Filomat 26:2 (2012), 389–395
DOI 10.2298/FIL1202389R

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Structure of shock wave for a viscous combustion model

Abdolrahman Razani[a]

[a]*Department of Mathematics, Faculty of Science, Imam Khomeini International University, Postal code: 34149-16818, Qazvin, Iran.*
School of Mathematics, Institute for Research in Fundamental Sciences, P. O. Box 19395-5746, Tehran, Iran.

Abstract. In this article, the existence of weak and strong detonation waves for a viscous combustion model is proved. In order to prove the existence of this kind of waves, it is necessary to consider a 3-dimensional system of ordinary differential equations and prove the existence of heteroclinic orbits of it. In order to prove this, topological arguments are mainly tools.

1. Introduction

Detonation waves were observed experimentally more than 100 years ago. Chapman and Jouguet (CJ) were the first to present a theory describing detonation (supersonic combustion wave), propagating at a unique velocity. The CJ theory (Fickett and Davis, 1979) treats the detonation wave as a discontinuity with infinite reaction rate. The conservation equations for mass, momentum and energy give a unique solution for the detonation velocity (CJ-velocity) and the state of combustion products immediately behind the detonation wave. During World War II, Zeldovich, von Neumann and Döring improved the CJ-model by taking the reaction rate into account (ZND) [18]. The ZND-model describes the detonation wave as a shock wave, immediately followed by a reaction zone (i.e. flame). The thickness of this zone is given by the reaction rate. The ZND-theory gives the same detonation velocities and pressures as the CJ-theory, the only difference between the two models is the thickness of the wave. Thus CJ-theory is replaced by ZND-theory.

Detonation is a process of supersonic combustion in which a shock wave is propagated forward due to energy release in a reaction zone. In this process, the shock compresses the material thus increasing the temperature to the point of ignition. The ignited material burns behind the shock and releases energy that supports the shock propagation. Detonation generates high pressures, so it is usually much more destructive than deflagrations. Typical velocities of detonation are of the order of 1,850 m/s for fuel/air mixtures and 3,000 m/s for fuel/oxygen mixtures. These velocities may be higher where hydrogen is the fuel. In fact, detonation is a rapid and violent processes of combustion generating a strong shock wave which is sustained by chemical reactions.

Detonations can be produced by explosives, reactive gaseous mixtures, certain dusts and aerosols. They are hard to control and are used primarily for demolition and in warfare. A great deal of research is conducted on achieving or preventing detonation in various materials to improve the performance of

2010 *Mathematics Subject Classification*. Primary 37C29; Secondary 34B10, 35K50, 54H20, 80A25
Keywords. Heteroclinic orbit, Majda's model, combustion, shock waves, travelling waves
Received: 18 December 2010; Accepted: 1 March 2011
Communicated by Jelena Manojlović
The author would like to thank the School of Mathematics, Institute for Research in Fundamental Sciences, Teheran, Iran, for supporting this research (Grant No. 84340112)
Email address: razani@ikiu.ac.ir (Abdolrahman Razani)

explosives and engines. An experimental form of jet propulsion, the pulse detonation engine, uses a series of well-timed detonations to generate thrust.

Hydrogen, acetylene, ethylene, ethylene oxide2, ethane and propane are some of the widely known materials that can detonate when mixed with air. Experimental work on confined detonations in air has been done on hydrogen, acetylene, ethylene and propane.

2. Majda's model

From mathematical point of view, the conservation equations for a one step irreversible chemical reaction Reactant→Product are a system of differential equations. The equilibrium points of this system can characterize two states. One state is related to the unburnt gas and the other one is related to burnt gas. Thus the existence of heteroclinic orbits of $\dot{X} = f(X)$, means a connection between these two states. One can study the behavior of the variables on this orbit. In other words, the existence of heteroclinic orbits for a system of differential equations can be investigated.
In order to give more details, consider a model which is proposed by Majda [9] and then is extended by Larrouturou [4], *i.e.*:

$$\begin{cases} (U + q_0 Z)_t + (f(U))_x = \beta U_{xx}, \\ Z_t = -\kappa \Phi(U) Z + D Z_{xx}, \end{cases} \tag{1}$$

From now on, we assume $U = T$ (the temperature). Also we encounter the well known cold boundary difficulty, that is, the unburned state is not a stationary point of (1) since the "reaction rate function" $\Phi(T) \neq 0$, for $T > 0$. The cold boundary difficulty lies in the fact that the governing equations modeling a steady planar premixed flame propagating in an infinite tube (that is, the simplest problem of flame propagation theory) admits no solution, whereas such solutions are expected to exit on an experimental basis: steady planar premixed flames are actually observed (although not in infinite tubes). The origin of the difficulty is the following: when modeled using the (wide accepted) Arrhenius law, the chemical reaction rate does not vanish in the fresh mixture. Therefore, the temperature of the fresh gases keeps increasing because of the small but nonzero reaction rate, and no steady state exists. This explains why the cold boundary difficulty has been "solved" by modifying the expression of the reaction term, for instance using an ignition temperature assumption. Beretycki et al [1] solved this problem mathematically. They show that the unmodified model (with the actual Arrhenius term) leads to a well-posed initial value problem, and that the unique time-dependent solution of the Arrhenius model remains close to a steady planar flame during a long time, before it diverges from the steady flame for even larger values of the time t. The long time limit here corresponds to the time interval which increase with the increase of the activation energy in the chemical reaction. Therefore the solution of the cold boundary difficulty can be based on activation energy asymptotic (see [17]).

In our analysis, we use the common mathematical idealization of an ignition temperature, according to which Φ is modified such that (see [11])

$$\Phi(T) = \begin{cases} 0 & \text{for } T < T_i, \\ \Phi_1(T) & \text{for } T \geq T_i, \end{cases} \tag{2}$$

where $\Phi_1(T)$ is a smooth positive function and T_i is the "ignition temperature" of the reaction. A typical example for $\Phi_1(T)$ is the Arrhenius law, *i.e.* $\Phi_1(T) = T^\gamma e^{-\frac{A}{T}}$ for some positive constants γ and A. Note that $\Phi(T)$ is discontinuous at the point T_i. A careful discussion of this assumption and its consequences for detonation and deflagration wave (with one-step chemistry) can be found in [11] and [3]. Finally, $f(T)$ is a convex strongly nonlinear function satisfying (see [9])

$$\frac{\partial f}{\partial T} = a(T) > 0, \ \frac{\partial^2 f}{\partial T^2} > \delta > 0,$$
$$\lim_{T \to +\infty} f(T) = +\infty,$$

and for example one can choose $f(T) = \frac{1}{2}aT^2$ ($a > 0$) (see [15] page 1100).

System (1), with $D = 0$, was proposed by Majda [9] as a model for dynamic combustion, i.e. the interaction between chemical reactions and compressible fluid dynamics. Larrouturou [4] considered existence, continuous dependence on data, and asymptotic properties of solutions of an extension of Majda's model for weak and strong detonation waves. This extension allows for mass transfer by diffusion. Li [5] studied (1) when $D = \beta = 0$. He established global existence of the solution to the problem and studied the asymptotic behavior of the solution. Liu and Ying [6] studied strong detonation waves for (1) and proved these waves are nonlinearly stable by using energy method for the fluid variable and a pointwise estimate for the reactant. Roquejoffre and Vila [14] proved uniform stability results for strong ZND detonation waves, in the context of singular perturbation theory, for Majda model with small viscosity. Also Liu and Yu [7] considered (1) when $D = 0$ and $\beta = 1$, and proved that the weak detonation waves of the model are nonlinear stable. Szepessy [16] studied the nonlinear stability of travelling weak detonation waves of (1) when $f(T) = \frac{1}{2}T^2$. Billingham and Mercer [2] investigated (1) when $f(T) = \frac{1}{2}\frac{hs}{V}(T - T_a)^2$ and $\phi(T) = e^{-\frac{E}{RT}}$. They used the method of matched asymptotic expansions to obtain asymptotic approximations for the permanent form travelling wave solutions and their results were confirmed numerically. Also, Razani [10, 11] considered (1) with $D = 0$ and proved the existence of weak, strong and CJ detonation waves and proved the existence of CJ detonation wave for (1) when $D \neq 0$ in [12]. In addition, he proved the existence of premixed laminar flames in [13]. Finally, Lyng and Zumbrun [8] developed a stability index for weak and strong detonation waves (for (1) when $D = 0$) yielding useful necessary conditions for stability.

In the next section, the main theorem is given. By this theorem, the existence of weak and strong detonation waves is proved.

3. Main results

In this section, the existence of weak and strong detonation waves is proved. In order to prove this, let $\xi = x - st$. Then (1) reduces to the following system of equations:

$$\begin{cases} -s(T + q_0 Z)_\xi + f(T)_\xi = \beta T_{\xi\xi}, \\ -sZ_\xi = -\kappa\Phi(T)Z + DZ_{\xi\xi}. \end{cases} \tag{3}$$

The first equation of (3) can be integrated once to give $\beta T_\xi = f(T) - s(T + q_0 Z) + C$, where C is the constant of integration. Let $W = sZ + DZ_\xi$ be an auxiliary variable. Using these relations, (3) becomes:

$$\begin{cases} \beta T_\xi &= f(T) - s(T + q_0 Z) + C := g_1(T, Z, W), \\ DZ_\xi &= W - sZ := g_2(T, Z, W), \\ W_\xi &= \kappa\Phi(T)Z := g_3(T, Z, W). \end{cases} \tag{4}$$

From mathematical point of view, the existence of weak (or strong) detonation wave corresponds to the existence of some complete orbits of (4) which are running from the rest point V_{00} (or V_{01}) to V_{m0} for some $0 < m \leq 1$. Such an orbit is called a travelling wave solution of (1). In order to do this, for fixed positive viscosity parameters β, D and κ, we are looking for some orbits of (4) which are defined for all $\xi \in \mathbb{R}$ and connect two different rest points of this system. Therefore in the first step we must determine the rest points of (4). Similar to [12], one can prove these rest points are:

$$\begin{cases} V_{00} = & (T_{00}, 0, 0), \\ V_{01} = & (T_{01}, 0, 0), \\ V_{m0} = & (T_m, m, sm), 0 < m \leq 1, T_m \leq T_i, \end{cases} \tag{5}$$

where $T_i < T_{0j}$, $j = 0, 1$.

In the present work it is assumed that the rest points V_{00} and V_{01} exist and are distinct. See [12] when they coincide with each other. The proof of the next theorem is simple and it is omitted.

Theorem 3.1. *If the rest points V_{00} or V_{01} exist, then the rest point V_{m0} exists for some $0 < m \leq 1$.*

Definition 3.2. A combustion shock wave between V_{00} and V_{m0} is called *a weak detonation wave*, and a combustion shock wave between V_{01} and V_{m0} is called *a strong detonation wave*.

Now, we define $b = \frac{D}{s}\kappa \sup_{g_1(V)\leq 0} \Phi(T)$ and $\Omega = \{V \in \mathbb{R}^3 :\ g_1(V) < 0,\ 0 < Z < 1,\ T < T_{01},\ 0 < W - sZ < b\}$. Note that the rest points V_{00} and V_{01}, are located on $\partial\Omega$. Moreover, similar to Lemma 4.3 in [10], it can be observed that the unstable manifolds at V_{00} and V_{01} intersect Ω on a curve and on a two dimensional manifold, respectively. Now, consider the following system of ordinary differential equations:

$$\begin{cases} \beta\dot{T} = & f(T) - s(T + q_0 Z) + C = g_1(V), \\ D\dot{Z} = & W - sZ = g_2(V), \\ \dot{W} = & \kappa\Phi_1(T)Z := g_4(V), \end{cases} \tag{6}$$

where $\Phi_1(T)$ is defined by (2) and $V = (T, Z, W)^T$.

Lemma 3.3. *Let Ω be defined as above. There is a unique orbit of (6) which lies in Ω, its α-limit set is V_{00}, and this orbit intersects the set $\Delta = \{V \in \overline{\Omega} : g_1(V) < 0,\ g_2(V) > 0,\ T < T_{01}$ and $Z = 1\}$. Besides, there are infinitely many orbits of (6) which lie in D, and their α-limit sets are V_{01}. Each of these orbits intersects the above set Δ. Along all of these orbits $T(\xi)$ is decreasing, but $Z(\xi)$ and $W(\xi)$ are increasing.*

Proof. The proof is given in six steps (the sketch of the proof is given, see [13] for more details).

Step 1. System (6) is gradient like with respect to $h(V) = Z$ in Ω, and is locally Lipschitz in a neighborhood of $\overline{\Omega}$.

Step 2. Note that Ω is homeomorphic to the parabola $\{X \in \mathbb{R}^3 : x_1^2 + x_2^2 < x_3,\ 0 < x_3 < 1\}$ and so $\{V \in \overline{\Omega} : h(V) = c\}$ corresponds to the set $\{X \in \mathbb{R}^3 : x_1^2 + x_2^2 < x_3,\ x_3 = c\}$ for $c \in [0, 1]$ under this homeomorphism.

Step 3. V_{00} and V_{01} are the only rest points of (6) which lies in $\{V \in \overline{\Omega} : h(V) = 0\}$.

Step 4. Let $E = \{V \in \partial\Omega : h(V) < 1\}$. For $p \in \overline{E} - \{V_{0j}$, for $j = 0, 1\}$, $p.\xi \in \Omega$ for small positive ξ, and $\notin \partial\Omega$ for small $|\xi| \neq 0$.

Step 5. Let $F = \partial\Omega - \overline{E} = \{V \in \partial\Omega : Z = 1\}$. For $p \in F$ and small $\xi > 0$, we have $p.\xi \notin H$.

Step 6. By steps 1-5, there is a point $p \in F$ such that $p.\xi \in \Omega$ for small $\xi < 0$ and $\lim_{\xi\to-\infty} p.\xi = V_{0j}$, for $j = 0, 1$.

Proof of Step 6: Suppose such a point p does not exist. Then for each $y \in F$ there is a $\xi < 0$ such that $y.\xi \notin \overline{\Omega}$. By step 4, there is $\xi(y) < 0$ such that $y.(\xi(y), 0) \subset \Omega$, $y.\xi(y) \in \overline{E}$ and $y.(\xi(y) - \varepsilon_1) \notin \overline{\Omega}$ for some $\varepsilon_1 > 0$. Now, define $\varphi : \overline{F} \to \overline{E}$, by $\varphi(y) = y.\xi(y)$. It follows from the continuity of $y.\xi$ with respect to the initial condition y that $\xi(y)$ and $\phi(y)$ are continuous. Next we show that $\varphi : \overline{F} \to \varphi(\overline{F})$ is a homeomorphism. By definition, φ is one to one. So it suffices to show that if V is open in $\overline{F}$ then $\varphi(V)$ is open in $\varphi(\overline{F})$. If V is open in $\overline{F}$, then $\overline{F}\backslash V$ is compact and so $\varphi(\overline{F}\backslash V)$ is closed. Hence $\varphi(V)$ is open in $\varphi(\overline{F})$.

Now we show that $\varphi : \overline{F} \to \overline{E}$ is onto. In fact, note that $\varphi(\partial F) = \partial E$ and $\varphi(F) = \text{int}(\overline{E})$, where $\text{int}(\overline{E})$ means the interior of $\overline{E} \subset \partial\Omega$. Since F is open in $\partial\Omega$, $\varphi(F)$ is open in $\partial\Omega$ by Brouwer's Theorem on the Invariance of Domain. Therefore it is open in $\text{int}(\overline{E})$ with respect to $\partial\Omega$. On the other hand, we have $\varphi(F) = \varphi(\overline{F}) \cap \text{int}(\overline{E})$ and $\varphi(\overline{F})$ is closed in $\overline{E}$. Hence $\varphi(F)$ is closed in $\text{int}(\overline{E})$. Since $\varphi(F) \neq \emptyset$, $\varphi(F) = \text{int}(\overline{E})$. Also for $y \in \partial F$, we have $\varphi(y) = y$. It then follows that $\varphi : \overline{F} \to \overline{E}$ is onto. This means that there is a point $y_0 \in F$ with $\xi(y_0) < 0$ such that $y_0.\xi(y_0) = V_{0j}$, for $j = 0, 1$. This is impossible since V_{0j}, for $j = 0, 1$, is a rest point of (6). Therefore there is a point $p \in \Omega$ such that $p.\xi$ is defined for all $\xi \leq 0$ and lies in Ω for all $\xi < 0$. The α-limit set of $p.\xi$ must be V_{0j}, for $j = 0, 1$, as the flow is gradient-like and V_{0j}, for $j = 0, 1$, is the only rest point of (6) in $\overline{\Omega}$. This completes the proof of Step 6.

It follows from Step 6, there is a unique orbit of (6) which lies in Ω, its α-limit set is V_{00}, and this orbit intersects the set $\Delta = \{u \in \overline{\Omega} : g_1(V) < 0, g_2(V) > 0, T < T_{01}$ and $Z = 1\}$.

Besides, there are infinitely many orbits of (6) which lie in Ω, and their common α-limit set is V_{01}. Each of these orbits intersects the above set Δ. Along all of these orbits $T(\xi)$ is decreasing and $Z(\xi)$ and $W(\xi)$ are increasing. □

Let $\tilde{V}(\xi), \xi \in (-\infty, \xi_0]$ be the orbit which is given by the above discussion. Then $\tilde{V}(\xi_0) \in \{V \in \overline{\Omega} : Z = 1\}$ and $\lim\limits_{\xi\to-\infty} \tilde{V}(\xi) = V_{00}$ or V_{01} . About the orbit $\tilde{V}(\xi)$ we have the following lemma.

Lemma 3.4. *Let $\tilde{V}(\xi)$ be as above. There is $0 < \tilde{Z} \leq 1$, such that for $\kappa\beta$ small enough, the orbit $\tilde{V}(\xi)$ meets the plane $T = T_i$, at*

$$\tilde{V}(\tilde{\xi}) = (T_i, \tilde{Z}, \tilde{W})^T,$$

for some $\tilde{\xi} \in (-\infty, \xi_0)$.

Proof. Let V_{00} and V_{01} be as (5). Choose the plane $P : T - T_{01} = 0$, where T_{01} is the first component of V_{01}, and we know that V_{00} is in $\{V | T < T_{01}\}$, because $T_{00} < T_{01}$. Let $(T_i, Z_i, W_i)^T$ be the unique solution of the equation

$$g_1(V) = 0, \ g_2(V) = 0, \ T = T_i.$$

Thus we obtain $Z_i = \frac{1}{sq_0}(f(T_i) - sT_i + C)$ and $W_i = sZ_i$. Also from $T_m < T_i < T_{01}$, it follows that $0 < Z_i < 1$, $0 < W_i < s$ and $\{V \in \overline{\Omega} : \ g_1(V) = 0, Z_i < Z < 1$ and $W_i < W < s\} \subset \{V \in \overline{\Omega} : T \leq T_i\}$.
Now consider the plane $P' : T - T_i = 0$, since $V_{m0} \in \{V \in \overline{\Omega} : (T - T_i) < 0\}$, we can choose $Z_i < Z_0 < 1$ such that

$$\{V \in \overline{\Omega} : g_1(V) = 0, Z_0 < Z < 1, \ W_0 < W < s\} \subset \{V \in \overline{\Omega} : T - T_i < 0\},$$

where $W_0 = sZ_0$.
Let $\Omega_0 = \{V \in \Omega : T_i < T < T_{01}, \ Z_0 < Z < 1\} \cup \{V \in \Omega : T_i < T < T_{01}, \ W_0 < W < s\}$ and $\delta = \max\limits_{V\in\overline{\Omega}_0} g_1(V)$. Then $\delta < 0$.
Now suppose the orbit $\tilde{V}(\xi), \xi \in (-\infty, \xi_0]$, does not meet the set $\{V \in \Omega : T = T_i, 0 < Z \leq 1$ and $0 < W < s\}$. Let $\xi_1 < \xi_0$ be the solution of the equation $\tilde{W}(\xi) = W_0$, where $\tilde{W}(\xi)$ is the third component of $\tilde{V}(\xi)$. Since $\frac{d}{d\xi}(T) = \frac{1}{\beta} g_1(V) < 0$, T is decreasing along the orbit $\tilde{V}(\xi)$, it follows that $\tilde{V}(\xi)$ remains in Ω_0 for $\xi_1 < \xi < \xi_0$. Now along the orbit $\tilde{V}(\xi)$ in Ω_0 we must have:

$$-\frac{dT}{dW} = \frac{1}{\frac{dW}{d\xi}}\left(-\frac{dT}{d\xi}\right) = \frac{1}{\kappa Z \Phi_1(T)}\left(\frac{-1}{\beta} g_1(V)\right) \geq \frac{\sigma(-\delta)}{\kappa\beta} > 0,$$

where $\frac{1}{\sigma} = \max\limits_{V\in\overline{\Omega}} \Phi_1(T)Z$. Let $T_0 = T(\xi_0)$, then $T_i < T_0$, if $\tilde{V}(\xi)$ does not meet T_i. Therefore

$$\begin{aligned} T_{00} - T_i \geqslant T_{00} - T_0 &= -\int_{-\infty}^{\xi_0} \frac{1}{\beta} g_1(V) d\xi = \int_{-\infty}^{\xi_0} \frac{1}{\beta}(-g_1(V)) d\xi \geqslant \int_{\xi_1}^{\xi_0} \frac{1}{\beta}(-g_1(V)) d\xi \\ &= \int_{W_0}^{W(\xi_0)} \frac{1}{\kappa Z \Phi_1(T)}\left(\frac{-1}{\beta} g_1(V)\right) dW \geqslant \frac{\sigma(-\delta)}{\kappa\beta}(W(\xi_0) - W(\xi_1)), \end{aligned}$$

which is impossible for $\kappa\beta$ small enough. Thus there is a $\tilde{\xi} \in (-\infty, \xi_0)$ such that the orbit $\tilde{V}(\xi)$ meets the plane $T = T_i$ at the point $\tilde{V}_i = (\tilde{T}_i, \tilde{Z}_i, \tilde{W}_i)^T$, where $\tilde{T}_i = T_i$, $\tilde{Z}_i = \tilde{Z}(\tilde{\xi})$ and $\tilde{W}_i = \tilde{W}(\tilde{\xi})$. □

From now on, we assume that $\kappa\beta$ is small enough, or the orbit $\tilde{V}(\xi)$ meets the line $T = T_i$ at the point $\tilde{V}_i = (\tilde{T}_i, \tilde{Z}_i, \tilde{W}_i)^T$. We call the point $\tilde{V}_i$ the ignition point. Note that, this point for weak detonation is unique, but for strong detonation there is a curve of ignition points.

Theorem 3.5. *Suppose (4) admits the rest points V_{00}, V_{01} and V_{m0}, for some $0 < m \leq 1$. If $\kappa\beta$ is small enough, then there is a unique orbit of (4) which is running from V_{00} to V_{m0}, for some $0 < m \leq 1$. Similarly, there are infinitely many orbits of this system which are running from V_{01} to V_{m0}, for some $0 < m \leq 1$.*

Proof. In the region $T < T_i$, the last equation of (4) becomes $\dot{W} = 0$. Thus, in this region, along the orbits of this system $W(\xi)$ is constant. Here, we let $W(\xi) = \tilde{W}_i$, where $\tilde{W}_i$ is the third component of $\tilde{V}_i$ (the ignition point). On the surface $W = \tilde{W}_i$, (4) reduces to the following two dimensional system of ordinary differential equations:

$$\begin{cases} \beta\dot{T} &= f(T) - s(T + q_0 Z) + C := F_1(T, Z), \\ D\dot{Z} &= \tilde{W}_i - sZ := F_2(T, Z). \end{cases} \tag{7}$$

By solving the last equation of (7), we obtain:

$$Z(\xi) = \frac{1}{s}[\tilde{W}_i - (\tilde{W}_i - s\tilde{Z}_i)e^{-\frac{s}{D}(\xi-\xi_0)}].$$

Note that

$$Z(\xi_0) = \tilde{Z}_i \text{ and } \lim_{\xi\to+\infty} Z(\xi) = \frac{\tilde{W}_i}{s}.$$

Thus (7) is reduced to:

$$\beta\dot{T} = f(T) - s(T + \frac{q_0}{s}[\tilde{W}_i - (\tilde{W}_i - s\tilde{Z}_i)e^{-\frac{s}{D}(\xi-\xi_0)}]) + C,$$

or

$$\beta\dot{T} = f(T) - sT - q_0[\tilde{W}_i - (\tilde{W}_i - s\tilde{Z}_i)e^{-\frac{s}{D}(\xi-\xi_0)}] + C := G(T). \tag{8}$$

Now, we show that any trajectory starting within the plane $T = T_i$ with $Z > \tilde{Z}_i$ converges to V_{m0} for some $0 < m \leq 1$. In order to do this, note that W remains constant along trajectories and $W - sZ$ decays exponentially with rate $\frac{1}{D}$. In addition, consider the region

$$H' = \{T \in \mathbb{R} : G(T) < 0, T < T_i\}.$$

Note that $T - T_i \in \partial H'$. Also it is trivial to see that any orbit of (8) initiating at a point on $\partial H' \cap \{T : T = T_i\}$ approaches the unique rest point of (8) which is located in the region $T < T_i$, as ξ tends to $+\infty$. We denote this rest point by $\overline{T}_i$. Now consider again the ignition point $\tilde{V}_i = (\tilde{T}_i, \tilde{Z}_i, \tilde{W}_i)$. By the above argument, there is a unique orbit of (4), say

$$\tilde{\tilde{V}}(\xi) = (\tilde{\tilde{T}}(\xi), \tilde{\tilde{Z}}(\xi), \tilde{\tilde{W}}(\xi)), \quad \tilde{\xi} < \xi < +\infty,$$

with

$$\tilde{\tilde{V}}(\tilde{\xi}) = (T_i, \tilde{Z}_i, \tilde{W}_i),$$
$$\tilde{\tilde{W}}(\xi) = \tilde{W}_i, \quad \text{for} \quad \xi \geq \tilde{\xi},$$

and

$$\lim_{\xi\to+\infty} \tilde{\tilde{V}}(\xi) = (\overline{T_i}, \overline{Z_i}, \tilde{W}_i).$$

Along this orbit $T(\xi)$ is decreasing, $Z(\xi)$ is increasing and $W(\xi)$ is constant. This orbit lies in Ω, the domain which is given after Definition 3.2.
Now define,

$$V(\xi) = \begin{cases} \tilde{V}(\xi) & \xi < \tilde{\xi}, \\ \tilde{\tilde{V}}(\xi) & \xi \geq \tilde{\xi}. \end{cases}$$

Then $V(\xi)$ is a complete orbit of (8) lying in Ω and is running from V_{00} (or V_{01}) to V_{m0} for some $0 < m \leq 1$. This completes the proof. □

4. Concluding

In this paper, the system (1) is considered in general case and the existence of weak and strong detonation waves for (1) is studied. Results are obtained without any necessary conditions on the problem. Moreover, one can study the behavior of the variables along the orbit (as it is mentioned in the main theorem). Finally, the simplicity of this technique can be noticed.

References

[1] H. Berestycki, B. Larrouturou, J.-M. Roquejoffre, Mathematical investigation of the cold boundary difficulty in flame propagation theory, Dynamical issues in combustion theory (Minneapolis, MN, 1989), 37–61, IMA Vol. Math. Appl., 35, Springer, New York, 1991.
[2] J. Billingham, G.N. Mercer, The effect of heat loss on the propagation of strongly exothermic combustion waves, Combust. Theory Modelling 5 (2001) 319–342.
[3] A.K. Kapila, B.J. Mathowsky, A. Van Harten, An asymptotic theory of deflagration and detonations 1. The steady solutions, SIAM J. Appl. Math. 43 (1983) 491–519.
[4] B. Larrouturou, Remarks on a model for combustion waves, Nonlinear Anal. 9 (1985) 905–935.
[5] T. Li, On the initiation problem for a combustion model, J. Differential Equations 112 (1994) 351–373.
[6] T.P. Liu, L.A. Ying, Nonlinear stability of strong detonation for a viscous combustion model, SIAM J. Math. Anal. 26 (1995) 519–528
[7] T.P. Liu, S.H. Yu, Nonlinear stability of weak detonation waves for a combustion model, Comm. Math. Phys. 204 (1999) 551–586.
[8] G. Lyng, K. Zumbrun, A stability index for detonation waves in Majda's model for reacting flow, Physica D 194 (2004) 1–29.
[9] A. Majda, A qualitative model for dynamic combustion, SIAM J. Appl. Math. 41 (1981) 70–93.
[10] A. Razani, Weak and strong detonation profiles for a qualitative model, J. Math. Anal. Appl. 276 (2002) 868–881.
[11] A. Razani, Chapman-Jouguet detonation for a qualitative model, Bull. Austral. Math. Soc. 66 (2002) 393–403.
[12] A. Razani, Existence of Chapman-Jouguet detonation for a viscous combustion model, J. Math. Anal. Appl. 293 (2004) 551–563.
[13] A. Razani, On the existence of premixed laminar flames, Bull. Austral. Math. Soc. 66 (2004) 393–403.
[14] J.-M. Roquejoffre, J.-P. Vila, Stability of ZND detonation waves in the Majda combustion model, Asymptot. Anal. 18 (1998) 329–348.
[15] R. Rosales, A. Majda, Weakly nonlinear detonation waves, SIAM J. Appl. Math. 43 (1983) 1086–1118.
[16] A. Szepessy, Dynamics and stability of a weak detonation wave, Comm. Math. Phys. 202 (1999) 547–569.
[17] F.A. Williams, Combustion Theory, Benjamin/Cummings, Melon Park, CA, 1985.
[18] K. van Wingerden, D. Bjerketvedt, J.R. Bakke, Detonations in pipes and in the open, Proceedings of the Petro-Chemical Congress, June 2324, (1999) 15 pages.