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On a semi-symmetric non-metric connection

S. K. Chaubey[a], R. H. Ojha[b]

[a]*Department of Mathematics, Dr. Virendra swarup group of institutions, Ragendra Swarup Knowledge city, Kanpur-Lucknow highway, Post Box No. 13, Unnao, Uttar Pradesh, India*
[b]*Department of Mathematics, Faculty of Science, Banaras Hindu University, Varanasi-221005, Uttar Pradesh, India*

Abstract. Yano [10] defined and studied semi-symmetric metric connection in a Riemannian manifold and this was extended by De and Senguta [4] and many other geometers. Recently, the present authors [2], [3] defined semi-symmetric non-metric connections in an almost contact metric manifold. In this paper, we studied some properties of a semi-symmetric non-metric connection in a Kenmotsu manifold.

1. Introduction

An n-dimensional Riemannian manifold (M_n, g) of class C^∞ with a 1-form η, the associated vector field ξ and a $(1,1)$ tensor field ϕ satisfying

$$\phi^2 X + X = \eta(X)\xi, \tag{1}$$

$$\phi\xi = 0, \qquad \eta(\phi X) = 0, \qquad \eta(\xi) = 1, \tag{2}$$

is called an almost contact manifold and the system (ϕ, ξ, η) is called an almost contact stucture to M_n [9]. If the associated Riemannian metric g in M_n satisfy

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \tag{3}$$

for arbitrary vector fields X, Y in M_n, then (M_n, g) is said to be an almost contact metric manifold. Putting ξ for X in (3) and using (2), we obtain

$$g(\xi, Y) = \eta(Y). \tag{4}$$

Also,

$$'F(X, Y) \stackrel{\text{def}}{=} g(\phi X, Y) \tag{5}$$

gives

$$'F(X, Y) +' F(Y, X) = 0. \tag{6}$$

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Email addresses: sk22_math@yahoo.co.in (S. K. Chaubey), rh_ojha@rediffmail.com (R. H. Ojha)

The Nijenhuis tensor $N(X,Y)$ of ϕ in an (M_n, g) is a vector valued bilinear function such that

$$\begin{aligned} N(X,Y) &= (D_{\phi X}\phi)(Y) - (D_{\phi Y}\phi)(X) \\ &- \phi((D_X\phi)(Y)) + \phi((D_Y\phi)(X)). \end{aligned} \tag{7}$$

If we define

$$'N(X,Y,Z) \stackrel{\text{def}}{=} g(N(X,Y),Z), \tag{8}$$

then in consequence of

$$g((D_X\phi)(Y),Z) = (D_X{}'F)(Y,Z) \tag{9}$$

and (8), (7) becomes

$$\begin{aligned} 'N(X,Y,Z) &= (D_{\phi X}{}'F)(Y,Z) - (D_{\phi Y}{}'F)(X,Z) \\ &+ (D_X{}'F)(Y,\phi Z) - (D_Y{}'F)(X,\phi Z). \end{aligned} \tag{10}$$

If moreover

$$(D_X\phi)(Y) = g(\phi X,Y)\xi - \eta(Y)\phi X, \tag{11}$$

$$D_X\xi = X - \eta(X)\xi, \tag{12}$$

hold in (M_n, g), where D being Riemannian connection of g, then (M_n, g) is called a Kenmotsu manifold [8]. Also the following relations hold in a Kenmotsu manifold

$$K(X,Y,\xi) = \eta(X)Y - \eta(Y)X, \tag{13}$$

$$K(\xi,X,Y) = \eta(Y)X - g(X,Y)\xi, \tag{14}$$

$$Ric(X,\xi) = -(n-1)\eta(X), \tag{15}$$

$$(D_X\eta)(Y) = g(X,Y) - \eta(X)\eta(Y) \tag{16}$$

for arbitrary vector fields X, Y, Z, where K be the Riemannian curvature tensor and Ric is the Ricci-tensor of the connection D.

2. Semi-symmetric non-metric connection

In 1992, Agashe and Chafle [1] defined and studied a semi-symmetric non-metric connection on Riemannian manifold.

A linear connection $\tilde{B}$ defined as

$$\tilde{B}_X Y = D_X Y +' F(X,Y)\xi \tag{17}$$

is said to be a semi-symmetric non-metric connection [2], if the torsion tensor $\tilde{S}$ of $\tilde{B}$ defined

$$\tilde{S}(X,Y) = \tilde{B}_X Y - \tilde{B}_Y X - [X,Y]$$

for arbitrary vector fields X, Y which satisfies

$$\tilde{S}(X,Y) = 2'F(X,Y)\xi \tag{18}$$

and

$$(\tilde{B}_X g)(Y,Z) = -\eta(Y)'F(X,Z) - \eta(Z)'F(X,Y). \tag{19}$$

The properties of such connection have been studied by Jaiswal and Ojha [7] and many others.
It is known [2],

$$(\tilde{B}_X\phi)(Y) = (D_X\phi)(Y) + g(\phi X, \phi Y)\xi, \tag{20}$$

$$(\tilde{B}_X\eta)(Y) = (D_X\eta)(Y) - g(\phi X, Y), \tag{21}$$

$$(\tilde{B}_X{}'F)(Y, Z) = (D_X{}'F)(Y, Z). \tag{22}$$

The curvature tensor R of $\tilde{B}$ defined as

$$R(X, Y, Z) = \tilde{B}_X\tilde{B}_Y Z - \tilde{B}_Y\tilde{B}_X Z - \tilde{B}_{[X,Y]}Z$$

which satisfies

$$\begin{aligned} R(X, Y, Z) &= K(X, Y, Z) +' F(Y, Z)D_X\xi -' F(X, Z)D_Y\xi \\ &+ g((D_X\phi)(Y) - (D_Y\phi)(X), Z)\xi, \end{aligned} \tag{23}$$

where

$$K(X, Y, Z) = D_X D_Y Z - D_Y D_X Z - D_{[X,Y]}Z \tag{24}$$

is the Riemannian curvature tensor of the Riemannian connection D [2].

Theorem 1. *If a Kenmotsu manifold admits a semi-symmetric non-metric connection $\tilde{B}$, then we have*

$$\begin{aligned} &(i) && (\tilde{B}_X{}'F)(Y, Z) + (\tilde{B}_X{}'F)(Z, Y) = 0, \\ &(ii) && (\tilde{B}_\xi{}'F)(Y, Z) = 0, \\ &(iii) && (\tilde{d}\eta)(X, Y) = 2'F(X, Y), \\ &(iv) && 'N(X, Y, Z) = 4\eta(Z)g(X, \phi Y), \end{aligned} \tag{25}$$

where

$$(\tilde{d}\eta)(X, Y) = (\tilde{B}_X\eta)(Y) - (\tilde{B}_Y\eta)(X). \tag{26}$$

Proof. Covariant differentiation of (6) gives (25) (i). Again, in view of (11), (20) becomes

$$(\tilde{B}_X\phi)(Y) = g(\phi X, Y)\xi - \eta(Y)\phi X + g(\phi X, \phi Y)\xi. \tag{27}$$

By the virtue of (2), (9), (11) and (22), we obtain

$$(\tilde{B}_X{}'F)(Y, Z) = \eta(Z)g(\phi X, Y) - \eta(Y)g(\phi X, Z). \tag{28}$$

Putting $X = \xi$ in (28) and using (2), we obtain (25) (ii). Next, (25) (iii) follows in consequence of (21), (16) and (26). Again, using (20) in (7) and then by virtue of (1), (2) and (4), we obtain

$$\begin{aligned} N(X, Y) &= (\tilde{B}_{\phi X}\phi)(Y) - (\tilde{B}_{\phi Y}\phi)(X) \\ &- \phi((\tilde{B}_X\phi)(Y)) + \phi((\tilde{B}_Y\phi)(X)) + 2g(X, \phi Y)\xi. \end{aligned} \tag{29}$$

In view of (8), (20) and (22), (29) becomes

$$\begin{aligned} 'N(X, Y, Z) &= (\tilde{B}_{\phi X}{}'F)(Y, Z) - (\tilde{B}_{\phi Y}{}'F)(X, Z) \\ &+ (\tilde{B}_X{}'F)(Y, \phi Z) - (\tilde{B}_Y{}'F)(X, \phi Z) + 4\eta(Z)g(X, \phi Y). \end{aligned} \tag{30}$$

In consequence of (1) and (28), (30) gives (25)(iv). □

Lemma 1. *If a Kenmotsu manifold admits a semi-symmetric non-metric connection $\tilde{B}$, then the following relations hold*

$$\begin{aligned} &(i) \quad 'N(\xi, Y, Z) =' N(X, \xi, Z) = 0, \\ &(ii) \quad 'N(X, Y, \phi Z) = 0. \end{aligned} \tag{31}$$

Proof. Relation (31) (i) follows by putting $X = \xi$, $Y = \xi$ in (25) (iv) respectively and by the use of (2) and (4). Again replacing Z by ϕZ in (25) (iv) and then using (2), we obtain (31) (ii). □

Theorem 2. *If a Kenmotsu manifold admits a semi-symmetric non-metric connection $\tilde{B}$, then the scalar curvature of $\tilde{B}$ coincide with that of D.*

Proof. By the virtue of (4), (5), (11) and (12), (23) becomes

$$\begin{aligned} R(X, Y, Z) &= K(X, Y, Z) +' F(Y, Z)X - 2\eta(X)'F(Y, Z)\xi \\ &- 'F(X, Z)Y + 2\eta(Y)'F(X, Z)\xi - 2\eta(Z)'F(X, Y)\xi. \end{aligned} \tag{32}$$

Contracting (32) with respect to X, we have

$$\tilde{Ric}(Y, Z) = Ric(Y, Z) + (n - 3)'F(Y, Z). \tag{33}$$

Again

$$\tilde{R}Y = RY + (n - 3)\phi Y \tag{34}$$

and

$$\tilde{r} = r, \tag{35}$$

where $\tilde{Ric}(Y, Z) \overset{\text{def}}{=} g(\tilde{R}Y, Z)$; $Ric(Y, Z) \overset{\text{def}}{=} g(RY, Z)$ and $\tilde{r}$; r are the Ricci tensors and scalar curvatures of the connections $\tilde{B}$ and D respectively. □

Theorem 3. *If a Kenmotsu manifold admits a semi-symmetric non-metric connection $\tilde{B}$, then the necessary and sufficient condition for the Ricci tensor of $\tilde{B}$ to be skew-symmetric is that the manifold is Ricci flat.*

Proof. In view of (6), (33) and

$$Ric(X, Y) = Ric(Y, X), \tag{36}$$

we obtain

$$\tilde{Ric}(Y, Z) + \tilde{Ric}(Z, Y) = 2Ric(Y, Z). \tag{37}$$

If Ricci tensor of $\tilde{B}$ is skew-symmetric, then we have from (37)

$$Ric(Y, Z) = 0.$$

Converse is obvious. □

Theorem 4. *If a Kenmotsu manifold admits a semi-symmetric non-metric connection $\tilde{B}$, then a necessary and sufficient condition for the Ricci-tensor of $\tilde{B}$ to be symmetric is that the dimension of the manifold is 3.*

Proof. In consequence of (6), (33) and (36), we have

$$\tilde{Ric}(Y, Z) - \tilde{Ric}(Z, Y) = 2(n - 3)'F(Y, Z). \tag{38}$$

If the Ricci-tensor of $\tilde{B}$ is symmetric, then (38) gives

$$(n - 3)'F(Y, Z) = 0. \tag{39}$$

Since $'F(Y, Z) \neq 0$, in general, therefore $n = 3$. Converse part is obvious from (38). □

Corollary 1. *If a Kenmotsu manifold equipped with a semi-symmetric non-metrc connection $\tilde{B}$ is of dimension not equal to* 3*, then the Ricci-tensor of $\tilde{B}$ is not symmetric.*

Proof. If we take $\tilde{Ric}(Y,Z) = \tilde{Ric}(Z,Y)$ and $n \neq 3$, then (39) gives $'F(Y,Z) = 0$, which is not possible in general. □

Corollary 2. *If a Kenmotsu manifold of dimension* 3 *admits a semi-symmetric non-metric connection $\tilde{B}$, then the Ricci-tensor of $\tilde{B}$ coincide with that of the Riemannian connection D.*

If we define

$$'R(X,Y,Z,W) \overset{\text{def}}{=} g(R(X,Y,Z),W) \tag{40}$$

then by view of (32), (40) and $'K(X,Y,Z,W) +' K(Y,X,Z,W) = 0$, we obtain

$$'R(X,Y,Z,W) +' R(Y,X,Z,W) = 0,$$

where

$$'K(X,Y,Z,W) \overset{\text{def}}{=} g(K(X,Y,Z),W). \tag{41}$$

Theorem 5. *If a Kenmotsu manifold admits a semi-symmetric non-metric connection $\tilde{B}$ whose curvature tensor vanishes, then the curvature tensor of the manifold satisfies the following relations*

$$\begin{aligned}
&(i) && K(X,Y,\xi) = \tilde{S}(X,Y) = (\tilde{d}\eta)(X,Y),\\
&(ii) && K(X,Y,\xi) = 2K(\xi,X,Y),\\
&(iii) && 'K(\xi,X,Y,Z) -' K(\xi,X,Z,Y) = (\tilde{B}_X{}'F)(Y,Z),\\
&(iv) && 'K(\xi,X,\xi,Y) = 0,\\
&(v) && 'K(\phi X,Y,\xi,Z) + 2\eta(Z)(D_X\eta)(Y) = 0.
\end{aligned} \tag{42}$$

Proof. In consequence of $R(X,Y,Z) = 0$, (32) becomes

$$\begin{aligned}
K(X,Y,Z) &= g(\phi X,Z)Y + 2\eta(X)g(\phi Y,Z)\xi\\
&- g(\phi Y,Z)X - 2\eta(Y)g(\phi X,Z)\xi + 2\eta(Z)g(\phi X,Y)\xi.
\end{aligned} \tag{43}$$

Putting $Z = \xi$ in (43) and then using (2), (4), (18) and (25) (iii), we get (42) (i). Again, substituting $X = \xi$ in (43) and then use of (2) gives

$$K(\xi,Y,Z) =' F(Y,Z)\xi. \tag{44}$$

In view of (18), (42) (i) and (44), we obtain (42) (ii). Next, by virtue of (2), (41) and (44), we have

$$'K(\xi,Y,Z,W) = \eta(W)'F(Y,Z). \tag{45}$$

From (28) and (45), we get (42) (iii). Also, putting $Z = \xi$ in (45) and then using (2), (4) and (5), we at-once obtain (42) (iv). Further, in consequence of (41), (43), (1) and (2), we have

$$\begin{aligned}
'K(X,Y,Z,W) &= -g(X,Z)g(Y,W) - g(\phi Y,Z)g(\phi X,W) + \eta(X)\eta(Z)g(Y,W)\\
&+ 2\eta(Y)\eta(W)g(X,Z) - 2\eta(Z)\eta(W)g(X,Y).
\end{aligned} \tag{46}$$

Replacing X by ϕX and putting $Z = \xi$ in (46) and then using (2), (3) and (16), (42) (v) follows immediately. □

Theorem 6. *If a Kenmotsu manifold admits a semi-symmetric non-metric connection $\tilde{B}$ whose scalar curvature vanishes, then it is necessary and sufficient condition that the concircular curvature tensor and curvature tensor of the manifold coincide.*

Proof. The concircular curvature tensor [5], [9] of the connection D is

$$C(X,Y,Z) = K(X,Y,Z) - \frac{r}{n(n-1)}\left(g(Y,Z)X - g(X,Z)Y\right). \tag{47}$$

In view of (34), (47) and $\tilde{r} = 0$, we obtain the necessary part of the theorem. Sufficient part of the theorem is obvious by (47). □

Theorem 7. *If a Kenmotsu manifold admits a semi-symmetric non-metric connection $\tilde{B}$, then the necessary and sufficient condition for the conformal curvature tensor of $\tilde{B}$ coincide with that of D is that the conharmonic curvature tensor of $\tilde{B}$ is equal to that of D.*

Proof. The conformal curvature tensor of $\tilde{B}$ is defined as

$$\begin{aligned} \tilde{V}(X,Y,Z) &= R(X,Y,Z) - \frac{1}{n-2}(\tilde{Ric}(Y,Z)X \\ &- \tilde{Ric}(X,Z)Y - g(X,Z)\tilde{R}Y + g(Y,Z)\tilde{R}X) \\ &+ \frac{\tilde{r}}{(n-1)(n-2)}(g(Y,Z)X - g(X,Z)Y). \end{aligned} \tag{48}$$

If we define

$$'\tilde{V}(X,Y,Z,W) = g(\tilde{V}(X,Y,Z),W) \tag{49}$$

then by virtue of (32), (33), (34), (35) and (49), we obtain

$$\begin{aligned} '\tilde{V}(X,Y,Z,W) &= {}'V(X,Y,Z,W) - \frac{1}{(n-2)}('F(Y,Z)g(X,W) \\ &- {}'F(X,Z)g(Y,W)) + 2\eta(Y)\eta(W)'F(X,Z) \\ &- 2\eta(X)\eta(W)'F(Y,Z) - 2\eta(Z)\eta(W)'F(X,Y) \\ &+ \frac{(n-3)}{(n-2)}(g(Y,Z)g(X,W) - g(X,Z)g(Y,W)), \end{aligned} \tag{50}$$

where

$$\begin{aligned} 'V(X,Y,Z,W) &= {}'K(X,Y,Z,W) - \frac{1}{(n-2)}(Ric(Y,Z)g(X,W) \\ &- Ric(X,Z)g(Y,W) - g(X,Z)Ric(Y,W) + g(Y,Z)Ric(X,W)) \\ &+ \frac{r}{(n-1)(n-2)}(g(Y,Z)g(X,W) - g(X,Z)g(Y,W)) \end{aligned}$$

is the conformal curvature tensor of D [5]. Again, we define conharmonic curvature tensor $\tilde{L}$ of $\tilde{B}$ as

$$\begin{aligned} '\tilde{L}(X,Y,Z,W) &= {}'R(X,Y,Z,W) - \frac{1}{(n-2)}(\tilde{Ric}(Y,Z)g(X,W) \\ &- \tilde{Ric}(X,Z)g(Y,W) - g(X,Z)\tilde{Ric}(Y,W) \\ &+ g(Y,Z)\tilde{Ric}(X,W)). \end{aligned} \tag{51}$$

In view of (32), (33), (40) and (41), (51) becomes

$$\begin{aligned} '\tilde{L}(X,Y,Z,W) &= {}'L(X,Y,Z,W) - \frac{1}{(n-2)}('F(Y,Z)g(X,W) \\ &- {}'F(X,Z)g(Y,W)) + 2\eta(Y)\eta(W)'F(X,Z) \\ &- 2\eta(X)\eta(W)'F(Y,Z) - 2\eta(Z)\eta(W)'F(X,Y) \\ &+ \frac{(n-3)}{(n-2)}(g(Y,Z)g(X,W) - g(X,Z)g(Y,W)), \end{aligned} \tag{52}$$

where

$$'\tilde{L}(X,Y,Z,W) \stackrel{\text{def}}{=} g(\tilde{L}(X,Y,Z),W)$$

and $'L(X,Y,Z,W)$ is a conharmonic curvature tensor of D [6], defined as

$$\begin{aligned} 'L(X,Y,Z,W) &= {}'K(X,Y,Z,W) - \frac{1}{(n-2)}(Ric(Y,Z)g(X,W) \\ &- Ric(X,Z)g(Y,W) - g(X,Z)Ric(Y,W) + g(Y,Z)Ric(X,W)). \end{aligned}$$

From (50) and (52), we have

$$'\tilde{V}(X,Y,Z,W) -' V(X,Y,Z,W) = '\tilde{L}(X,Y,Z,W) -' L(X,Y,Z,W).$$

□

Remark 1. Conharmonic curvature tensor $'L$ of a Riemannian manifold M_n has been introduced by Y. Ishii [6], as an invariant with respect to conformal transformations which preserve in a sense real harmonic functions on M_n. Here we define the conformal and conharmonic curvature tensors $'\tilde{V}$ and $'\tilde{L}$ with respect to the semi-symmetric non-metric connection $\tilde{B}$ in the formal sense.

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