Selection properties in fuzzy metric spaces

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Abstract. We introduce and study some boundedness properties in fuzzy metric spaces. These properties are related to the classical covering properties of Menger, Hurewicz and Rothberger.

1. Introduction

In the literature there are several different definitions of fuzzy metric spaces [11], [7], [3], [4]. In particular, Kramosil and Michalek [11] introduced fuzzy metric spaces based on the notion of continuous triangular norms that were the first time applied in [16] to modify the definition of probabilistic metric spaces introduced by K. Menger [2]. By a slight modification of the Kramosil-Michalek definition, George and Veeramani [3], [4] introduced and studied fuzzy metric spaces and topological spaces induced by fuzzy metric (see also [5], [6]). The notion of intuitionistic fuzzy metric spaces was introduced by Park in [12] (see also [14] where completeness and precompactness of these spaces have been studied). Both fuzzy metric and intuitionistic fuzzy metric spaces have many applications in different areas of mathematics as well as in engineering and in many branches of the quantum particle physics. In [6] (see also [15]) it was shown that Park's definition of intuitionistic fuzzy metric spaces contains extra conditions and can be derived, in an equivalent manner, from the definition of fuzzy metric spaces.

We investigate fuzzy metric spaces in connection with several kinds of boundedness properties related to selection principles and studied already in other mathematical structures, such as uniform spaces [8] and topological groups [1].

The paper is organized in such a way that after this introduction in Section 2 we give basic definitions concerning fuzzy metric spaces. In Section 3 we define FM-, FH- and FR-boundedness properties which are the central objects of this article and give some examples. Section 4 is devoted to the main results describing the considered properties, in particular under basic operations: subspaces and products. Finally, in Section 5 we mention some possible directions of investigation related to game theory and $\mathcal{L}$-fuzzy metric spaces.

2. Preliminaries

We begin with basic definitions about fuzzy metric spaces following [3].
First of all recall that if X is a nonempty set, then a fuzzy set A in X is a function from X into $[0, 1]$.

Definition 2.1. ([16]) A binary operation $* : [0,1] \times [0,1] \rightarrow [0,1]$ is a *continuous t-norm* if the following conditions are satisfied:

(1) $*$ is commutative and associative;

(2) $*$ is continuous;

(3) $a * 1 = a$ for all $a \in [0,1]$;

(4) $a * b \leq c * d$ whenever $a \leq c$ and $b \leq d$, $(a, b, c, d \in [0,1])$.

Definition 2.2. ([3]) A 3-tuple $(X, M, *)$ is said to be a *fuzzy metric space* if X is an arbitrary nonempty set, $*$ is a continuous t-norm, and M is a fuzzy set on $X^2 \times (0, \infty)$ satisfying for all $x, y \in X$ and all $s, t > 0$ the following conditions:

(M.1) $M(x, y, t) > 0$;

(M.2) $M(x, y, t) = 1$ if and only if $x = y$;

(M.3) $M(x, y, t) = M(y, x, t)$;

(M.4) $M(x, y, t) * M(y, z, s) \leq M(x, z, t + s)$;

(M.5) $M(x, y, \cdot) : (0, \infty) \rightarrow (0, 1]$ is a continuous function.

The pair $(M, *)$ (or only M) is called a *fuzzy metric* on X.

The function $M(x, y, t)$ denotes the degree of nearness between x and y with respect to t. Note also that $M(x, y, \cdot)$ is a non-decreasing function (with respect to t) for all $x, y \in X$.

Let $(X, M, *)$ be a fuzzy metric space. Given $x \in X$, $\varepsilon \in (0, 1)$ and $t > 0$, the set

$$B(x, \varepsilon, t) := \{y \in X : M(x, y, t) > 1 - \varepsilon\}$$

is called the *open ball* with center x and radius ε with respect to t.

The collection

$$\{B(x, \varepsilon, t) : x \in X, \varepsilon \in (0, 1), t > 0\}$$

is a base of a topology on X; denote this topology by τ_M. Notice that the collection

$$\left\{ B\left(x, \frac{1}{n}, \frac{1}{n}\right) : x \in X, n \in \mathbb{N} \right\}$$

is also a base for τ_M. The topology τ_M is Hausdorff and metrizable.

3. Definitions and examples

The following definitions are motivated by definitions of the classical Menger, Rothberger and Hurewicz covering properties (for details see the survey papers [9] and [10]) and considerations in [1] and [8].

Recall that a topological space has the Menger (Rothberger) [Hurewicz] covering property if for each sequence $(\mathcal{U}_n : n \in \mathbb{N})$ of open covers of X there is a sequence $(\mathcal{V}_n : n \in \mathbb{N})$ $((U_n : n \in \mathbb{N}))$ $[(\mathcal{W}_n : n \in \mathbb{N})]$ such that for each $n \in \mathbb{N}$, $\mathcal{V}_n$ is a finite subset of $\mathcal{U}_n$ $(U_n \in \mathcal{U}_n)$ $[\mathcal{W}_n$ is finite subset of $\mathcal{U}_n]$ and $X = \bigcup_{n \in \mathbb{N}} \bigcup \mathcal{V}_n$ $(X = \bigcup_{n \in \mathbb{N}} U_n)$ [each $x \in X$ belongs to $\cup \mathcal{W}_n$ for all but finitely many n].

Definition 3.1. A fuzzy metric space $(X, M, *)$ is said to be:

FM : F-*Menger-bounded* (or FM-bounded);

FR : F-*Rothberger-bounded* (or FR-bounded);

FH : F-*Hurewicz-bounded* (or FH-bounded)

if for each sequence $(\varepsilon_n : n \in \mathbb{N})$ of elements of $(0, 1)$ and each $t > 0$ there is a sequence

FM : $(A_n : n \in \mathbb{N})$ of finite subsets of X such that $X = \bigcup_{n \in \mathbb{N}} \bigcup_{a \in A_n} B(a, \varepsilon_n, t)$;

FR : $(x_n : n \in \mathbb{N})$ of elements of X such that $X = \bigcup_{n \in \mathbb{N}} B(x_n, \varepsilon_n, t)$;

FH : $(A_n : n \in \mathbb{N})$ of finite subsets of X such that for each $x \in X$ there is $n_0 \in \mathbb{N}$ such that $x \in \bigcup_{a \in A_n} B(a, \varepsilon_n, t)$ for all $n \geq n_0$.

Recall that a fuzzy metric space is said to be *precompact* (respectively, *pre-Lindelöf*) if for every $\varepsilon \in (0, 1)$ and every $t > 0$ there is a finite (respectively, countable) set $A \subset X$ such that $X = \bigcup_{a \in A} B(a, \varepsilon, t)$.

Evidently,

$$\mathsf{F-precompact} \Rightarrow \mathsf{FH-bounded} \Rightarrow \mathsf{FM-bounded} \Rightarrow \mathsf{F-pre\text{-}Lindel\ddot{o}f}$$

and

$$\mathsf{FR-bounded} \Rightarrow \mathsf{FM-bounded}.$$

Example 3.2. Let (X, d) be a metric space with the Menger (Rothberger, Hurewicz) property. Consider the standard fuzzy metric M_d on X induced by d defined by

$$M_d(x, y, t) = \frac{t}{t + d(x, y)}, \quad (x, y \in X, t > 0)$$

and denote

$$a * b = ab, \quad (a, b \in [0, 1]).$$

Then the fuzzy metric space $(X, M_d, *)$ is FM-bounded (FR-bounded, FH-bounded).

Consider only the FM-bounded case because the other two are shown quite similarly.

Let $(\varepsilon_n : n \in \mathbb{N})$ be a sequence in $(0, 1)$ and let $t > 0$. As (X, d) has the Menger covering property, there is a sequence $(A_n : n \in \mathbb{N})$ of finite subsets of X such that

$$X = \bigcup_{n \in \mathbb{N}} \bigcup_{a \in A_n} K(a, \varepsilon_n),$$

where $K(a, \varepsilon) = \{y \in X : d(a, y) < \varepsilon\}$.

Let $x \in X$. There is $n \in \mathbb{N}$ and a point $a_n \in A_n$ satisfying $d(x, a_n) < \varepsilon_n$. Then

$$M_d(x, a_n, t) = \frac{t}{t + d(x, a_n)} > \frac{t}{t + \varepsilon_n} = 1 - \frac{\varepsilon_n}{t + \varepsilon_n} > 1 - \varepsilon_n.$$

Therefore we have $x \in B(a_n, \varepsilon_n, t)$, i.e. $X = \bigcup_{n \in \mathbb{N}} \bigcup_{a \in A_n} B(a, \varepsilon_n, t)$ which means that $(X, M_d, *)$ is FM-bounded.

Example 3.3. Let $X = \mathbb{R}$ and $d = |\cdot|$, and let $*$ be as in Example 3.2. Then the fuzzy metric space $(X, M_d, *)$ is FM-bounded by the previous example and the fact that $(\mathbb{R}, |\cdot|)$ has the Menger property [9], [10]. On the other hand, $(X, M_d, *)$ is not FR-bounded.

Indeed, if we take the sequence $(2^{-n} : n \in \mathbb{N}) \subset (0, 1)$ and $t = 2^{-1}$, then X cannot be covered by the open balls $B(x_n, 2^{-n}, 2^{-1})$ for any choice of elements $x_n, n \in \mathbb{N}$, from X. Otherwise, we would have that for every $x \in \mathbb{R}$ there is $n \in \mathbb{N}$ such that

$$\frac{2^{-1}}{2^{-1} + |x - x_n|} > 1 - 2^{-n},$$

i.e.

$$\frac{2^{-1} + |x - x_n|}{2^{-1}} < \frac{2^n}{2^n - 1}.$$

From here we obtain that for each $x \in X$ there is $n \in \mathbb{N}$ with

$$|x - x_n| < \frac{1}{2(2^n - 1)} < \frac{1}{2^n},$$

which means

$$\mathbb{R} = \bigcup_{n \in \mathbb{N}} (x_n - 2^{-n}, x_n + 2^{-n}).$$

However, it is impossible.

Example 3.4. Let $X = (0, \infty)$ with the fuzzy metric $(M, *)$ defined by

$$a * b = ab, (a, b \in [0, 1]); \quad M(x, y, t) = \begin{cases} \frac{x}{y}, & \text{if } x \leq y; \\ \frac{y}{x}, & \text{if } x \geq y, \text{ for all } x, y \in X, t > 0. \end{cases}$$

It is easy to see that for $x \in X$, $\varepsilon \in (0, 1)$, $t > 0$.

$$B(x, \varepsilon, t) = ((1 - \varepsilon)x, x/(1 - \varepsilon)) \subset X,$$

i.e. the open balls in $(X, M, *)$ are the usual open intervals in X equipped with the metric topology on $\mathbb{R}$.

Thus $(X, M, *)$ is not FR-bounded. Consider the sequence $(2^{-n} : n \in \mathbb{N}) \subset (0, 1)$, and $t = 2^{-1}$. For any sequence $(x_n : n \in \mathbb{N})$ of points of X the open balls $B(x_n, 2^{-n}, 2^{-1})$, $n \in \mathbb{N}$, cannot cover X because the sum of lengths of intervals $((1 - 2^{-n})x_n, x_n/(1 - 2^{-n}))$, $n \in \mathbb{N}$, is finite.

Notice that if $X = \mathbb{N}$, and M and $*$ are as above, then $(\mathbb{N}, M, *)$ is FR-bounded.

4. Results

4.1. Subspaces

If $(X, M, *)$ is a fuzzy metric space and $Y \subset X$, then $(Y, M_Y, *)$, where $M_Y = M \upharpoonright Y^2 \times (0, \infty)$, is also a fuzzy metric space and it is called the *fuzzy metric subspace* (or shortly *fm*-subspace) of $(X, M, *)$.

Theorem 4.1. *Every fm-subspace of an* FM-*bounded space* $(X, M, *)$ *is also* FM-*bounded.*

Proof. Let $(Y, M_Y, *)$ be an *fm*-subspace of $(X, M, *)$ and let $(\varepsilon_n : n \in \mathbb{N})$ be a sequence of elements of $(0, 1)$ and $t > 0$. Because of continuity of $*$ for each $n \in \mathbb{N}$ there is $\delta_n \in (0, 1)$ such that $(1 - \delta_n) * (1 - \delta_n) > 1 - \varepsilon_n$. Apply now to the sequence $(\delta_n : n \in \mathbb{N})$ and $\frac{t}{2}$ the assumption on $(X, M, *)$. There is a sequence $(P_n : n \in \mathbb{N})$ of finite subsets of X such that

$$X = \bigcup_{n \in \mathbb{N}} \bigcup_{p \in P_n} B(p, \delta_n, t/2).$$

For each $n \in \mathbb{N}$ let

$$Q_n = \{q \in P_n : \exists y \in Y \text{ with } y \in B(q, \delta, t/2)\}.$$

Further, for each $q \in Q_n$ pick an element $y_q \in Y$ such that $y_q \in B(q, \delta, t/2)$ and set

$$S_n = \{y_q : q \in Q_n\}.$$

Let us show that the sequence $(S_n : n \in \mathbb{N})$ of finite subsets of Y witnesses for $(\varepsilon_n : n \in \mathbb{N})$ and t that $(Y, M_Y, *)$ is FM-bounded.

Let y be an arbitrary element of Y. There exist $n \in \mathbb{N}$ and $p \in P_n$ such that $y \in B(p, \delta_n, t/2)$, and from the definition of Q_n it follows $p \in Q_n$. Therefore, there exists $y_p \in S_n$ such that $y_p \in B(p, \delta_n, t/2)$, hence $p \in B(y_p, \delta_n, t/2)$. So, we have

$$M(p, y, t/2) > 1 - \delta_n \text{ and } M(p, y_p, t/2) > 1 - \delta_n.$$

According to (M.4) we have

$$M(y, y_p, t) \geq M(y, p, t/2) * M(p, y_p, t/2) > (1 - \delta_n) * (1 - \delta_n) > 1 - \varepsilon_n,$$

which means $y \in B(y_p, \varepsilon_n, t)$. As $y \in Y$ was arbitrary we conclude

$$Y = \bigcup_{n \in \mathbb{N}} \bigcup_{y_p \in S_n} B(y_p, \varepsilon_n, t),$$

i.e. Y is FM-bounded. $\square$

The proof of the following theorem is similar to the proof of Theorem 4.1 and thus it is omitted.

Theorem 4.2. *Every fm-subspace of an* FH*-bounded space* $(X, M, *)$ *is also* FH*-bounded.*

Theorem 4.3. *Let a fuzzy metric space* $(Y, M_Y, *)$ *be dense in a fuzzy metric space* $(X, M, *)$. *If Y is* F*-Hurewicz-bounded, then X is also* F*-Hurewicz-bounded.*

Proof. Let a sequence $(\varepsilon_n : n \in \mathbb{N})$ of elements from $(0, 1)$ and $t > 0$ be given. Choose a sequence $(\delta_n : n \in \mathbb{N})$ of elements of $(0, 1)$ such that $(1 - \delta_n) * (1 - \delta_n) > 1 - \varepsilon_n$ for each $n \in \mathbb{N}$. Applying to $(\delta_n : n \in \mathbb{N})$ and $t/2$ the fact that $(Y, M_Y, *)$ is FH-bounded we find a sequence $(A_n : n \in \mathbb{N})$ of finite subsets of Y such that each $y \in Y$ belongs to $\bigcup_{a \in A_n} B(a, \delta_n, t/2)$ for all but finitely many n. We claim that $(A_n : n \in \mathbb{N})$ witnesses for $(\varepsilon_n : n \in \mathbb{N})$ and t that $(X, M, *)$ is FH-bounded.

Let $x \in X$. Since Y is dense in X the intersection $Y \cap B(x, \delta_n, t/2)$ is non-empty for each $n \in \mathbb{N}$; let $y_n \in Y \cap B(x, \delta_n, t/2)$. But $(Y, M_Y, *)$ is FH-bounded, so that there is $n_0 \in \mathbb{N}$ such that $y_n \in \bigcup_{a \in A_n} B(a, \delta_n, t/2)$ for each $n \geq n_0$, i.e. for each $n \geq n_0$ there is $a_n \in A_n$ with $y_n \in B(a_n, \delta_n, t/2)$. Therefore, for each $n \geq n_0$ we have

$$M(x, a_n, t) \geq M(x, y_n, t/2) * M(y_n, a_n, t/2) > (1 - \delta_n) * (1 - \delta_n) > 1 - \varepsilon_n,$$

i.e. for each $n \geq n_0$, $x \in \bigcup_{a \in A_n} B(a, \varepsilon_n, t)$ which means that X is FH-bounded. $\square$

4.2. Products

Let $(X, M_X, *)$ and $(Y, M_Y, *)$ be fuzzy metric spaces and let $Z = X \times Y$. For $z_1 = (x_1, y_1), z_2 = (x_2, y_2) \in Z$ and $t > 0$ define

$$M_Z(z_1, z_2, t) = M_X(x_1, y_1, t) * M_Y(x_2, y_2, t).$$

Then $(M_Z, *)$ is a fuzzy metric on Z, and the triple $(Z, M_Z, *)$ is called the *product metric space* of X and Y.

Theorem 4.4. *The product* $(Z, M_Z, *)$ *of two* FH*-bounded spaces* $(X, M_X, *)$ *and* $(Y, M_Y, *)$ *is also* FH*-bounded.*

Proof. Suppose the sequence $(\varepsilon_n : n \in \mathbb{N}) \subset (0, 1)$ and $t > 0$ are given. Choose for each $n \in \mathbb{N}$ an element δ_n in $(0, 1)$ such that $(1 - \delta_n) * (1 - \delta_n) > 1 - \varepsilon_n$. By assumption on X and Y there are sequences $(S_n : n \in \mathbb{N})$ and $(T_n : n \in \mathbb{N})$ of finite sets of X and Y, respectively and natural numbers n_1 and n_2 such that each $x \in X$ belongs to $\bigcup_{a \in S_n} B(a, \delta_n, t/2)$ for all $n \geq n_1$, and each $y \in Y$ belongs to $\bigcup_{c \in T_n} B(c, \delta_n, t/2)$ for all $n \geq n_2$. We claim that the sequence $(S_n \times T_n : n \in \mathbb{N})$ of finite subsets of Z witnesses for $(\varepsilon_n : n \in \mathbb{N})$ and t that $(Z, M_Z, *)$ is FH-bounded.

Let $z = (x, y) \in Z$. Pick $n_1, n_2 \in \mathbb{N}$ such that for each $n \geq n_1$ and each $k \geq n_2$

$x \in B(a_n, \delta_n, t/2)$ for some $a_n \in S_n$

and

$y \in B(c_k, \delta_k, t/2)$ for some $c_k \in T_k$.

Then for each $n \geq n_0 = \max\{n_1, n_2\}$ there is $z_n = (a_n, c_n) \in S_n \times T_n$ such that

$$M_Z(z, z_n, t) = M_X(x, a_n, t) * M_Y(y, c_n, t) > (1 - \delta_n) * (1 - \delta_n) > 1 - \varepsilon_n.$$

This means that $z \in B(z_n, \varepsilon_n, t)$ as required to be shown. $\square$

In a similar way one can prove the following assertion.

Theorem 4.5. *The product* $(Z, M_Z, *)$ *of an* FM-*bounded space* $(X, M_X, *)$ *and an* FH-*bounded space* $(Y, M_Y, *)$ *is* FM-*bounded.*

Example 4.6. The product of an FR-bounded fuzzy metric space and a (pre)compact fuzzy metric space need not be FR-bounded.

Let $X = \mathbb{N}$ with the fuzzy metric $(M_X, *)$ defined by

$$a * b = a \cdot b; \quad M_X(x, y, t) = \begin{cases} \frac{x}{y}, & \text{if } x \leq y; \\ \frac{y}{x}, & \text{if } x \geq y, \end{cases}$$

and $Y = [0, 1]$ with the fuzzy metric $(M_Y, *)$

$$M_Y(x, y, t) = \frac{t}{t + |x - y|},$$

for all $x, y \in Y$ and $t > 0$, We observed that X is FR-bounded; also, Y is (pre)compact. However, the product space $Z = X \times Y$ is not FR-bounded.

Take for each $n \in \mathbb{N}$, $\varepsilon_n = 2^{-n}$ and consider the sequence $(2^{-n} : n \in \mathbb{N}) \subset (0, 1)$ and $t = 1$. An easy calculation shows that for $z_n = (n, y_n) \in \{n\} \times [0, 1]$ we have $B(z_n, 2^{-n}, 1) = \{(n, y) : y \in [0, 1], |y - y_n| < (2^n - 1)^{-1}\}$. Therefore, for any sequence $(z_n : n \in \mathbb{N})$ of elements of Z the balls $B(z_n, 2^{-n}, 1)$, $n \in \mathbb{N}$, do not cover Z.

4.3. Two theorems more

Let S be a subset of X, $\varepsilon \in (0, 1)$, $t > 1$. Then we denote

$$B(S, \varepsilon, t) := \bigcup_{x \in S} B(x, \varepsilon, t).$$

Theorem 4.7. *For a fuzzy metric space* $(X, M, *)$ *the following are equivalent:*

(a) *For each sequence* $(\varepsilon_n : n \in \mathbb{N}) \subset (0, 1)$ *and each* $t > 0$ *there is a sequence* $(S_n : n \in \mathbb{N})$ *of finite subsets of* X *such that each finite subset* $E \subset X$ *is contained in* $B(S_n, \varepsilon_n, t)$ *for some* S_n;

(b) *For each sequence* $(\varepsilon_n : n \in \mathbb{N}) \subset (0, 1)$ *and each* $t > 0$ *there is a sequence* $(S_n : n \in \mathbb{N})$ *of finite subsets of* X *and an increasing sequence* $n_1 < n_2 < \cdots$ *of natural numbers such that each finite subset* $E \subset X$ *is contained in* $\bigcup_{n_k \leq i < n_{k+1}} B(S_i, \varepsilon_i, t)$ *for some* $k \in \mathbb{N}$.

Proof. Evidently (a) implies (b). We prove $(b) \Rightarrow (a)$. Let $(\varepsilon_n : n \in \mathbb{N})$ be a sequence of elements from $(0, 1)$ and $t > 0$. For each $n \in \mathbb{N}$ let $\delta_n = \min\{\varepsilon_i : i \leq n\}$ and apply (b) to $(\delta_n : n \in \mathbb{N})$ and t. There is an increasing sequence $n_1 < n_2 < \cdots$ in $\mathbb{N}$ such that each finite set $S \subset X$ is contained in $\bigcup_{n_k \leq i < n_{k+1}} B(S_i, \delta_i, t)$ for some $k \in \mathbb{N}$. Define now

$T_n = \bigcup_{i < n_1} S_i$, for each $n < n_1$,

$T_n = \bigcup_{n_k \leq i < n_{k+1}} S_i$, for each n such that $n_k \leq n < n_{k+1}$.

We claim that the sequence $(T_n : n \in \mathbb{N})$ of finite subsets of X witnesses for $(\varepsilon_n : n \in \mathbb{N})$ and t that (a) is satisfied.

Let F be a finite subset of X. Choose $k \in \mathbb{N}$ such that $F \subset \bigcup_{n_k \leq i < n_{k+1}} B(S_i, \delta_i, t)$. For (each) n with $n_k \leq n < n_{k+1}$ put $T_n = \bigcup_{n_k \leq i < n_{k+1}} S_i$. We have that for each $x \in F$ there is j, $n_k \leq j < n_{k+1}$ and $y \in S_j$ with $x \in B(y, \delta_j, t)$. Further, we have $B(y, \delta_j, t) \subset B(y, \varepsilon_j, t)$ and since $y \in T_n$ we have $x \in B(T_j, \varepsilon_j, t)$, and thus $F \subset B(T_j, \varepsilon_j, t)$. $\square$

With a small modification in the previous proof one can prove the following.

Theorem 4.8. *For a fuzzy metric space $(X, M, *)$ the following are equivalent:*

(a) *For each sequence $(\varepsilon_n : n \in \mathbb{N}) \subset (0, 1)$ and each $t > 0$ there is a sequence $(S_n : n \in \mathbb{N})$ of finite subsets of X such that each finite subset $E \subset X$ is contained in $B(S_n, \varepsilon_n, t)$ for all but finitely many n;*

(b) *For each sequence $(\varepsilon_n : n \in \mathbb{N}) \subset (0, 1)$ and each $t > 0$ there is a sequence $(S_n : n \in \mathbb{N})$ of finite subsets of X and an increasing sequence $n_1 < n_2 < \cdots$ of natural numbers such that each finite subset $E \subset X$ is contained in $\bigcup_{n_k \leq i < n_{k+1}} B(S_i, \varepsilon_i, t)$ for all but finitely many $k \in \mathbb{N}$.*

5. Conclusion

In this paper we demonstrated an application of the idea of fuzzy metric spaces in the classical field of covering properties defined in terms of selection principles of the Menger, Rothberger and Hurewicz types. It is reasonable to expect that this investigation, like other studies on fuzzy metric and intuitionistic fuzzy metric spaces, may have some applications in other mathematical disciplines.

At the end, let us emphasize the following two facts.

1. The results obtained in this paper can be obtained, by minor modifications, for generalized fm-spaces, i.e. $\mathcal{L}$-fuzzy metric spaces as considered in [13] and [15].

2. To each boundedness property considered here one can correspond an infinitely long two-person game. Let us explain this scenario for the FM-boundedness; it is easy to apply the same idea to the other two cases.

Consider the game G_{FM} on a fuzzy metric space $(X, M, *)$ defined as follows. Let $t > 0$ be fixed. There are two players, ONE and TWO, who play a round for each positive integer n. In the n-th round ONE chooses $\varepsilon_n \in (0, 1)$, and TWO responds by choosing a finite subset A_n of X. TWO wins a play $\varepsilon_1, A_1; \varepsilon_2, A_2; \cdots; \varepsilon_n, A_n; \cdots$ if and only if $X = \bigcup_{n \in \mathbb{N}} \bigcup_{a \in A_n} B(a, \varepsilon_n, t)$.

Evidently, if the player TWO has a winning strategy (or even, ONE has no winning strategy) in the game G_{FM}, then $(X, M, *)$ is FM-bounded.

Call a fuzzy metric space X *strongly* FM-*bounded* if TWO has a winning strategy in the game G_{FM}. (In a similar way one defines *strong* FR-*boundedness* and *strong* FH-*boundedness*.)

Conjecture 5.1. *There are fuzzy metric spaces which are* FM-*bounded* (*respectively,* FR-*bounded,* FH-*bounded*), *but not strongly* FM-*bounded* (*respectively, strongly* FR-*bounded, strongly* FH-*bounded*).

Problem 5.2. *Characterize strongly* FM-*bounded, strongly* FR-*bounded and strongly* FH-*bounded fuzzy metric spaces.*

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