +(\lambda + 2(\alpha^2 - \beta^2) - \beta(\xi f))((\nabla_X\eta)(Y)\xi - (\nabla_Y\eta)(X)\xi) \\ &\quad +(\lambda + 2(\alpha^2 - \beta^2) - \beta(\xi f))((\nabla_X\xi)\eta(Y) - (\nabla_Y\xi)\eta(X)). \end{aligned} \tag{35}$$

Taking inner product with ξ in (35),we get

$$0 = g(R(X,Y)Df,\xi) = 2\alpha(\lambda + 2(\alpha^2 - \beta^2) - \beta(\xi f))g(\phi Y, X).$$

Thus we have $2\alpha(\lambda + 2(\alpha^2 - \beta^2) - \beta(\xi f)) = 0$.

Now we consider the following cases:

Case i) $\alpha = 0$, or
Case ii) $\lambda + 2(\alpha^2 - \beta^2) - \beta(\xi f) = 0$,
Case iii) $\alpha = 0$ and $\lambda + 2(\alpha^2 - \beta^2) - \beta(\xi f) = 0$.

Case i) If $\alpha = 0$, then the manifold reduces to a β-Kenmotsu manifold.

Case ii) Let $\lambda + 2(\alpha^2 - \beta^2) - \beta(\xi f) = 0$. If we use this in (31) we get $Y(\xi f) = \beta(\xi f)\eta(Y)$. Substitute this value in (33) we obtain$S(X,Y) + \lambda g(X,Y) = \beta(\xi f)g(X,Y)$. Contracting this equation, we get $\tau + 3\lambda = 3\beta(\xi f)$, which implies that

$$(\xi f) = \frac{\tau}{3\beta} + \frac{\lambda}{\beta}.$$

If τ =constant, then (ξf) =constant= c(say). Therefore from (29) we have $Df = (\xi f)\xi = c\xi$. Thus we can write from this equation $g(Df,X) = c\eta(X)$, which means that $df(X) = c\eta(X)$. Applying d this, we get $cd\eta = 0$. Since $d\eta \neq 0$, we have $c = 0$. Hence we get $Df = 0$. This means that f =constant. Therefore equation (23) reduces to$S(X,Y) = 2(\alpha^2 - \beta^2)g(X,Y)$, that is, M is an Einstein manifold.

Case iii) Using $\alpha = 0$ and $\lambda + 2(\alpha^2 - \beta^2) - \beta(\xi f) = 0$ in (31) we obtain $Y(\xi f) = \beta(\xi f)\eta(Y)$. Now as in *Case ii)* we conclude that the manifold is an Einstein manifold.

Thus we have the following:

Theorem 5.1. *If a 3-dimensional trans-Sasakian manifold with constant scalar curvature admits gradient Ricci soliton, then the manifold is either a β-Kenmotsu manifold or an Einstein manifold provided α, β =constant.*

In a recent paper De and Sarkar [22] proved that if a 3-dimensional compact connected trans-Sasakian manifold is of constant curvature, then it is either α-Sasakian or β-Kenmotsu. Since for a 3-dimensional Riemannian manifold constant curvature and Einstein manifold are equivalent, therefore from the Theorem 3 we state the following:

Corollary 5.2. *If a compact 3-dimensional trans-Sasakian manifold with constant scalar curvature admits Ricci soliton, then the manifold is either α-Sasakian or β-Kenmotsu.*

Also in [22] authors proved that a 3-dimensional connected trans-Sasakian manifold is locally ϕ-symmetric if and only if the scalar curvature is constant provided α and β are constants. Hence from Theorem 3 we obtain the following:

Corollary 5.3. *If a locally ϕ-symmetric 3-dimensional connected trans-Sasakian manifold admits gradient Ricci soliton, then the manifold is either β-Kenmotsu manifold or an Einstein manifold provided α, β =constant.*

Acknowledgement

The authors are thankful to the referee for his valuable suggestions towards the improvement of this paper.

References

[1] A. Derdzinski, Compact Ricci solitons, preprint.
[2] A. Gray, L.M. Hervella, The sixteen classes of almost Hermitian manifolds and their linear invariants, Ann. Mat. Pura Appl. 123 (1980) 35–58.
[3] B. Chow, D. Knopf, The Ricci flow: An introduction, Mathematical Surveys and Monographs 110, American Math. Soc., 2004.
[4] D. E. Blair, Contact manifolds in Riemannian geometry, Lecture Notes in Mathematics, Vol. 509, Springer-Verlag, Berlin-New York, 1976.
[5] D. E. Blair, Riemannian geometry of contact and symplectic manifolds, Progress in Mathematics, Vol. 203, Birkhauser Boston, Inc., Boston, 2002.
[6] D. E. Blair, J. A. Oubina, Conformal and related changes of metric on the product of two almost contact metric manifolds, Publications Matematiques 34 (1990) 199–207.
[7] D. E. Blair, The theory of quasi-Sasakian structures, J. Differ. Geometry 1 (1967) 331–345.
[8] D. Janssens, L. Vanhecke, Almost contact structures and curvature tensors, Kodai Math. J. 4 (1981) 1–27.
[9] D. Chinea, C. Gonzales, A classification of almost contact metric manifolds, Ann. Mat. Pura Appl. 156 (1990) 15–30.
[10] D. Friedan, Non linear models in 2+ϵ-dimensions, Ann. Phys. 163 (1985) 318–419.
[11] F. Cantrijnt, M. De Leon, E. A. Lacomha, Gradient vector fields in cosymplectic manifolds, J. Phys. A 25 (1992) 175–188, .elv. Phys. Acta. 69 (1996) 344–347.
[12] H. Geiges, A brief history of contact geometry and topology, Expo. Math. 19 (2001) 25–53.
[13] J. C. Marrero, The local structure of trans-Sasakian manifolds, Ann. Mat. Pura Appl. 162 (1992) 77–86.
[14] J. C. Marrero, D. Chinea, On trans-Sasakian manifolds, (Spanish), in "Proceedings of the XIVth Spanish-Portuguese Conference on Mathematics, Vol. I-III, (Spanish), Puerto de la Cruz, 1989", Univ. La Laguna, La Laguna, 655–659, 1990.
[15] J. A. Oubina, New classes of almost contact metric structures, Publ. Mat. Debrecen 32 (1985) 187–193.
[16] R. Sharma, Certain results on K-contact and (k,μ)-contact manifolds, Journal of Geometry 89 (2008) 138–147.
[17] R. S. Hamilton, The Ricci flow on surfaces, Mathematics and general relativity (Santa Cruz, CA, 1986), 237–262, Contemp. Math. 71, American Math. Soc., 1988.
[18] S. Maclane, Geometrical mechanics II, Lecture notes, University of Chicago, 1968.
[19] T. Chave, G. Valent, Quasi-Einstein metrics and their renoirmalizability properties, Helv. Phys. Acta. 69 (1996) 344–347.
[20] T. Chave, G. Valent, On a class of compact and non-compact quasi-Einstein metrics and their renoirmalizability properties, Nuclear Phys. B 478 (1996), 758–778.
[21] T. Ivey, Ricci solitons on compact 3-manifolds, Differential Geo. Appl. 3 (1993) 301–307.
[22] U. C. De, A. Sarkar, On three-dimensional Trans-Sasakian Manifolds, Extracta Math. 23 (2008) 265–277.
[23] U.C. De, M. M. Tripathi, Ricci Tensor in 3-dimensional Trans-Sasakian Manifolds, Kyungpook Math. J. 43 (2003) 247–255.
[24] V. E. Nazaikinskii, V. E. Shatalov, B. Y. Sternin, Contact geometry and linear differential equations, Walter de Gruyter, Berlin, 1992.
[25] V. I. Arnold, Contact geometry: the geometrical method of Gibb's thermodynamics, Proceedings of the Gibb's Symp., Yale University (May 15-17, 1989), American Math. Soc., 163-179, 1990.