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On the boundedness of solutions of a kind of non-autonomous differential equations of second order with finitely many deviating arguments

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Abstract. We study the boundedness of the solutions to a non-autonomous differential equation of second order with finitely many deviating arguments. We give two examples to illustrate the main results. By this work, we improve some boundedness results obtained for a differential equation with a deviating argument in the literature to the boundedness of the solutions of a differential equation with finitely many deviating arguments.

1. Introduction

Differential equations of second order with and without deviating arguments are essential tools in scientific modeling of problems arisen in many fields of sciences and technologies, such as biology, chemistry, physics, mechanics, electronics, engineering, economy, control theory, medicine, atomic energy, information theory, and so on. Hence, the qualitative behaviors of solutions of differential equations of second order have extensively been discussed and are still being investigated by numerous authors in the literature. In particular, for some works performed on the boundedness of solutions of some certain second order nonlinear differential equations, the reader can refer to the book of Ahmad and Rama Mohana Rao [1] and the papers of Baker [2], Cheng and Xu [3], Kato ([4], [5]), Malyseva [6], Muresan [7], Nápoles Valdés [8], Saker [9], Tunç [10–17], C. Tunç and E. Tunç [18], Zhao [19], Wang [21] and the references cited in these sources.

Meanwhile, in 2007, Zhao et al. [20] considered the nonlinear differential equation of second order with a deviating argument

$$(r(t)x'(t))' + p(t)x'(t) + q_1(t)x(t) + q_2(t)x(h(t)) = f(t, x(t)). \tag{1}$$

In [20], the authors established two theorems which include some sufficient conditions and guarantee that all solutions of Eq. (1) are bounded. Namely, Zhao et al. [20] first proved the following theorem.

Theorem 1.1. (Zhao et al. [20,Theorem 1]) *Assume that*

(i) $q_1(t) \in C^1[a, \infty)$, $q_1(t) > 0$, $r(t) > 0$, $h(t) \le t$, $h'(t) > 0$ *for all* $t \in [a, \infty)$,

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(ii) $Q(t) = \frac{1}{2}[r'(t)q_1(t) + 2p(t)q_1(t) + r(t)q_1'(t)] > 0,\ b_1(t) = \frac{q_2^2(t)}{Q(t)h'(t)}$ *for all* $t \in [a, \infty)$,

(iii) $\int\limits_a^\infty b_1(t)dt < \infty,\ \int\limits_a^\infty \frac{g^2(t)}{Q(t)}\,dt < \infty,\ \int\limits_a^\infty \frac{k(t)}{\sqrt{q_1(t)r(t)}}\,dt < \infty.$

Then, every solution of Eq. (1) together with its derivative satisfy

$$|x(t)| = O(1),\ |x'(t)| = O\left(\sqrt{\frac{q_0(t)}{r(t)}}\right).$$

Zhao et al. [20] second proved the following theorem.

Theorem 1.2. (Zhao et al. [20, Theorem 2]) *Assume that*

(i) $q_1(t) \in C^1[a, \infty),\ q_1(t) > 0,\ r(t) > 0,\ h(t) \le t,\ h'(t) > 0$ *for all* $t \in [a, \infty)$,

(ii) $Q(t) = p(t)q_1(t) + \frac{1}{2}r'(t)q_1(t) + \frac{1}{4}r(t)q_1'(t) > 0,\ b_2(t) = \frac{q_2^2(t)q_1^{\frac{1}{2}}(t)}{Q(t)h'(t)}$ *for all* $t \in [a, \infty)$,

(iii) $\int\limits_a^\infty \frac{g^2(t)q_1^{\frac{1}{2}}(t)}{Q(t)}\,dt < \infty,\ \int\limits_a^\infty b_2(t)q_1^{-\frac{1}{2}}(t)dt < \infty,\ \int\limits_a^\infty q_1^{-1}(t)q_1'(t)dt < \infty,\ \int\limits_a^\infty k(t)q_1^{\frac{-1-\alpha}{4}}(t)r^{-\frac{1}{2}}(t)dt < \infty.$

Then, every solution of Eq. (1) together with its derivative satisfy

$$|x(t)| = O(1),\ |x'(t)| = O\left(\sqrt{\frac{q_0(t)}{r(t)}}\right).$$

In this paper, instead of Eq. (1), we consider the following non-autonomous and non-linear differential equation of second order with multiple deviating arguments of the form

$$(r(t)x'(t))' + p(t)x'(t) + q_0(t)x(t) + \sum_{i=1}^{n} q_i(t)x(h_i(t)) = f(t, x(t)), \tag{2}$$

where $t \in [a, \infty)$, $(a > 0,\ a \in \Re)$, p, q_0 and q_i are continuous functions and r and h_i are differentiable functions on $[a, \infty)$ with $r(t) > 0$, $0 \le h_i(t) \le t$, $\lim\limits_{t\to\infty} h_i(t) \to \infty$, f is a continuous function on $[a, \infty) \times (-\infty, \infty)$ with

$$\left|f(t, x)\right| \le g(t) + k(t)\,|x|^\alpha,\ g(t) \ge 0, k(t) \ge 0, 0 \le \alpha \le 1,\ \alpha \in \Re. \tag{3}$$

This work is motivated by the paper of Zhao et al. [20]. We here establish two new results on the boundedness of the solutions of Eq. (2) and also give two examples to show the feasibility of our results. It should be noted that Eq. (2) and the assumptions will be established here are different from that in the papers mentioned above. It should be also noted that to the best of our knowledge there is not any paper based on the results in [20] in the literature. Our results in this paper improve the results in [20], Theorem 1.1, Theorem 1.2, in the sense that our results do not require only a deviating argument, but finitely many deviating arguments.

2. Main results

Our first main problem is given by the following theorem.

Theorem 2.1. *In addition to the basic assumptions imposed on the functions r, p, q_0, q_i, h_i and f, we assume that the following conditions hold:*

(i) $q_0(t), q_i(t) \in C^1[a,\infty)$, $q_0(t) > 0$, $q_i(t) > 0$, $h_i(t) \leq t$, $h_i'(t) > 0$ *for all* $t \in [a,\infty)$,

(ii) $Q(t) = \frac{1}{2n}[r'(t)q_0(t) + 2p(t)q_0(t) + r(t)q_0'(t)] > 0$, $n \in N$, $b_i(h_i(t)) = \frac{q_i^2(t)}{Q(t)h_i'(t)}$ *for all* $t \in [a,\infty)$,

(iii) $\int\limits_a^\infty b_i(t)dt < \infty$, $\int\limits_a^\infty \frac{g^2(t)}{Q(t)}\, dt < \infty$, $\int\limits_a^\infty \frac{k(t)}{\sqrt{q_0(t)r(t)}}\, dt < \infty$.

Then, every solution of Eq. (2) together with its derivative satisfy

$$|x(t)| = O(1),\ |x'(t)| = O\left(\sqrt{\frac{q_0(t)}{r(t)}}\right).$$

Proof. Define the Lyapunov functional

$$V(t) = x^2(t) + \frac{r(t)}{q_0(t)}(x'(t))^2 + \sum_{i=1}^{n} \int\limits_{h_i(t)}^{t} b_i(s)x^2(s)ds. \tag{4}$$

It follows from (2) that

$$r(t)x''(t) = f(t,x(t)) - r'(t)x'(t) - p(t)x'(t) - q_0(t)x(t) - \sum_{i=1}^{n} q_i(t)x(h_i(t)). \tag{5}$$

Calculating the time derivative of the Lyapunov functional $V(t)$, we obtain

$$\begin{aligned} \frac{dV}{dt} &= 2xx' + \frac{r'(t)(x')^2+2r(t)x'x''}{q_0(t)} - \frac{r(t)(x')^2q_0'(t)}{q_0^2(t)} + \sum_{i=1}^{n} b_i(t)x^2 - \sum_{i=1}^{n} b_i(h_i(t))x^2(h_i(t))h_i'(t) \\ &= -\frac{r'(t)}{q_0(t)}(x')^2 - \frac{2p(t)}{q_0(t)}(x')^2 - \frac{r(t)q_0'(t)}{q_0^2(t)}(x')^2 + \frac{2x'f(t,x)}{q_0(t)} + \sum_{i=1}^{n} b_i(t)x^2 - \frac{2x'}{q_0(t)}\sum_{i=1}^{n} q_i(t)x(h_i(t)) - \sum_{i=1}^{n} b_i(h_i(t))x^2(h_i(t))h_i'(t). \end{aligned}$$

In view of (5), the assumption $\left|f(t,x)\right| \leq g(t) + k(t)\left|x\right|^\alpha$ in (3) and the above estimate, we have

$$\begin{aligned} \frac{dV}{dt} &\leq -\frac{r'(t)q_0(t)+2p(t)q_0(t)+r(t)q_0'(t)}{q_0^2(t)}(x')^2 + \frac{2g(t)|x'|+2k(t)|x|^\alpha|x'|}{q_0(t)} + \sum_{i=1}^{n} b_i(t)x^2 - \frac{2x'}{q_0(t)}\sum_{i=1}^{n} q_i(t)x(h_i(t)) - \sum_{i=1}^{n} b_i(h_i(t))x^2(h_i(t))h_i'(t) \\ &= -\frac{2nQ(t)}{q_0^2(t)}(x')^2 + \frac{2g(t)}{q_0(t)}|x'| + \frac{2k(t)}{q_0(t)}\ |x|^\alpha|x'| + \sum_{i=1}^{n} b_i(t)x^2 - \frac{2x'}{q_0(t)}\sum_{i=1}^{n} q_i(t)x(h_i(t)) - \sum_{i=1}^{n} b_i(h_i(t))x^2(h_i(t))h_i'(t). \end{aligned}$$

On the other hand, from the inequality $0 \leq (ax-b)^2$, it is clear that $-ax^2 + bx \leq -\frac{a}{2}x^2 + \frac{b^2}{2a}$, where $a\ (>0)$, $b,\ x \in \Re$.

The foregoing estimate implies that

$$\frac{dV}{dt} \leq -\frac{nQ(t)}{q_0^2(t)}(x')^2 + \frac{g^2(t)}{nQ(t)} + \frac{2k(t)}{q_0(t)}\ |x|^\alpha|x'| + \sum_{i=1}^{n} b_i(t)x^2 - \frac{2x'}{q_0(t)}\sum_{i=1}^{n} q_i(t)x(h_i(t)) - \sum_{i=1}^{n} b_i(h_i(t))x^2(h_i(t))h_i'(t).$$

Besides, it follows that

$$-\tfrac{nQ(t)}{q_0^2(t)}(x')^2-\tfrac{2x'}{q_0(t)}\sum_{i=1}^{n}q_i(t)x(h_i(t))=-\tfrac{Q(t)}{q_0^2(t)}\left[(x')^2+2\tfrac{q_0(t)q_1(t)}{Q(t)}x'x(h_1(t))\right]-\tfrac{Q(t)}{q_0^2(t)}\left[(x')^2+2\tfrac{q_0(t)q_2(t)}{Q(t)}x'x(h_2(t))\right]$$

$$-...-\tfrac{Q(t)}{q_0^2(t)}\left[(x')^2+2\tfrac{q_0(t)q_n(t)}{Q(t)}x'x(h_n(t))\right]$$

$$=-\tfrac{Q(t)}{q_0^2(t)}\left[x'+\tfrac{q_0(t)q_1(t)}{Q(t)}x(h_1(t))\right]^2-\tfrac{Q(t)}{q_0^2(t)}\left[x'+\tfrac{q_0(t)q_2(t)}{Q(t)}x(h_2(t))\right]^2$$

$$-...-\tfrac{Q(t)}{q_0^2(t)}\left[x'+\tfrac{q_0(t)q_n(t)}{Q(t)}x(h_n(t))\right]^2+\tfrac{1}{Q(t)}\sum_{i=1}^{n}q_i^2(t)x^2(h_i(t))$$

$$\leq\tfrac{1}{Q(t)}\sum_{i=1}^{n}q_i^2(t)x^2(h_i(t)).$$

Hence, we get

$$\frac{dV}{dt}\leq\frac{g^2(t)}{nQ(t)}+\frac{2k(t)}{q_0(t)}\;|x|^\alpha\,|x'|+\sum_{i=1}^{n}b_i(t)x^2+\sum_{i=1}^{n}\left(\frac{q_i^2(t)}{Q(t)}-b_i(h_i(t))h_i'(t)\right)x^2(h_i(t)).$$

Let

$$b_i(h_i(t))=\frac{q_i^2(t)}{Q(t)h_i'(t)},\ (i=1,2,...,n).$$

This choice leads that

$$\frac{dV}{dt}\leq\frac{g^2(t)}{nQ(t)}+\frac{2k(t)}{q_0(t)}\;|x|^\alpha\,|x'|+\sum_{i=1}^{n}b_i(t)x^2.$$

On the other hand, form the definition of the Lyapunov functional in (4), it follows that

$$x^2(t)\leq V(t),|x(t)|^\alpha\leq V^{\frac{\alpha}{2}}(t),|x'(t)|\leq\frac{\sqrt{q_0(t)}}{\sqrt{r(t)}}V^{\frac{1}{2}}(t).$$

Integrating the above inequality from a to t and using the last estimates, we get

$$\begin{aligned}V(t)\;&\leq V(a)+\tfrac{1}{n}\int_a^t\tfrac{g^2(s)}{Q(s)}ds+2\int_a^t\tfrac{k(s)}{q_0(s)}\;|x(s)|^\alpha\,|x'(s)|\,ds+\int_a^t\sum_{i=1}^{n}b_i(s)x^2(s)ds\\&\leq V_0+2\int_a^t\tfrac{k(s)}{\sqrt{q_0(s)r(s)}}\{V(s)\}^{\frac{1+\alpha}{2}}ds+\int_a^t\sum_{i=1}^{n}b_i(s)V(s)ds,\end{aligned}$$

where $V_0=V(a)+\frac{1}{n}\int_a^t\frac{g^2(s)}{Q(s)}ds\geq 0.$

Using Gronwall-Reid-Bellman inequality (see Ahmad and Rama Mohana Rao [1]), one can obtain

$$V(t)\leq\begin{cases}\exp\{\int_a^t\sum_{i=1}^{n}b_i(s)ds\}\left[V_0^{\frac{1-\alpha}{2}}+\tfrac{1-\alpha}{2}\int_a^t\tfrac{2k(s)}{\sqrt{q_0(s)r(s)}}\exp\left(\int_a^s\tfrac{\alpha-1}{2}\sum_{i=1}^{n}b_i(\tau)d\tau\right)ds\right]^{\frac{2}{1-\alpha}},\ 0\leq\alpha<1,\\ V_0\exp\left[\int_a^t\left(\sum_{i=1}^{n}b_i(s)+\tfrac{2k(s)}{\sqrt{q_0(s)r(s)}}\right)ds\right],\ \alpha=1.\end{cases}$$

Since all the integrals in the last inequality converge when $t \to \infty$, under the assumptions of Theorem 2.1, we can conclude for a positive constant M that $V(t) \le M$. This result leads that

$$|x(t)| = O(1),\ |x'(t)| = O\left(\sqrt{\frac{q_0(t)}{r(t)}}\right).$$

□

Example 2.2. Consider the following non-autonomous differential equation of second order with multiple deviating arguments

$$(t^3x'(t))' + t^2x'(t) + t^3x(t) + tx\left(\frac{t}{2}\right) + \frac{t}{2}x\left(\frac{3t}{4}\right) + \frac{t}{4}x\left(\frac{2t}{3}\right) = t + x^\alpha, t \ge 1 = a. \tag{6}$$

Comparing Eq. (6) with Eq. (2), we obtain

$$r(t) = t^3, r'(t) = 3t^2, p(t) = t^2, q_0(t) = t^3, q_0'(t) = 3t^2,$$

$$q_1(t) = t > 0, q_2(t) = \frac{t}{2} > 0, q_3(t) = \frac{t}{4} > 0,$$

$$h_1(t) = \frac{t}{2} \le t, h_2(t) = \frac{t}{4} \le t, h_3(t) = \frac{t}{3} \le t,$$

$$h_1'(t) = \frac{1}{2} > 0, h_2'(t) = \frac{1}{4} > 0, h_3'(t) = \frac{1}{3} > 0,$$

$$f(t, x(t)) = t + x^\alpha, g(t) = t, k(t) = 1,$$

$$Q(t) = \frac{1}{2n}[r'(t)q_0(t) + 2p(t)q_0(t) + r(t)q_0'(t)] = \frac{4}{3}t^5 > 0, \quad n = 3,$$

$$b_1(h_1(t)) = \frac{q_1^2(t)}{Q(t)h_1'(t)} \Rightarrow b_1\left(\frac{t}{2}\right) = \frac{3}{2t^3}, b_1(t) = \frac{3}{16t^3},$$

$$\int_1^\infty b_1(t)dt = \frac{3}{16}\int_1^\infty t^{-3}dt = \frac{3}{32} < \infty,$$

$$b_2(h_2(t)) = \frac{q_2^2(t)}{Q(t)h_2'(t)} \Rightarrow b_2\left(\frac{t}{4}\right) = \frac{96}{5t^4}, b_2(t) = \frac{3}{40t^4},$$

$$\int_1^\infty b_2(t)dt = \frac{3}{40}\int_1^\infty t^{-4}dt = \frac{1}{40} < \infty,$$

$$b_3(h_3(t)) = \frac{q_3^2(t)}{Q(t)h_3'(t)} \Rightarrow b_3\left(\frac{t}{3}\right) = \frac{9}{64t^4}, b_3(t) = \frac{1}{576t^4},$$

$$\int_1^\infty b_3(t)dt = \frac{1}{576}\int_1^\infty t^{-4}dt = \frac{1}{1728} < \infty,$$

$$\int_a^\infty \frac{g^2(t)}{Q(t)}dt = \frac{3}{4}\int_1^\infty \frac{1}{t^3}dt = \frac{3}{8} < \infty,$$

$$\int_a^\infty \frac{k(t)}{\sqrt{q_0(t)r(t)}}dt = \int_1^\infty \frac{1}{t^3}dt = \frac{1}{2} < \infty.$$

In view of the above estimates, it follows that all the assumptions of Theorem 2.1 hold. Hence, we conclude that all solutions of Eq. (6) satisfies

$$|x(t)| = O(1),\ |x'(t)| = O(1).$$

The second main problem is the following theorem.

Theorem 2.3. *In addition to the basic assumptions imposed on the functions r, p, q_0, q_i, h_i and f, we assume that the following conditions hold:*

(i) $q_i(t) \in C^1[a,\infty)$, $r(t) > 0$, $q_i(t) > 0$, $h_i(t) \le t$, $h_i'(t) > 0$ *for all* $t \in [a,\infty)$,

(ii) $Q(t) = \frac{1}{n}[p(t)q_0(t) + \frac{1}{2}r'(t)q_0(t) + \frac{1}{4}r(t)q_0'(t)] > 0$, $n \in N$, $b_i(h_i(t)) = \frac{q_i^2(t)q_0^{\frac{1}{2}}(t)}{Q(t)h_i'(t)}$ *for all* $t \in [a,\infty)$,

(iii) $\int_a^\infty \frac{g^2(t)q_0^{\frac{1}{2}}(t)}{Q(t)}dt < \infty$, $\int_a^\infty b_i(t)q_0^{-\frac{1}{2}}(t)dt < \infty$, $\int_a^\infty q_0^{-1}(t)q_0'(t)dt < \infty$, $\int_a^\infty k(t)q_0^{\frac{-1-\alpha}{4}}(t)r^{-\frac{1}{2}}(t)dt < \infty$.

Then, every solution of Eq. (3) together with its derivative satisfy

$$|x(t)| = O\left(\frac{1}{\sqrt[4]{q_0(t)}}\right),\ |x'(t)| = O\left(\frac{\sqrt[4]{q_0(t)}}{\sqrt{r(t)}}\right).$$

Proof. Define the Lyapunov functional

$$V_1(t) = \sqrt{q_0(t)}x^2(t) + \frac{r(t)}{\sqrt{q_0(t)}}(x'(t))^2 + \sum_{i=1}^n \int_{h_i(t)}^t b_i(s)x^2(s)ds. \tag{7}$$

Calculating the time derivative of the Lyapunov functional $V_1(t)$ in (7), we obtain

$$\frac{dV_1}{dt} = \frac{1}{2}q_0^{-\frac{1}{2}}(t)q_0'(t)x^2 + \frac{2f(t,x)-2\sum_{i=1}^n q_i(t)x(h_i(t))-2p(t)x'-r'(t)x'}{\sqrt{q_0(t)}}x' - \frac{1}{2}q_0^{-\frac{3}{2}}(t)q_0'(t)r(t)(x')^2 + \sum_{i=1}^n b_i(t)x^2 - \sum_{i=1}^n b_i(h_i(t))x^2(h_i(t))h_i'(t).$$

Using the assumptions of Theorem 2.3, it follows that

$$\begin{aligned}\frac{dV_1}{dt} &\le \frac{1}{2}q_0^{-\frac{1}{2}}(t)q_0'(t)x^2 + 2\frac{f(t,x)}{\sqrt{q_0(t)}}x' - 2\frac{\sum_{i=1}^n q_i(t)x(h_i(t))}{\sqrt{q_0(t)}}x' - \frac{2p(t)q_0(t)+r'(t)q_0(t)+2^{-1}r(t)q_0'(t)}{\sqrt{(q_0(t))^3}}(x')^2 \\ &+ \sum_{i=1}^n b_i(t)x^2 - \sum_{i=1}^n b_i(h_i(t))x^2(h_i(t))h_i'(t).\end{aligned}$$

Making use of the assumption (ii) of Theorem 2.3 and the inequality $\left|f(t,x)\right| \le g(t) + k(t)\left|x\right|^\alpha$, we have

$$\begin{aligned}\frac{dV_1}{dt} &\le \frac{1}{2}q_0^{-\frac{1}{2}}(t)q_0'(t)x^2 + 2\frac{g(t)}{\sqrt{q_0(t)}}x' + 2\frac{k(t)}{\sqrt{q_0(t)}}\left|x\right|^\alpha\left|x'\right| - 2\frac{\sum_{i=1}^n q_i(t)x(h_i(t))}{\sqrt{q_0(t)}}x' - \frac{2nQ}{\sqrt{(q_0(t))^3}}(x')^2 \\ &+ \sum_{i=1}^n b_i(t)x^2 - \sum_{i=1}^n b_i(h_i(t))x^2(h_i(t))h_i'(t).\end{aligned}$$

On the other hand , from the inequality $0 \le (ax-b)^2$, it follows that $-ax^2+bx \le -\frac{a}{2}x^2+\frac{b^2}{2a}$, where $a(>0)$, b, $x \in \Re$.

The above inequality leads that

$$-\frac{2nQ(t)}{\sqrt{(q_0(t))^3}}x^2+\frac{2g(t)}{\sqrt{q_0(t)}}x \le -\frac{nQ(t)}{\sqrt{(q_0(t))^3}}x^2+\frac{\sqrt{q_0(t)}g^2(t)}{nQ(t)}.$$

Therefore,

$$\begin{aligned}\frac{dV_1}{dt} \quad &\le -\frac{nQ}{\sqrt{(q_0(t))^3}}(x')^2+\frac{\sqrt{q_0(t)}g^2(t)}{nQ(t)}+2\frac{k(t)}{\sqrt{q_0(t)}}|x|^{\alpha}|x'|+\frac{1}{2}q_0^{-\frac{1}{2}}(t)q_0'(t)x^2-2\frac{\sum_{i=1}^{n}q_i(t)x(h_i(t))}{\sqrt{q_0(t)}}x'\\ &+\sum_{i=1}^{n}b_i(t)x^2-\sum_{i=1}^{n}b_i(h_i(t))x^2(h_i(t))h_i'(t).\end{aligned}$$

Besides, it follows that

$$\begin{aligned}&-\frac{nQ(t)}{\sqrt{(q_0(t))^3}}(x')^2-2\frac{\sum_{i=1}^{n}q_i(t)x(h_i(t))}{\sqrt{q_0(t)}}x'\\ &=-\frac{Q(t)}{\sqrt{(q_0(t))^3}}\left[(x')^2+2\frac{q_0(t)q_1(t)}{Q(t)}x'x(h_1(t))\right]-\frac{Q(t)}{\sqrt{(q_0(t))^3}}\left[(x')^2+2\frac{q_0(t)q_2(t)}{Q(t)}x'x(h_2(t))\right]\\ &-...-\frac{Q(t)}{\sqrt{(q_0(t))^3}}\left[(x')^2+2\frac{q_0(t)q_n(t)}{Q(t)}x'x(h_n(t))\right]\\ &=-\frac{Q(t)}{\sqrt{(q_0(t))^3}}\left[x'+\frac{q_0(t)q_1(t)}{Q(t)}x(h_1(t))\right]^2-\frac{Q(t)}{\sqrt{(q_0(t))^3}}\left[x'+\frac{q_0(t)q_2(t)}{Q(t)}x(h_2(t))\right]^2\\ &-...-\frac{Q(t)}{\sqrt{(q_0(t))^3}}\left[x'+\frac{q_0(t)q_n(t)}{Q(t)}x(h_n(t))\right]^2+\frac{\sqrt{q_0(t)}}{Q(t)}\sum_{i=1}^{n}q_i^2(t)x^2(h_i(t))\\ &\le \frac{\sqrt{q_0(t)}}{Q(t)}\sum_{i=1}^{n}q_i^2(t)x^2(h_i(t)).\end{aligned}$$

Hence, we get

$$\frac{dV_1}{dt} \quad \le \frac{\sqrt{q_0(t)}g^2(t)}{nQ(t)}+2\frac{k(t)}{\sqrt{q_0(t)}}|x|^{\alpha}x'+\frac{1}{2}q_0^{-\frac{1}{2}}(t)q_0'(t)x^2+\sum_{i=1}^{n}b_i(t)x^2+\sum_{i=1}^{n}\{\frac{\sqrt{q_0(t)}}{Q(t)}q_i^2(t)-b_i(h_i(t))h_i'(t)\}x^2(h_i(t)).$$

Let

$$b_i(h_i(t))=\frac{\sqrt{q_0(t)}\,q_i^2(t)}{Q(t)h_i'(t)},\ (i=1,2,...,n).$$

The above choice gives that

$$\frac{dV_1}{dt} \le \frac{\sqrt{q_0(t)}g^2(t)}{nQ(t)}+2\frac{k(t)}{\sqrt{q_0(t)}}|x|^{\alpha}|x'|+\frac{1}{2}q_0^{-\frac{1}{2}}(t)q_0'(t)x^2+\sum_{i=1}^{n}b_i(t)x^2.$$

On the other hand, form the definition of the Lyapunov functional in (7), it follows that

$$x^2(t) \le q_0^{-\frac{1}{2}}(t)V_1(t), |x(t)|^{\alpha} \le q_0^{-\frac{\alpha}{4}}(t)V_1^{\frac{\alpha}{2}}(t), |x'(t)| \le \frac{q_0^{\frac{1}{4}}(t)}{r^{\frac{1}{2}}(t)}V_1^{\frac{1}{2}}(t).$$

Using the above estimates, we obtain

$$\frac{dV_1}{dt} \le \frac{\sqrt{q_0(t)}g^2(t)}{nQ(t)} + 2k(t)q_0^{-\frac{1+\alpha}{4}}(t)\, r^{-\frac{1}{2}}(t)V_1^{\frac{1+\alpha}{2}}(t) + \left(\frac{1}{2}q_0^{-1}(t)q_0'(t) + q_0^{-\frac{1}{2}}(t)\sum_{i=1}^{n} b_i(t)\right)V_1(t).$$

Integrating the foregoing estimate from a to t, we find

$$\begin{aligned} V_1(t) &\le V_1(a) + \frac{1}{n}\int_a^t \frac{\sqrt{q_0(s)}g^2(s)}{Q(s)}ds + \int_a^t \left(2k(s)q_0^{-\frac{1+\alpha}{4}}(s)\, r^{-\frac{1}{2}}(t)V_1^{\frac{1+\alpha}{2}}(s)\right)ds + \int_a^t \left(\frac{1}{2}q_0^{-1}(s)q_0'(s) + q_0^{-\frac{1}{2}}(s)\sum_{i=1}^{n} b_i(s)\right)V_1(s)ds \\ &\le V_0 + \int_a^t \left(2k(s)q_0^{-\frac{1+\alpha}{4}}(s)\, r^{-\frac{1}{2}}(t)V_1^{\frac{1+\alpha}{2}}(s)\right)ds + \int_a^t \left(\frac{1}{2}q_0^{-1}(s)q_0'(s) + q_0^{-\frac{1}{2}}(s)\sum_{i=1}^{n} b_i(s)\right)V_1(s)ds, \end{aligned}$$

where $V_0 = V_1(a) + \int_a^t \frac{\sqrt{q_0(s)}g^2(s)}{nQ(s)}ds.$

Using Gronwall-Reid-Bellman inequality (see Ahmad and Rama Mohana Rao [1]), one can obtain

$$V_1(t) \le \sqrt{\frac{q_0(t)}{q_0(a)}}\, V_0 \exp\left(\int_a^t \left(\frac{1}{2}q_0^{-1}(s)\sum_{i=1}^{n} b_i(s) + 2k(s)q_0^{-\frac{1+\alpha}{4}}(s)r^{-\frac{1}{2}}(s)\right)ds\right),\ \alpha = 1,$$

and

$$V_1(t) \le \sqrt{\frac{q_0(t)}{q_0(a)}}\ \exp\left(\int_a^t \left(\frac{1}{2}q_0^{-1}(s)\sum_{i=1}^{n} b_i(s)ds\right)\right)\times$$

$$\left[V_0^{\frac{1-\alpha}{2}} + \frac{1-\alpha}{2}\int_a^t 2k(s)q_0^{-\frac{1+\alpha}{4}}(s)r^{-\frac{1}{2}}(s)\exp\left(\frac{\alpha-1}{2}\int_a^\tau \left[2^{-1}q_0^{-1}(s)q_0'(s) + q_0^{-\frac{1}{2}}(s)\sum_{i=1}^{n} b_i(s)\right]ds\right)d\tau\right]^{\frac{2}{1-\alpha}},$$

where $0 \le \alpha < 1$.

Under the assumptions of Theorem 2.3, all the integrals in the above last inequalities converge when $t \to \infty$. Therefore, we can conclude that

$$|x(t)| = O\left(\frac{1}{\sqrt[4]{q_0(t)}}\right),\ |x'(t)| = O\left(\frac{\sqrt[4]{q_0(t)}}{\sqrt{r(t)}}\right).$$

□

Example 2.4. Consider the following non-autonomous second order differential equation

$$(t^3x'(t))' + t^2x'(t) + x(t) + \frac{1}{t}x\left(\frac{t}{2}\right) + \frac{2}{t}x\left(\frac{3t}{4}\right) + \frac{4}{t}x\left(\frac{2t}{3}\right) = \sqrt[3]{t} + x^\alpha, t \ge 1 = a. \tag{8}$$

Comparing Eq. (8) with Eq. (2), we obtain

$$r(t) = t^3, r'(t) = 3t^2, p(t) = t^2, q_0(t) = 1, q_0'(t) = 0,$$

$$q_1(t) = \frac{1}{t} > 0, q_2(t) = \frac{2}{t} > 0, q_3(t) = \frac{4}{t} > 0,$$

$$h_1(t) = \frac{t}{2} \le t, h_2(t) = \frac{t}{4} \le t, h_3(t) = \frac{t}{3} \le t,$$

$$h_1'(t) = \frac{1}{2} > 0, h_2'(t) = \frac{1}{4} > 0, h_3'(t) = \frac{1}{3} > 0,$$

$$f(t,x(t)) = \sqrt[3]{t} + x^{\alpha}, g(t) = \sqrt[3]{t}, k(t) = 1,$$

$$Q(t) = \frac{1}{2n}[r'(t)q_0(t) + 2p(t)q_0(t) + r(t)q_0'(t)] = \frac{5}{6}t^2 > 0,$$

$$b_1(h_1(t)) = \frac{q_1^2(t)}{Q(t)h_1'(t)} \Rightarrow b_1\left(\frac{t}{2}\right) = \frac{12}{5t^4}, b_1(t) = \frac{3}{20t^4},$$

$$\int_1^{\infty} b_1(t)dt = \frac{3}{20}\int_1^{\infty} t^{-4}dt = \frac{1}{20} < \infty,$$

$$b_2(h_2(t)) = \frac{q_2^2(t)}{Q(t)h_2'(t)} \Rightarrow b_2\left(\frac{t}{4}\right) = \frac{3}{4t^3}, b_2(t) = \frac{3}{256t^3},$$

$$\int_1^{\infty} b_2(t)dt = \frac{3}{256}\int_1^{\infty} t^{-3}dt = \frac{3}{512} < \infty,$$

$$b_3(h_3(t)) = \frac{q_3^2(t)}{Q(t)h_3'(t)} \Rightarrow b_3\left(\frac{t}{3}\right) = \frac{9}{64t^4}, b_3(t) = \frac{32}{45t^4},$$

$$\int_1^{\infty} b_3(t)dt = \frac{32}{45}\int_1^{\infty} t^{-4}dt = \frac{32}{1355} < \infty,$$

$$\int_a^{\infty} \frac{|g(t)|^2}{Q(t)}dt = \frac{5}{6}\int_1^{\infty} \frac{1}{\sqrt[3]{t^4}}dt = \frac{5}{2} < \infty,$$

$$\int_a^{\infty} \frac{k(t)}{\sqrt{q_0(t)r(t)}}dt = \int_1^{\infty} \frac{1}{\sqrt{t^3}}dt = 2 < \infty.$$

In view of the above estimates, it follows that all the assumptions of Theorem 2.3 hold. Hence, we conclude that all solutions of Eq. (8) satisfies $|x(t)| = O(1)$, $|x'(t)| = O(\sqrt{t^{-3}})$.

Remark 2.5. When we take into account Eq. (1), Eq. (2), the assumptions of Theorem 1.1, Theorem 1.2, Theorem 2.1 and Theorem 2.3, respectively, it can be seen the following:

- (i) The equation discussed in Zhao et al. [20], Eq. (1), includes only a deviating argument. However, our equation, Eq. (2), includes finitely many deviating arguments. This case is an extension and improvement on the topic of the work in [20].
- (ii) For the case $n = i = 1$, the assumptions of Theorem 2.1 and Theorem 2.3 reduce to that of Theorem 1.1 and Theorem 1.2 in [20], respectively.
- (iii) When $n = i = 1$, Theorem 2.3 corrects the result of Theorem 1.2 of Zhao et al. [20]. Because the result of Theorem 1.2 in [20] has to be

$$|x(t)| = O\left(\frac{1}{\sqrt[4]{q_0(t)}}\right), |x'(t)| = O\left(\frac{\sqrt[4]{q_0(t)}}{\sqrt{r(t)}}\right)$$

but not

$$|x(t)| = O(1), |x'(t)| = O\left(\sqrt{\frac{q_0(t)}{r(t)}}\right)$$

(see Zhao et al. [120]).

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