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Generalized topological function spaces and a classification of generalized computer topological spaces

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Abstract. We introduce several kinds of generalized continuities and homeomorphisms in computer topology and investigate some properties of function spaces of these generalized continuous maps and classify generalized computer topological spaces up to each of these generalized homeomorphisms.

1. Introduction

Since many results in computer topology (or digital topology) are recently shown, the paper starts with a brief review of computer topological researches for continuities, homeomorphisms and digital connectivity. By $\mathbf{N}$ and $\mathbf{Z}$ we denote the sets of all natural numbers and integer numbers, respectively. In digital topology several approaches have been proposed for the study of a set $X \subseteq \mathbf{Z}^n$, as follows.

(1) The digital topological approach was introduced in [24] with k-adjacency relations of $\mathbf{Z}^n, n \in \mathbf{N}$.

(2) The connected order topological space was introduced in [18], which recovers the structure of a topology.

(3) The complex cell approach was developed in [20], by which an object is recognized as a structure consisting of different dimensional cells. This approach can recover the structure of a topology.

(4) The Alexandroff topological approach was established in [1], by which gives a link between a T_0-Alexandroff topology and a partially ordered set.

This paper follows the Khalimsky product topology for the study of a subspace $(X, T^n_X) \subseteq (\mathbf{Z}^n, T^n)$ induced by the Khalimsky n-space $(\mathbf{Z}^n, T^n)$. This approach provides a sound mathematical basis for digital geometry such as image thinning, border tracking, contour filling, and object counting in which topological problems issue. For instance, in [4], the θ-generalized homeomorphism was studied and further, in [21], an extension problem of a Khalimsky continuous map $f : A \to \mathbf{Z}$ for $A \subseteq \mathbf{Z}^n$ was treated under Khalimsky topology. Meanwhile, the Khalimsky continuity has some limitations [16] related to both a preservation of digital connectivity and a translation of digital objects. Consider a set $X \subseteq \mathbf{Z}^n$ with one of the k-adjacency

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relation of $\mathbf{Z}^n$, denoted by (X, k). Studying a map $f : (X, k_0) \to (Y, k_1)$, we may consider that for two points x_0 and x_1 are k_0-adjacent their images by f are required to be k_1-adjacent or equal to each other [2]. Thus, the notion of digital (k_0, k_1)-continuity was introduced [2, 6].

Meanwhile, we observe that continuity of maps between two Khalimsky topological spaces $(X, k_0, T_X^{n_0})$ with a k_0-adjacency and $(Y, k_1, T_Y^{n_1})$ with a k_1-adjacency, denoted by $f : (X, k_0, T_X^{n_0}) \to (Y, k_1, T_Y^{n_1})$, need not satisfy the requirement of preserving the k_0-connectivity of $(X, k_0, T_X^{n_0})$ into the k_1-one of $(Y, k_1, T_Y^{n_1})$ [12, 16] (see also Remark 3.5). This is a reason why we use the (k_0, k_1)-continuity of Definition 3.3. To be specific, if a space $(X, T_X^{n_0})$ is not connected in Khalimsky topology, then a Khalimsky continuous map $f : (X, k_0, T_X^{n_0}) \to (Y, k_1, T_Y^{n_1})$ cannot preserve the k_0-connectivity into the k_1-one of $(Y, k_1, T_Y^{n_1})$ (see Figure 1). In relation to the study a set $X \subseteq \mathbf{Z}^n$ from the viewpoint of digital topology, the preservation of the k_0-connectivity into the k_1-connectivity should be considered. If not, the map cannot preserve lots of information of $(X, k_0, T_X^{n_0})$ into $(Y, k_1, T_Y^{n_1})$. Thus, studying (X, k, T_X^n), we strongly need *KD*-(k_0, k_1)-continuity [12, 15, 16]. This is one of the reasons why we study computer topological space by using several kinds of generalized continuities(homeomorphisms) including *KD*-(k_0, k_1)-continuity(homeomorphism). In [12] several kinds of continuities and homeomorphisms in computer topology such as *KD*-(k_0, k_1)-, (k_0, k_1)-and *K*-(k_0, k_1)-continuities (or homeomorphisms) were introduced and compared with each other. In [16] a generalized *KD*-(k_0, k_1)-continuity was established and computer topological function space consisting of generalized *KD*-(k_0, k_1)-continuous maps was studied. In [23], the notion of δ-continuity was introduced. Finally, in [5, 22, 25] various continuities including super continuity were established. Motivated from the above-mentioned continuities, we can study various continuities in computer topology.

This paper is organized as follows. Section 2 gives basic notions which underpin our work. Section 3 studies the notion of *GKD*-(k_0, k_1)-continuity and its properties. Section 4 shows some topologies on the set of all *GKD*-(k_0, k_1)-continuous functions. Section 5 studies various properties of *GKD*-(k_0, k_1)-continuity. Section 6 establishes several kinds of generalized homeomorphisms and investigates their various properties from the viewpoint of computer topology and further, classifies generalized computer topological spaces up to each of generalized homeomorphisms. Finally, Section 7 concludes the paper with a summary.

2. Preliminaries

For basic concepts of this section see [9–13, 19, 24]. For $\{a, b\} \subseteq \mathbf{Z}$ with $a \leq b$, $[a, b]_{\mathbf{Z}} = \{n | a \leq n \leq b, n \in \mathbf{Z}\}$ is a *digital interval* considered as a discrete topological subspace of $\mathbf{Z}$ [2] or a subspace $([a, b]_{\mathbf{Z}}, T_{[a,b]_{\mathbf{Z}}})$ of the Khalimsky line topology $(\mathbf{Z}, T)$ depending on the situation.

Let $n \in \mathbf{N}$, $\mathbf{Z}^n = \underbrace{\mathbf{Z} \times \cdots \times \mathbf{Z}}_{n-times}$, and $p = (p_1, \ldots, p_n)$ and $q = (q_1, \ldots, q_n) \in \mathbf{Z}^n$. We say that the points p and q are k-(or $k(m, n)$-) adjacent according to m, where $m \in \mathbf{N}$ with $m \in [1, n]_{\mathbf{Z}}$, if

(1) there are at most m indices i such that $|p_i - q_i| = 1$, and

(2) for all other indices i such that $|p_i - q_i| \neq 1$, we have $p_i = q_i$.

Let the number k be the cardinality of the set of points k-adjacent to given a point of $\mathbf{Z}^n$ corresponding to the positive integer m with $m \in [1, n]_{\mathbf{Z}}$. This operator consisting of these two items (1) and (2) is called $k(m, n)$ (briefly, k_m or k)-adjacency of $\mathbf{Z}^n$. Finally, we obtain the adjacency relations of $\mathbf{Z}^n$ [6] (see also [8, 14, 15]), as follows.

$$k := k(m, n) = \sum_{i=n-m}^{n-1} 2^{n-i} C_i^n, \tag{2.1}$$

where $C_i^n = \frac{n!}{(n-i)!\, i!}$. Hereafter, each space $X \subseteq \mathbf{Z}^n$ is assumed with one of the k-adjacency relations of $\mathbf{Z}^n$ in (2.1) and is denoted by (X, k).

In order to study a set $A \subseteq \mathbf{Z}^n$ with a k-adjacency, we recall the following notions. A *digital picture* is represented as a quadruple $(\mathbf{Z}^n, k, \bar{k}, X)$, where X is a subset of $\mathbf{Z}^n$, k is an adjacency relation for X, and $\bar{k}$ is an adjacency relation for $\mathbf{Z}^n \setminus X$ [24], and $n \in \mathbf{N}$. The pair (X, k) in $(\mathbf{Z}^n, k, \bar{k}, X)$ is called a *(binary) space with a k-adjacency* (or briefly, a space) [19, 20, 24]. In this paper we are concerned with only the k-adjacency of X.

For the basic concepts of this section, see [11, 12, 19, 24]. For $\{a, b\} \subseteq \mathbf{Z}$ with $a \leq b$, $[a, b]_{\mathbf{Z}} = \{a \leq n \leq b | n \in \mathbf{Z}\}$ is considered in $(\mathbf{Z}, 2, 2, [a, b]_{\mathbf{Z}})$ [2]. But in this paper we will not concern with the $\bar{k}$-adjacency of $\mathbf{Z}^n \setminus X$.

We say that a set of lattice points is *k-connected* if it is not a union of two disjoint non-empty sets that are not k-adjacent to each other [19]. Thus a singleton set with a k-adjacency is k-connected. For a space (X, k) in $\mathbf{Z}^n$, two distinct points $x, y \in X$ are called *k-connected* if there is a sequence $(x_0 = x, x_1, \cdots, x_m = y) \subseteq X$ such that x_i and x_{i+1} are k-adjacent, $i \in [0, m-1]_{\mathbf{Z}}, m \in \mathbf{N} \setminus \{1\}$ [19]. Then we call it a *k-path*. The *length* of a k-path is called the number m [19]. A *simple k-curve* is considered as a k-path $(x_0, x_1, \cdots, x_m) \subseteq \mathbf{Z}^n$ such that x_i and x_j are k-adjacent if and only if $j = i \pm 1$ [19]. Furthermore, a *simple closed k-curve* with l elements in $\mathbf{Z}^n$ is a k-path $(w_0, w_1, \cdots, w_{l-1})$ derived from a simple k-curve $(w_0, w_1, \cdots, w_{l-1}, w_l)$ with $w_0 = w_l$, where w_i and w_j are k-adjacent if and only if $j = i + 1 (mod\, l)$ or $i = j + 1 (mod\, l)$ [19]. The *length* of the simple k-path, denoted by $l_k(x_0, x_{l-1})$, is the number l. Let $SC_k^{n,l}$ denote a simple closed k-curve with l elements in $\mathbf{Z}^n$ [9].

Let us now recall basic notions of Khalimsky topology as follows. The *Khalimsky line topology* on $\mathbf{Z}$ is generated by the following subbasis $\{[2n-1, 2n+1]_{\mathbf{Z}} | n \in \mathbf{Z}\}$ [4] and is denoted by $(\mathbf{Z}, T)$. Furthermore, the product topology on $\mathbf{Z}^n$ derived from $(\mathbf{Z}, T)$ is called the *Khalimsky product topology* on $\mathbf{Z}^n$ (or *the Khalimsky n-space*), $n \geq 2$, and is denoted by $(\mathbf{Z}^n, T^n)$ [3, 4]. Indeed, in the Khalimsky line $(\mathbf{Z}, T)$, since the singletons $\{2n | n \in \mathbf{Z}\}$ and $\{2n+1 | n \in \mathbf{Z}\}$ are closed and open, respectively, we can see that the union of any subsets of the closed sets is also closed. Furthermore, for a subset $X \subseteq \mathbf{Z}^n$, we consider the subspace (X, T_X^n) induced from the Khalimsky n-space $(\mathbf{Z}^n, T^n)$. Moreover, the topological research area of the space (X, T_X^n) with some k-adjacency relations is called *computer topology* in this paper.

Let us examine the structure of the Khalimsky n-space. A point $x = (x_1, x_2, \cdots, x_n) \in \mathbf{Z}^n$ is *open* if all coordinates are odd, and *closed* if each of the coordinates is even [4, 14, 21]. These points are called *pure* and the other points in $\mathbf{Z}^n$ is called *mixed*. In all subspaces in $(\mathbf{Z}^n, T^n), n \geq 2$, of Figure 1, black big circle means a pure open point, and the symbols ■ and • mean a pure closed point and a mixed point, respectively.

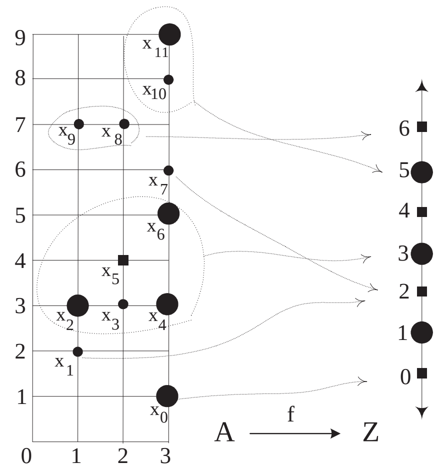

Figure 1: Comparison between Khalimsky continuity and KD-(8, 2)-continuity

3. GKD-(k_0, k_1)-continuity

For a set X in $\mathbf{Z}^n$ with a k-adjacency (X, k), in order to establish the notion of *KD*-(k_0, k_1)-continuity, we now recall the *digital (topological) k-neighborhood* in [6], as follows.

Definition 3.1. ([6], see also [7–10]) Let (X, k) be a space in $\mathbf{Z}^n$ with one of the k-adjacency of $\mathbf{Z}^n$, $x, y \in X$, and $\varepsilon \in \mathbf{N}$. By $N_k(x, \varepsilon)$ we denote the set

$$\{y \in X | l_k(x, y) \leq \varepsilon\} \cup \{x\}, \varepsilon \in \mathbf{N},$$

where $l_k(x, y)$ is the length of a shortest simple k-path x to y in X. Besides, we say that $l_k(x, y) = \infty$ if there is no k-path from x to y.

Thus, if the k-component of x is the singleton $\{x\}$, then we see that $N_k(x, \varepsilon) = \{x\}$ for any $\varepsilon \in \mathbf{N}$. For the Khalimsky n-space $(\mathbf{Z}^n, T^n)$ and a subset $X \subseteq \mathbf{Z}^n$, we obtain the subspace (X, T^n_X) induced by $(\mathbf{Z}^n, T^n)$, where $T^n_X = \{O \cap X | O \in T^n\}$.

Definition 3.2. ([6], see also [12, 16]) We say that a space (X, T^n_X) with a k-adjacency is a *(computer topological) space* and use the notation (X, k, T^n_X)(briefly, $X_{n,k}$).

Let us now recall that $f : X_{n_0,k_0} \to Y_{n_1,k_1}$ is *Khalimsky continuous* at a point $x_0 \in X$ if for any $O_{f(x_0)} \in T^{n_1}_Y$ there is $O_{x_0} \in T^{n_0}_X$ satisfying $f(O_{x_0}) \subseteq O_{f(x_0)}$ as usual.

Definition 3.3. ([12]) For two spaces X_{n_0,k_0} and Y_{n_1,k_1}, we say that a function $f : X \to Y$ is Khalimsky continuous with digital (k_0, k_1)-continuity (briefly, KD-(k_0, k_1)-continuous) at a point $x_0 \in X$ if
(1) f is Khalimsky continuous at the point x_0; and
(2) for any $N_{k_1}(f(x_0), \varepsilon) \subseteq Y$, there is $N_{k_0}(x_0, \delta) \subseteq X$ such that $f(N_{k_0}(x_0, \delta)) \subseteq N_{k_1}(f(x_0), \varepsilon)$, where $\varepsilon, \delta \in \mathbf{N}$.
Furthermore, we say that a map $f : X \to Y$ is *KD*-(k_0, k_1)-continuous if the map f is *KD*-(k_0, k_1)-continuous at any point $x \in X$.

Remark 3.4. The current condition (2) of Definition 3.3 is the digital (k_0, k_1)-continuity in [6] (see also [7–10]) which is exactly focused on the preservation of the k_0-connectivity of (X, k_0) into the k_1-connectivity of (Y, k_1). Thus in Definition 3.3 we may write $\delta = 1 = \varepsilon$ (see [13]).

Indeed, owing to the condition (1) of Definition 3.3 (see Remark 3.5), we observe that there is a big difference between the digital (k_0, k_1)-continuity in [6–10] and the current KD-(k_0, k_1)-continuity.

Remark 3.5. ([12], see also [16]) Neither of the conditions (1) and (2) of Definition 3.3 implies the other. To be specific, consider a space $(A, 8, T^2_A)$ in Figure 1 and the map $f : A \to \mathbf{Z}$ with the mapping in Figure 1. Then, while the map f is a Khalimsky continuous map, f cannot satisfy the condition (2) of Definition 3.3 at the points x_7 and x_8, which means that the condition (1) of Definition 3.3 does not imply the condition (2) of Definition 3.3.

Meanwhile, in general, since the digital k-neighborhood $N_k(x, \varepsilon)$ need not be a Khalimsky topological neighborhood in $X_{n,k}$, the condition (2) may not imply the condition (1) of Definition 3.3.

In terms of the condition (2) of Definition 3.3 with $\delta = 1 = \varepsilon$, the paper [2] (see also [10, 11]) define the notion of (k_0, k_1)-isomorphism. Precisely, a discrete topological space with a k-adjacency (X, k) can be recognized to be a digital k-graph G_k [10]. To be specific, the vertex set of G_k can be considered as the set of points of X. Besides, two points $x_1, x_2 \in X$ determine a k-edge of G_k if and only if x_1 and x_2 are k-adjacent in X [10]. Indeed, a space (X, k) can be considered as a simplicial complex via a geometric realization of (X, k) [10]. Thus digital graph versions of (k_0, k_1)-continuity and (k_0, k_1)-homeomorphism in [2] were established [10].

Definition 3.6. ([2], see also [10, 13]) For two digital spaces (X, k_0) in $\mathbf{Z}^{n_0}$ and (Y, k_1) in $\mathbf{Z}^{n_1}$, a map $h : X \to Y$ is called a (k_0, k_1)-*isomorphism* if h is a digitally (k_0, k_1)-continuous bijection and further, $h^{-1} : Y \to X$ is digitally (k_1, k_0)-continuous. Then we use the notation $X \approx_{(k_0,k_1)} Y$. If $n_0 = n_1$ and $k_0 = k_1$, then we call it a k_0-*isomorphism* and use the notation $X \approx_{k_0} Y$.

Obviously, we have the computer topological category consisting of a collection $Ob(C)$ of computer topological spaces with k-adjacency relations of $\mathbf{Z}^n$ and a class $Mor(X, Y)$ of *KD*-(k_0, k_1)-continuous maps for each pair X_{n_0,k_0} and Y_{n_1,k_1} in $Ob(C)$, denoted by *KDTC* [12].

Motivated by the *cartesian product adjacency* of [17], we obtain an *N-compatible* adjacency of a digital product [16]. This can strongly contribute to the study a digital function space. In order to study related to some product properties of a function space in Sections 4, 5, and 6, we take the following notion which is a convenient presentation of the N-compatible of a digital product in [16].

Definition 3.7. ([16]) For two digital spaces (X, k_1) in $\mathbf{Z}^{n_1}$ and (Y, k_2) in $\mathbf{Z}^{n_2}$ we say that two distinct points $(x, y), (x', y') \in X \times Y$ are normally compatible (briefly, N-compatible) k-adjacent with the k_i-adjacency, $i \in \{1, 2\}$, if

(1) $(x', y') \in N_k((x, y), 1) \Rightarrow x' \in N_{k_1}(x, 1), y' \in N_{k_2}(y, 1)$, and

(2) the k- (or $k(m, n_1 + n_2)$−) adjacency of $X \times Y$ is determined by some number m with $m \geq \max\{m_1, m_2\}$, where the number m_i is taken from the k_i-(or $k_i(m_i, n_i)$-)adjacency, $i \in \{1, 2\}$.

Theorem 3.8. ([16]) *For $(X_i, k_i), i \in \{1, 2\}$, assume $X_1 \times X_2$ with an N-compatible adjacency. Then, the natural projection map $p_i : X_1 \times X_2 \to X_i$ is a (k, k_i)-continuous map.*

Hereafter, we study that for two computer topological spaces $(X_i)_{n_i,k_i}, i \in \{1, 2\}$, the cartesian product space $(X_1 \times X_2, k, T^{n_1+n_2}_{X_1\times X_2})$ is assumed to have an N-compatible k-adjacency in relation with the k_1-and the k_2-adjacency. Indeed, an N-compatible adjacency of a cartesian product plays an important role in studying a computer topological product space in relation with the study of a computer topological function space in Sections 4 and 5.

In general, on $\mathbf{Z}$ we can consider many topologies. For example, we can consider the Scott topology and the upper topology(for more details, see [5]).

Precisely, we recall the following notion [16].

(1) Let $z \in \mathbf{Z}$ and $\uparrow z = \{x \in \mathbf{Z} | z \leq x\}$. The family consisting of $\mathbf{Z}$ and the set $\{\uparrow z | z \in \mathbf{Z}\}$, as a base, defines a topology on $\mathbf{Z}$, denoted here by τ_{up}.

Furthermore, we can consider the product topology τ^n_{up} on $\mathbf{Z}^n$ derived from the topology τ_{up}.

(2) Let $z \in \mathbf{Z}$ and $\downarrow z = \{x \in \mathbf{Z} | x \leq z\}$. The family consisting of $\mathbf{Z}$ and the set $\{\downarrow z | z \in \mathbf{Z}\}$, as a base, defines a topology on $\mathbf{Z}$, denoted here by τ_{lo}, which is called *lower topology*.

Besides, we can consider the product topology τ^n_{lo} on $\mathbf{Z}^n$ derived from the topology τ_{lo}.

Definition 3.9. ([16]) Let τ be an arbitrary topology on $\mathbf{Z}$ and let τ^n be a product topology on $\mathbf{Z}^n$ induced from $(\mathbf{Z}, \tau)$. For $X \subseteq \mathbf{Z}^n$, consider the subspace (X, τ^n_X) induced from $(\mathbf{Z}^n, \tau^n)$. Furthermore, considering the topological space (X, τ^n_X) with a k-adjacency, we call it *a generalized computer topological space with a k-adjacency* and use the notation (X, k, τ^n_X).

If τ^n is the Khalimsky product topology T^n, then the notion of generalized computer topological space coincides with the notion of computer topological space (X, T^n_X).

Definition 3.10. ([16]) Let $(X, k_0, \tau^{n_1}_X)$ and $(Y, k_1, \tau^{n_2}_Y)$ be two generalized computer topological spaces and $x_0 \in X$. A function $f : X \to Y$ is called generalized (k_0, k_1)-continuous at the point x_0 if

(1) f is topologically continuous at the point x_0 and

(2) for $N_{k_1}(f(x_0), 1) \subseteq Y$, there is $N_{k_0}(x_0, 1) \subseteq X$ such that $f(N_{k_0}(x_0, 1)) \subseteq N_{k_1}(f(x_0), 1)$.

Besides, we say that the map $f : X \to Y$ is *generalized KD-(k_0, k_1)-continuous* (briefly, *GKD-(k_0, k_1)-continuous*) if the map f is generalized KD-(k_0, k_1)-continuous at any point $x \in X$.

If τ is the Khalimsky line topology T, then the notion of *GKD*-(k_0, k_1)-continuity coincides with the notion of *KD*-(k_0, k_1)-continuity.

As an analog of computer topological category we obtain the following theorem.

Theorem 3.11. ([16]) *(1) Let (X, k, τ^n_X) be a generalized computer topological space. Then, the identity map 1_X is GKD-(k, k)-continuous.*

(2) Let $(X, k_0, \tau^{n_1}_X)$, $(Y, k_1, \tau^{n_2}_Y)$, and $(Z, k_2, \tau^{n_3}_Z)$ be three generalized computer topological spaces. If a map $f_1 : X \to Y$ is GKD-(k_0, k_1)-continuous and a map $f_2 : Y \to Z$ is GKD-(k_1, k_2)-continuous, then the composite map $f_2 \circ f_1 : X \to Z$ is GKD-(k_0, k_2)-continuous.

(3) For a GKD-(k_0, k_1)-continuous map $f : (X, k_0, \tau^{n_0}_X) \to (Y, k_1, \tau^{n_1}_Y)$, $f \circ 1_X = 1_Y \circ f = f$, where "$\circ$" stands for the composition. Moreover, for a KD-(k_0, k_1)-continuous map $f : (X, k_0, \tau^{n_0}_X) \to (Y, k_1, \tau^{n_1}_Y)$, a KD-$(k_1, k_2)$-continuous map $g : (Y, k_1, \tau^{n_1}_Y) \to (Z, k_2, \tau^{n_2}_Z)$, and a KD-$(k_2, k_3)$-continuous map $h : (Z, k_2, \tau^{n_2}_Z) \to (W, k_3, \tau^{n_3}_W)$, we see that $h(\circ g \circ f) = (h \circ g) \circ f$.

Consequently, we have the category consisting of a collection $Ob(C)$ of computer topological spaces with a k-adjacency relations of $\mathbf{Z}^n$ and a class $Mor(X,Y)$ of GKD-(k_0,k_1)-continuous maps for each pair $(X,k_0,\tau_X^{n_0})$ and $(Y,k_1,\tau_Y^{n_1})$ in $Ob(C)$.

By Theorem 3.8, we can establish the category of set of all *GKD*-(k_0,k_1)-continuous maps denoted by *GKDTC*.

Theorem 3.12. ([16]) *Let $(X,k_1,\tau_X^{n_1})$, $(Y,k_2,\tau_Y^{n_2})$, and $(Z,k_3,\tau_Z^{n_3})$ be three generalized computer topological spaces. Assume that each of the product spaces $(X\times Z,k,\tau_{X\times Z}^{n_1+n_3})$ and $(Y\times Z,k',\tau_{Y\times Z}^{n_2+n_3})$ is well defined with an N-compatible k-adjacency. If a map $f: X\to Y$ is a GKD-(k_1,k_2)-continuous map, then the product map $f\times id: X\times Z\to Y\times Z$ is GKD-(k,k')-continuous, where $id: Z\to Z$ is the identity map.*

The following simple closed 8-curve on $\mathbf{Z}^2$ in [8–10] and a simple closed 26-curves on $\mathbf{Z}^3$ will be often used later in this paper.

$$\left\{\begin{array}{l} SC_4^{2,8} \approx_4 ((0,0),(0,1),(0,2),(1,2),(2,2),(2,1),(2,0),(1,0)),\\ SC_8^{2,6} \approx_8 ((0,0),(1,1),(1,2),(0,3),(-1,2),(-1,1)),\\ SC_{18}^{3,6} := ((0,0,0),(1,-1,0),(1,-1,1),(2,0,1),(1,1,1),(1,1,0)).\\ SC_8^{2,4} \approx_8 ((0,0),(1,1),(0,2),(-1,1)). \end{array}\right\} \tag{3.1}$$

Remark 3.13. Contrary to the hypothesis of Theorem 3.12, if the existence of an *N*-compatible *k*-adjacency of $X\times Z$ or $Y\times Z$ is not allowed, the assertion of Theorem 3.12 may not be successful with the following example.

Consider the following two maps with several simple closed 4- and 8-curves in (3.1)
$f: SC_4^{2,8} := (a_i)_{i\in[0,7]_{\mathbf{Z}}} \to SC_8^{2,4} := (b_j)_{j\in[0,3]_{\mathbf{Z}}}$ given by $f(a_i)=b_{i(mod\,4)}$ and
$1_{SC_8^{2,6}}: SC_8^{2,6}\to SC_8^{2,6}$ which is the identity map of $SC_8^{2,6} := (c_t)_{t\in[0,5]_{\mathbf{Z}}}$ in (3.1).
Let us now consider the cartesian product map

$$f\times 1_{SC_8^{2,6}} : SC_4^{2,8}\times SC_8^{2,6}\to SC_8^{2,4}\times SC_8^{2,6}$$

given by

$$f\times 1_{SC_8^{2,6}}(a_i,c_t) = (f(a_i),c_t).$$

Then, we observe that $SC_4^{2,8}\times SC_8^{2,6}\subseteq \mathbf{Z}^4$ cannot have an *N*-compatible *k*-adjacency, $k\in\{8,32,64,80\}$. Thus, we observe that the product map $f\times 1_{SC_8^{2,6}}$ cannot be a *KD*-(or *GKD*-)(k,k')-continuous map.

Theorem 3.14. ([16]) *Let $(X,k_1,\tau_X^{n_1})$, $(Y,k_2,\tau_Y^{n_2})$, and $(Z,k_3,\tau_Z^{n_3})$ be generalized computer topological spaces such that the product space $(X\times Y,k,\tau_{X\times Y}^{n_1+n_3})$ is well defined with an N-compatible k-adjacency. If a map $F: X\times Y\to Z$ is GKD-(k,k_3)-continuous and $x\in X$, then the map $F_x: Y\to Z$ for which $F_x(y)=F(x,y)$ for every $y\in Y$ is GKD-(k_2,k_3)-continuous.*

4. Topologies on the set of all GKD-(k_0,k_1)-continuous functions

We recall that a topological space is said to be a $T_{\frac{1}{2}}$-space if each singleton is either open or closed [3]. Indeed, $(\mathbf{Z},T)$ is obviously a $T_{\frac{1}{2}}$-space and the Khalimsky *n*-space $(\mathbf{Z}^n,T^n)$ is not a $T_{\frac{1}{2}}$-space but a T_0-space if $n\geq 2$ [3].

Notation 1.([16]) Let $(Y,k_1,\tau_Y^{n_1})$ and $(Z,k_2,\tau_Z^{n_2})$ be two generalized computer topological spaces. By $C(Y,Z)$ we denote the set of all GKD-(k_1,k_2)-continuous functions from Y into Z. Let $f\in C(Y,Z)$. We can suppose that f is the set of all pairs $((y_1,\dots,y_{n_1}),(z_1,\dots,z_{n_2}))$ such that $f(y_1,\dots,y_{n_1})=(z_1,\dots,z_{n_2})$, $(y_1,\dots,y_{n_1})\in Y$, and $(z_1,\dots,z_{n_2})\in Z$.

We put

$$((y_1, \dots, y_{n_1}), (z_1, \dots, z_{n_2})) \equiv (y_1, \dots, y_{n_1}, z_1, \dots, z_{n_2}).$$

Then, we can consider every map $f \in C(Y, Z)$ as a subset of $\mathbf{Z}^{n_1+n_2}$.

In what follows by

$$C\langle Y, Z\rangle$$

we denote the set

$$\cup\{f : f \in C(Y, Z)\}.$$

Clearly, $C\langle Y, Z\rangle \subseteq \mathbf{Z}^{n_1+n_2}$.

Besides, we consider the space $(C\langle Y, Z\rangle, k)$, where $k \in \{3^{n_1+n_2} - 1(n_1 + n_2 \geq 2), 3^{n_1+n_2} - \sum_{t=0}^{r-2} C_t^{n_1+n_2} 2^{n_1+n_2-t} - 1(2 \leq r \leq n_1 + n_2 - 1, n_1 + n_2 \geq 3), 2(n_1 + n_2)(n_1 + n_2 \geq 1)\}$ (see (2.1).

Notation 2. Let $(Y, k_1, \tau_Y^{n_1})$ and $(Z, k_1, \tau_Z^{n_2})$ be generalized computer topological spaces.

(1) If $\bar{y} \equiv (y_1, \dots, y_{n_1}) \in Y$ and $f \in C(Y, Z)$, then by $f(\bar{y})$ we denote the point $\bar{z} = (z_1, \dots, z_{n_2}) \in Z$ such that $(\bar{y}, \bar{z}) \in f$.

(2) If $f \in C\langle Y, Z\rangle$, $K \subseteq Y$, and $L \subseteq Z$, then by $f(K)$ we denote the set of all points $(z_1, \dots, z_{n_2}) \in Z$ such that there is $(y_1, \dots, y_{n_1}) \in K$ with $(y_1, \dots, y_{n_1}, z_1, \dots, z_{n_2}) \in f$. By $f^{-1}(L)$ we denote the set

$$\cup\{(y_1, \dots, y_{n_1}) \in Y | f(y_1, \dots, y_{n_1}) \in L\}.$$

A topology t on $(C\langle Y, Z\rangle, k_0)$ is called $\mathcal{A}$-*splitting* if for every $(X, k, \tau_X^n) \in \mathcal{A}$, the *GKD*-$(k_3, k_2)$-continuity of the map $F : X \times Y \to Z$ implies the *GKD*-(k, k_0)-continuity of the map

$$\hat{F} : (X, k, \tau_X^n) \to (C\langle Y, Z\rangle, k_0, t).$$

A topology t on $(C\langle Y, Z\rangle, k_0)$ is called $\mathcal{A}$-*admissible* if for every space $(X, k, \tau_X^n) \in \mathcal{A}$, the *GKD*-$(k, k_0)$-continuity of the map

$$G : (X, k, \tau_X^n) \to (C\langle Y, Z\rangle, k_0, t)$$

implies the *GKD*-(k_3, k_2)-continuity of the map

$$\tilde{G} : (X \times Y, k_3, \tau_{X\times Y}^{n+n_1}) \to (Z, k_2, \tau_Z^{n_2}).$$

Definition 4.1. The point-open topology on $(C\langle Y, Z\rangle, k_0)$, denoted here by t_{po}, is the topology for which the family of all set of the form

$$[\bar{y}, U] = \{f \in C\langle Y, Z\rangle | f(\bar{y}) \in U\}$$

composes a subbase, where $\bar{y} \in Y$ and $U \in \tau_Z^{n_2}$.

Definition 4.2. The set-open topology on $(C\langle Y, Z\rangle, k_0)$, denoted here by t_{so}, is the topology for which the family of all sets of the form

$$[K, U] = \{f \in C\langle Y, Z\rangle\} | f(K) \subseteq U\}$$

compose a subbase, where K is a subset of Y and $U \in \tau_Z^{n_2}$.

Definition 4.3. A generalized computer topological space $(Z, k_2, \tau_Z^{n_2})$ is called a T_i-space, $i \in \{0, 1, 2, 3\}$, if $(Z, \tau_Z^{n_2})$ is a T_i-space, where T_i is the separation axiom as usual.

Theorem 4.4. *If a generalized computer topological space $(Z, k_2, \tau_Z^{n_2})$ is a T_i-space, $i \in \{0, 1, 2\}$, then the computer topological function space $(C\langle Y, Z\rangle, k_0, t_{po}))$ is a T_i-space.*

Proof: We assume that a generalized computer topological space $(Z, k_2, \tau_Z^{n_2})$ is a T_0-space and prove that the computer topological function space $(C\langle Y, Z\rangle, k_0, t_{po}))$ is a T_0-space Indeed, let $f, g \in C\langle Y, Z\rangle$ and

$$(y_1, \cdots, y_{n_1}, z_1, \cdots, z_{n_2}) \in f \text{ and } (y_1, \cdots, y_{n_1}, z'_1, \cdots, z'_{n_2}) \in g.$$

Since the space Z is a T_0-space, there exists an open set $U \in \tau_Z^{n_2}$ such that $f(y_1, \cdots, y_n) = (z_1, \cdots, z_{n_2}) \in U$ and $g(y_1, \cdots, y_n) = (z'_1, \cdots, z'_{n_2}) \notin U$. □

We consider the subbasic open set $[\bar{y}, U]$. Clearly $f \in [\bar{y}, U]$ and $g \notin [\bar{y}, U]$. Thus, the generalized computer space $(C\langle Y, Z\rangle, k_0, t_{po})$ is also a T_0-space.

Similarly, if the space $(Z, k_2, \tau_Z^{n_2})$ is a T_i-space, $i \in \{1, 2\}$, then the computer topological function space $(C\langle Y, Z\rangle, k_0, t_{po})$ is also a T_i-space, respectively.

Notation 3. For $(C\langle Y, Z\rangle, k_0)$, by $t_{trivial}$ and t_{dis} we denote the trivial topology and the discrete topology $\{\phi, C\langle Y, Z\rangle\}$ and $\mathcal{P}(C\langle Y, Z\rangle)$, respectively, where $\mathcal{P}(X)$ means the power set of X.

Theorem 4.5. *For the topologies t_{tr}, t_{po}, and t_{dis}, we have*

$$t_{tr} \subseteq t_{po} \subseteq t_{dis}.$$

Proof: The proof of this theorem is clear. □

Example 4.6. Assume that $Y = \{1\} \subseteq \mathbf{Z}$ and $Z = \{1, 3\} \subseteq \mathbf{Z}$. We consider the spaces $(Y, 2)$ and $(Z, 2)$ with the Khalimsky topologies T_Y and T_Z, respectively. In this case, we observe $C(Y, Z) = \{f = \{(1, 1)\}, g = \{(1, 3)\}\}$. Clearly, $C\langle Y, Z\rangle = \{(1, 1), (1, 3)\} \subseteq \mathbf{Z}^2$. Thus, we can consider the computer topological spaces

$$(C\langle Y, Z\rangle, 4, T^2_{C\langle Y,Z\rangle})$$

and

$$(C\langle Y, Z\rangle, 8, T^2_{C\langle Y,Z\rangle}).$$

Let $(X, T_X) \subseteq (\mathbf{Z}, T)$ be a computer topological space such that the product $X \times Y$ is well defined.

Obviously, there is a *GKD*-$(4, 2)$-continuous map

$$F : (X \times Y, 4, T^2_{X\times Y}) \to (Z, 2, T_Z) := Z_{1,2}$$

which does not imply the GKD-$(2, k)$-continuity of the map $\hat{F} : (X, 2, T_X) := X_{1,2} \to (C\langle Y, Z\rangle, k, T^2_{C\langle Y,Z\rangle}), k \in \{4, 8\}$.

Questions. 1. Is the trivial topology $t_{trivial}$ on $(C\langle Y, Z\rangle, k_0)$ $\mathcal{A}$-splitting?

2. Is the point-open topology t_{po} on $(C\langle Y, Z\rangle, k_0)$ $\mathcal{A}$-splitting?

3. Is the discrete topology t_{dis} on $(C\langle Y, Z\rangle, k_0)$ $\mathcal{A}$-admissible?

5. Strong and weak forms of GKD-(k_0, k_1)-continuity

Let X be a set and let τ be a topology on X, and $A \subseteq X$. In what follows by $Cl(A)$ (respectively, $Int(A)$) we denote the *closure* (respectively, the *interior*) of A in the topological space (X, τ). Besides, the subset A of X called *regularly open*(respectively *regularly closed*) if $A = Int(Cl(A))$(respectively $A = (Cl(Int(A)))$.

Definition 5.1. Let $(X, k_0, \tau_X^{n_1})$ and $(Y, k_1, \tau_Y^{n_2})$ be two generalized computer topological spaces and $x_0 \in X$. A function $f : X \to Y$ is called *almost GKD-(k_0, k_1)-continuous* (briefly, *AGKD*-(k_0, k_1)-continuous) at the point x_0 if

(1) f is topologically almost continuous at the point x_0 (i.e. for every open neighborhood U of $f(x_0)$ in Y, there exists an open neighborhood V of x_0 in X such that $f(V) \subseteq Int(Cl(U))$ [25]) and

(2) $f(N_{k_0}(x_0, 1)) \subseteq N_{k_1}(f(x_0), 1)$.

We say that a map $f : X \to Y$ is *AGKD-(k_0, k_1)-continuous* if the map f is *AGKD*-(k_0, k_1)-continuous at any point $x \in X$.

Theorem 5.2. *GKD-(k_0, k_1)-continuity implies AGKD-(k_0, k_1)-continuity.*

Proof: The proof of this theorem follows by the fact that every topologically continuous map is topologically almost continuous. □

The converse of Theorem 5.2 may not be true, as the following example shows.

Example 5.3. Let $(X = \{0,1,2,3\}, 2, \tau_X)$ and $(Y = \{0,1,2,3,4\}, 2, \tau'_Y)$ be two generalized computer topological spaces, where

$$\tau = \{\emptyset, \mathbf{Z}, \{0,1,3\}, \{0,1\}\},\ \tau_X = \{\emptyset, \{0,1,3\}, \{0,1\}, X\}\},\ \text{and}$$

$$\tau' = \{\emptyset, \mathbf{Z}, \{0,1,2\}, \{0\}\}, \tau'_Y = \{\emptyset, \{0,1,2\}, \{0\}, Y\}\}.$$

We consider the map $f : (X, 2, \tau_X) \to (Y, 2, \tau'_Y)$ for which $f(x) = x$, for every $x \in X$. The map f is not *GKD*-$(2,2)$-continuous at the point $x = 0$ but it is *AGKD*-$(2,2)$-continuous at this point.

Let us now explain Example 5.3 as follows: For the open neighborhood $\{0\}$ of $f(0) = 0$, there is not an open neighborhood $V \in \tau_X$ of the point 0 such that $f(V) \subseteq \{0\}$.
Meanwhile, the map f is *AGKD*-$(2,2)$-continuous at the point 0. Indeed, for the smallest open neighborhood $\{0\}$ of $f(0) = 0$ in Y, there exists the open neighborhood $\{0,1\}$ of $x = 0$ in X(see Figure 2) such that

$$f(\{0,1\}) = \{0,1\} \subseteq \quad Int(Cl(\{0\})) = \{0,1,2,3\}.$$

We also have $f(N_2(0,1)) \subseteq N_2(f(0),1)$.

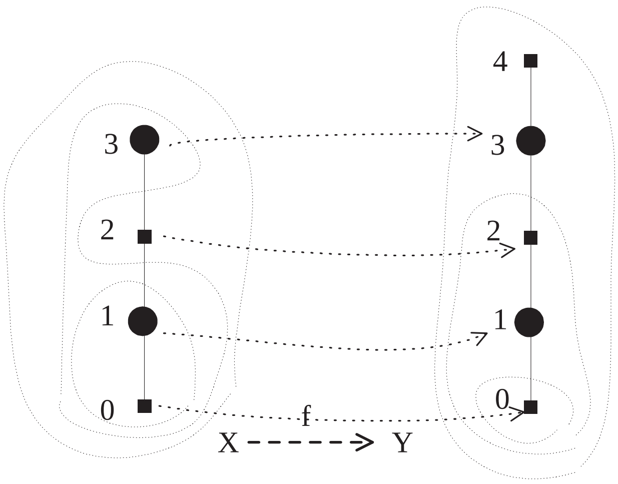

Figure 2: AGKD $(2,2)$-continuity of f

Theorem 5.4. *Let $(X, k_0, \tau_X^{n_1})$ and $(Y, k_1, \tau_Y^{n_2})$ be two generalized computer topological spaces and $f : X \to Y$. The following statements are equivalent:*

(1) f is almost GKD-(k_0, k_1)-continuous.
(2) a) Inverse image of every regularly-open set of Y is an open subset of X, and
b) For any point $x \in X$, $f(N_{k_0}(x,1)) \subseteq N_{k_1}(f(x),1)$.
(3) a) Inverse image of every regularly-closed set of Y is closed subset of X, and
b) For any point $x \in X$, $f(N_{k_0}(x,1)) \subseteq N_{k_1}(f(x),1)$.
(4) a) $f^{-1}(A) \subseteq Int(f^{-1}(Int(Cl(A)))$ for every open subset A of Y, and
b) For any point $x \in X$, $f(N_{k_0}(x,1)) \subseteq N_{k_1}(f(x),1)$.
(5) a) $Cl(f^{-1}(Int(Cl(B))) \subseteq f^{-1}(B)$ for every closed subset B of Y, and
b) For any point $x \in X$, $f(N_{k_0}(x,1)) \subseteq N_{k_1}(f(x),1)$.

Proof: The proof of this theorem follows by the Definition 5.1 and Theorem 2.2 of [25].□

Definition 5.5. Let $(X, k_0, \tau_X^{n_1})$ and $(Y, k_1, \tau_Y^{n_2})$ be two generalized computer topological spaces and $x_0 \in X$. A function $f : X \to Y$ is called *super GKD-(k_0, k_1)-continuous* (briefly, *SGKD*-(k_0, k_1)-continuous) at the point x_0 if

(1) f is topologically super continuous at the point x_0 (i.e. for every open neighborhood U of $f(x_0)$ in Y there exists an open neighborhood V of x_0 in X such that $f(Int(Cl(V))) \subseteq U$ [22]

and

(2) $f(N_{k_0}(x_0, 1)) \subseteq N_{k_1}(f(x_0), 1)$.

Besides, we say that a map $f : X \to Y$ is *SGKD-(k_0, k_1)-continuous* if the map f is *SGKD*-(k_0, k_1)-continuous at any point $x \in X$.

Theorem 5.6. *SGKD-(k_0, k_1)-continuity implies GKD-(k_0, k_1)-continuity and, therefore, AGKD-(k_0, k_1)-continuity.*

Proof: The proof of this theorem follows by the fact that every topologically super continuous map is topologically continuous and, therefore, topologically almost continuous.□

The converse of Theorem 5.6 may not be true, as the following example shows.

Example 5.7. Let $(X = \{0, 1, 2, 3\}, 2, \tau_X)$ and and $(Y = \{0, 1, 2, 3\}, 2, \tau'_Y)$ be generalized computer topological spaces, where

$$\tau = \{\emptyset, \mathbf{Z}, \{0, 1, 3\}, \{0, 1\}\}, \tau_X = \{\emptyset, \{0, 1, 3\}, \{0, 1\}, \{0, 1, 2, 3\}\}, \text{and}$$

$$\tau' = \{\emptyset, \mathbf{Z}, \{1, 3\}, \{1\}\}, \tau'_Y = \{\emptyset, \{1, 3\}, \{1\}, \{0, 1, 2, 3\}\}.$$

We consider the map $f : (X, 2, \tau_X) \to (Y, 2, \tau'_Y)$ for which $f(0) = f(1) = 1$, $f(2) = 2$, and $f(3) = 3$. The map f is not *SGKD*-$(2, 2)$-continuous at the point $x = 1$ but it is *GKD*-$(2, 2)$-continuous at the point $x = 1$.

Theorem 5.8. *Let $(X, k_0, \tau_X^{n_1})$ and $(Y, k_1, \tau_Y^{n_2})$ be two generalized computer topological spaces and $f : X \to Y$. The following statements are equivalent:*

(1) f is SGKD-(k_0, k_1)-continuous.

(2) a) Inverse image of every open set of Y is a δ-open subset of X. (A subset U of X is called δ-open if for each $x \in U$ there exists a regularly open set H such that $x \in H \subseteq U$).

b) For any point $x \in X$, $f(N_{k_0}(x, 1)) \subseteq N_{k_1}(f(x), 1)$.

(3) a) For each point $x \in X$ and each open neighborhood U of $f(x)$, there is a δ-open neighborhood V of x such that $f(V) \subseteq U$.

b) For any point $x \in X$, $f(N_{k_0}(x, 1)) \subseteq N_{k_1}(f(x), 1)$.

Proof: The proof of this theorem follows by the Definition 5.5 and Theorem 2.1 of [22].□

Remark 5.9. By the similar method as the above we can give many different forms of continuity. For example, we give the notions of generalized (k_0, k_1)-θ-continuity, *WGKD*-(k_0, k_1)-continuity, and *GKD*-(k_0, k_1)-δ-continuity (see below).

Definition 5.10. Let $(X, k_0, \tau_X^{n_1})$ and $(Y, k_1, \tau_Y^{n_2})$ be two generalized computer topological spaces and $x_0 \in X$. A function $f : X \to Y$ is called *GKD-(k_0, k_1)-θ-continuous* (briefly, *GKD*-(k_0, k_1)-θ-continuous) at the point x_0 if

(1) f is topologically θ-continuous at the point x_0 (i.e. for every open neighborhood U of $f(x_0)$ in Y there exists an open neighborhood V of x_0 in X such that $f(Cl(V)) \subseteq Cl(U)$) and

(2) $f(N_{k_0}(x_0, 1)) \subseteq N_{k_1}(f(x_0), 1)$.

Besides, we say that a map $f : X \to Y$ is *generalized (k_0, k_1)-θ-continuous* (briefly, *GKD*-(k_0, k_1)-θ-continuous) if the map f is *GKD*-(k_0, k_1)-θ-continuous at any point $x \in X$.

Definition 5.11. Let $(X, k_0, \tau_X^{n_1})$ and $(Y, k_1, \tau_Y^{n_2})$ be two generalized computer topological spaces and $x_0 \in X$. A function $f : X \to Y$ is called *weakly GKD-(k_0, k_1)-continuous* (briefly, *WGKD*-(k_0, k_1)-continuous) at the point x_0 if

(1) f is Khalimsky topologically continuous at the point x_0 (i.e. for every open neighborhood U of $f(x_0)$ in Y there exists an open neighborhood V of x_0 in X such that $f(V) \subseteq Cl(U)$ [23]) and

(2) $f(N_{k_0}(x_0, 1)) \subseteq N_{k_1}(f(x_0), 1)$.

We say that a map $f : X \to Y$ is *WGKD*-(k_0, k_1)*-continuous* if the map f is *WGKD*-(k_0, k_1)-continuous at any point $x \in X$.

Motivated by the notion of δ-continuity of [23], we obtain the following notion.

Definition 5.12. Let $(X, k_0, \tau_X^{n_1})$ and $(Y, k_1, \tau_Y^{n_2})$ be two generalized computer topological spaces and $x_0 \in X$. A function $f : X \to Y$ is called generalized *KD*-(k_0, k_1)-δ*-continuous* (briefly, *GKD*-(k_0, k_1)-δ-continuous) at the point x_0 if

(1) f is topologically δ-continuous at the point x_0 (i.e. for every open neighborhood U of $f(x_0)$ in Y there exists an open neighborhood V of x_0 in X such that $f(Int(Cl(V))) \subseteq Int(Cl(U))$ and

(2) $f(N_{k_0}(x_0, 1)) \subseteq N_{k_1}(f(x_0), 1)$.

We say that a map $f : X \to Y$ is *GKD*-(k_0, k_1)-δ*-continuous* if the map f is *GKD*-(k_0, k_1)-δ-continuous at any point $x \in X$.

Then we obviously obtain the following theorem.

Theorem 5.13. *(1) AGKD-(k_0, k_1)-continuity implies GKD-(k_0, k_1)-θ-continuity.*

(2) SGKD-(k_0, k_1)-continuity implies GKD-(k_0, k_1)-δ-continuity.

(3) GKD-(k_0, k_1)-δ-continuity implies AGKD-(k_0, k_1)-continuity.

(4) GKD-(k_0, k_1)-θ-continuity implies WGKD-(k_0, k_1)-continuity.

The converse of Theorem 5.13 need not be true, as the following examples show.

Example 5.14. Let $X = \{(1, 1), (1, 2), (1, 3), (2, 2), (2, 3), (3, 3), (3, 2), (3, 1)\}$ and $Y = \{0, 1, 2\}$. Let us now consider the map $f : (X, 4, T_X^2) \to (Y, 2, T_Y)$ for which

$$f(\{(1, 1), (1, 2)\}) = \{0\}, f(\{(1, 3), (2, 3), (3, 3), (2, 2)\}) = \{1\},$$

and

$$f(\{(3, 2), (3, 1)\}) = \{2\}.$$

Then, while the map f is not *GKD*-$(4, 2)$-continuous at the point $(2, 2)$, it is *GKD*-$(4, 2)$-θ-continuous.

Example 5.15. Let $(X = \{0, 1, 2, 3\}, 2, \tau_X)$ and $(Y = \{0, 1, 2, 3\}, 2, \tau'_Y)$ be two generalized computer topological spaces, where

$$\tau = \{\emptyset, \mathbf{Z}, \{0, 1, 2\}\},\ \tau_X = \{\emptyset, \{0, 1, 2\}, \{0, 1, 2, 3\}\}, \text{ and}$$

$$\tau' = \{\emptyset, \mathbf{Z}, \{0, 1\}, \{0, 1, 2\}\},\ \tau'_Y = \{\emptyset, \{0, 1\}, \{0, 1, 2\}, \{0, 1, 2, 3\}\}.$$

We consider the map $f : (X = \{0, 1, 2, 3\}, 2, \tau_X) \to (Y = \{0, 1, 2, 3\}, 2, \tau'_Y)$ for which $f(x) = x$ for every $x \in X$. Clearly, the map f is *GKD*-$(2, 2)$-δ-continuous at the point 0 but it is not *GKD*-$(2, 2)$-continuous and it is not *SGKD*-$(2, 2)$-continuous at this point either.

Example 5.16. Let $(X = \{0, 1, 2, 3\}, 2, \tau_X)$ and $(Y = \{0, 1, 2, 3\}, 2, \tau'_Y)$ be two generalized computer topological spaces, where

$$\tau = \{\emptyset, \mathbf{Z}, \{0, 1\}, \{0, 1, 2\}\},\ \tau_X = \{\emptyset, \{0, 1\}, \{0, 1, 2\}, \{0, 1, 2, 3\}\}, \text{ and}$$

$$\tau' = \{\emptyset, \mathbf{Z}, \{0\}, \{0, 1\}, \{2, 3\}, \{0, 1, 2, 3\}\},\ \tau'_Y = \{\emptyset, \{0\}, \{0, 1\}, \{2, 3\}, \{0, 1, 2, 3\}\},$$

We consider the map

$$f : (X = \{0, 1, 2, 3\}, 2, \tau_X) \to (Y = \{0, 1, 2, 3\}, 2, \tau'_Y)$$

for which $f(x) = x$ for every $x \in X$. Clearly, the map f is *AGKD*-$(2, 2)$-continuous at the point 0 but it is not *GKD*-$(2, 2)$-δ-continuous at this point.

Example 5.17. Let $(X = \{0, 1, 2, 3\}, 2, \tau_X)$ and $(Y = \{0, 1, 2, 3\}, 2, \tau'_Y)$ be two generalized computer topological spaces, where

$$\tau = \{\emptyset, \mathbf{Z}, \{0\}, \{0, 1\}, \{0, 1, 2\}\},\ \tau_X = \{\emptyset, \{0\}, \{0, 1\}, \{0, 1, 2\}, \{0, 1, 2, 3\}\}, \text{and}$$

$\tau' = \{\emptyset, \mathbf{Z}, \{0\}, \{1\}, \{2\}, \{3\}, \{0, 1\}, \{0, 2\}, \{0, 3\}, \{0, 1, 2\}, \{1, 2, 3\}, \{0, 1, 2, 3\}, \{2, 3\}, \{1, 3\}\}$,
$\tau'_Y = \{\emptyset, \{0\}, \{1\}, \{2\}, \{3\}, \{0, 1\}, \{0, 2\}, \{0, 3\}, \{0, 1, 2\}, \{1, 2, 3\}, \{2, 3\}, \{1, 3\}, \{0, 1, 2, 3\}\}$.
We consider the map

$$f : (X = \{0, 1, 2, 3\}, 2, \tau_X) \to (Y = \{0, 1, 2, 3\}, 2, \tau'_Y)$$

for which $f(x) = x$ for every $x \in X$. Then, the map f is obviously *WGKD*-$(2, 2)$-continuous at the point 0 but it is not *GKD*-$(2, 2)$-θ-continuous at this point.

6. Classification of generalized computer topological spaces up to each of generalized homeomorphisms in computer topology

On the basis of Definitions 5.1, 5.5, 5.10, 5.11, and 5.12, by the same method as the establishment of GKDTC in Theorem 3.11, several kinds of categories motivated by *AGKD*-(k_0, k_1)-, *SGKD*-(k_0, k_1)-, *WGKD*-(k_0, k_1)-, *GKD*-(k_0, k_1)-δ-, *GKD*-(k_0, k_1)-θ-continuous maps are established, which are denoted by *AGKDTC, SGKDTC, WGKDTC, GKD-δ-TC, and GKD-θ-TC*, respectively.

In *AGKDTC, SGKDTC, WGKDTC, GKD-δ-TC*, and *GKD-θ-TC*, their corresponding isomorphisms (or homeomorphisms) can be established in computer topology. For instance, let $(X, k_0, \tau_X^{n_1})$ and $(Y, k_1, \tau_Y^{n_2})$ be two generalized computer topological spaces and $x_0 \in X$. A function $f : X \to Y$ is called an almost *GKD*-(k_0, k_1)-homeomorphism (or almost *GKD*-(k_0, k_1)-isomorphism) (briefly, *AGKD*-(k_0, k_1)-homeomorphism (or *AGKD*-(k_0, k_1)-isomorphism) if

(1) f is a bijection and
(2) f is an *AGKD*-(k_0, k_1)-continuous map and f^{-1} is an *AGKD*-(k_1, k_0)-continuous map.

Remark 6.1. Isomorphisms between objects in the categories introduced are called homeomorphisms (see Figure 3).

By Theorem 5.2, we obtain the following theorem.

Theorem 6.2. *GKD-(k_0, k_1)-homeomorphism implies AGKD-(k_0, k_1)-homeomorphism.*

Similarly, let $(X, k_0, \tau_X^{n_1})$ and $(Y, k_1, \tau_Y^{n_2})$ be two generalized computer topological spaces and $x_0 \in X$. A function $f : X \to Y$ is called a super *GKD*-(k_0, k_1)-homeomorphism (briefly, *SGKD*-(k_0, k_1)-homeomorphism) if

(1) f is a bijection and
(2) f is an *SGKD*-(k_0, k_1)-continuous map and f^{-1} is an SGKD-(k_1, k_0)-continuous map.
In terms of Theorems 5.2, 5.6 and 5.13, we obtain the following theorem.

Theorem 6.3. *An SGKD-(k_0, k_1)-homeomorphism implies a GKD-(k_0, k_1)-homeomorphism and, therefore, an AGKD-(k_0, k_1)-homeomorphism.*

By using the same method of the construction of both an *AGKD*-(k_0, k_1)-homeomorphism and an *SGKD*-(k_0, k_1)-homeomorphism, we can consider the following:
Let $(X, k_0, \tau_X^{n_1})$ and $(Y, k_1, \tau_Y^{n_2})$ be two generalized computer topological spaces and $x_0 \in X$. A function $f : X \to Y$ is called a *GKD*-(k_0, k_1)-θ-homeomorphism (briefly, *GKD*-(k_0, k_1)-θ-homeomorphism) if

(1) f is a bijection and
(2) f is a generalized (k_0, k_1)-θ-continuous map and f^{-1} is a generalized (k_1, k_0)-θ-continuous map.

Besides, let $(X, k_0, \tau_X^{n_1})$ and $(Y, k_1, \tau_Y^{n_2})$ be two generalized computer topological spaces and $x_0 \in X$. A function $f : X \to Y$ is called a weakly *GKD*-(k_0, k_1)-homeomorphism (briefly, *WGKD*-(k_0, k_1)-homeomorphism) if

(1) f is a bijection and

(2) f is a *WGKD*-(k_0, k_1)-continuous map and f^{-1} is a *WGKD*-(k_1, k_0)-θ-continuous map.

Finally, let $(X, k_0, \tau_X^{n_1})$ and $(Y, k_1, \tau_Y^{n_2})$ be two generalized computer topological spaces and $x_0 \in X$. A function $f : X \to Y$ is called a generalized *KD*-(k_0, k_1)-δ-homeomorphism (briefly, *GKD*-(k_0, k_1)-δ-homeomorphism) at the point x_0 if

(1) f is a bijection and

(2) f is a *GKD*-(k_0, k_1)-δ-continuous map and f^{-1} is a a *GKD*-(k_1, k_0)-δ-continuous map.

In terms of Theorem 5.13, we obtain the following theorem (see Figure 3).

Theorem 6.4. *(1) AGKDTC is a subcategory of GKD-θ-TC.*

(2) SGKDTC is a subcategory of GKD-δ-TC.

(3) GKD-δ-TC is a subcategory of AGKDTC.

(4) GKD-θ-TC is a subcategory of WGKDTC.

In terms of Theorems 6.2, 6.3 and 6.4, we can establish a distribution of several generalized homeomorphisms(isomorphisms) in computer topology.

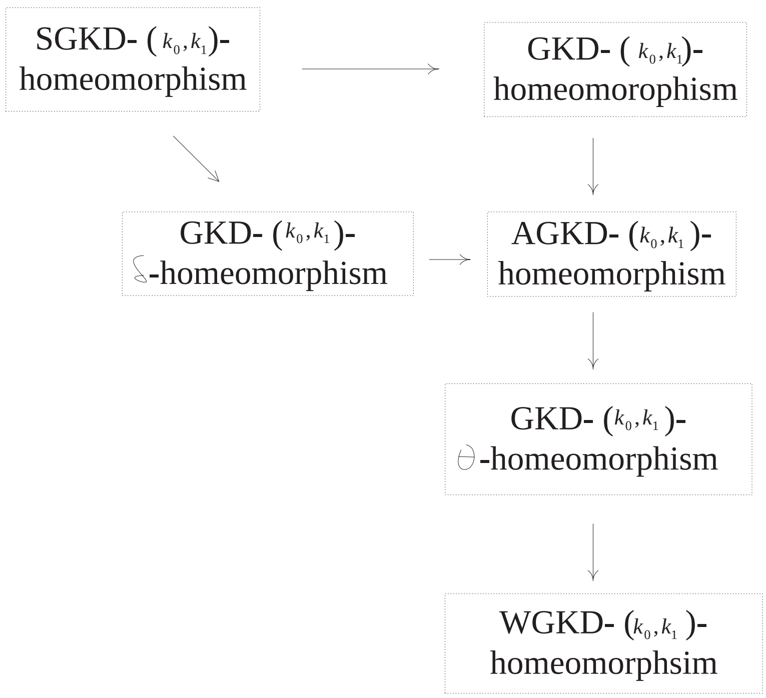

Figure 3: Distribution diagram of generalized isomorphisms (or homeomorphisms) from a computer topological point of view

Remark 6.5. In the categories *AGKDTC, SGKDTC, WGKDTC, GKD-δ-TC, and GKD-θ-TC*, using each of their corresponding homeomorphisms (or isomorphisms), we can classify generalized computer topological spaces.

7. Summary

We have compared among *GKD*-(k_0, k_1)-, *AGKD*-(k_0, k_1)-, *SGKD*-(k_0, k_1)-, *GKD*-(k_0, k_1)-δ-, *GKD*-(k_0, k_1)-θ- and *WGKD*-(k_0, k_1)-continuities (homeomorphisms). Besides, due to the *KD*-(k_0, k_1)-continuity from Definition 3.3, we obtain Theorem 4.5 comparing among trivial topology, point open topology, and discrete topology of given a computer topological function space. In Theorem 4.5 we see that the discrete topology of $C(Y, Z)$ is equivalent to the digital function space $C(Y, Z)$, where Y and Z are digital spaces in the digital topological category of digital spaces and digital (k_0, k_1)-continuous maps. Finally, we have shown the distribution diagram among several generalized homeomorphisms (or isomorphisms) from Theorem 6.3(see Figure 3).

References

[1] P. Alexandroff, Diskrete Räume, Mat. Sbornik 2 (1937) 501–518.
[2] L. Boxer, Digitally continuous functions Pattern Recognition Letters 15 (1994) 833–839.
[3] W. Dunham, $T_{\frac{1}{2}}$ spaces, Kyungpook Mathematical Journal 17 (1977) 161–169.
[4] J. Dontchev, H. Maki, Groups of θ-generalized homeomorphisms and the digital line, Topology and Its Applications 95 (1999) 113–128.
[5] G. Gierz, K.H. Hofmann, K. Keimel, J.D. Lawson, M. Mislove, D.S. Scott, A Compendium of Continuous Lattices, Springer, Berlin-Heidelberg-New York 1980.
[6] S.E. Han, Computer topology and its applications, Honam Math. Jour. 25 (2003) 153–162.
[7] S.E. Han, Comparison between digital continuity and computer continuity, Honam Math. Jour. 26 (2004) 331–339.
[8] S.E. Han, Digital coverings and their applications, Journal of Applied Mathematics and Computing 18 (2005) 487–495.
[9] S.E. Han, Non-product property of the digital fundamental group, Information Sciences 171 (2005) 73–91.
[10] S.E. Han, On the simplicial complex stemmed from a digital graph, Honam Mathematical Journal 27 (2005) 115–129.
[11] S.E. Han, Strong k-deformation retract and its applications, Journal of the Korean Mathematical Society 44 (2007) 1479–1503.
[12] S.E. Han, Continuities and homeomorphisms in computer topology, Journal of the Korean Mathematical Society 45 (2008) 923–952.
[13] S.E. Han, Equivalent (k_0, k_1)-covering and generalized digital lifting, Information Sciences 178 (2008) 550-561.
[14] S.E. Han, The k-homotopic thinning and a torus-like digital image in Z^n, Journal of Mathematical Imaging and Vision 31 (2008) 1–16.
[15] S.E. Han, KD-(k_0, k_1)-homotopy equivalence and its applications Journal of the Korean Mathematical Society 47 (2010) 1031–1054.
[16] S.E. Han, N.D. Georgiou, On computer topological function space, Journal of the Korean Mathematical Society 46 (2009) 841–857.
[17] F. Harary, Graph theory, Addison-Wesley Publishing, Reading, MA, 1969.
[18] E. Khalimsky, R. Kopperman, P.R. Meyer, Computer graphics and connected topologies on finite ordered sets, Topology and Its Applications 36 (1991), 1–17.
[19] T.Y. Kong, A. Rosenfeld, Topological Algorithms for the Digital Image Processing, Elsevier Science, Amsterdam, 1996.
[20] V. Kovalevsky, Finite topology as applied to image analysis, Computer Vision, Graphic, and Image processing 46 (1989) 141–161.
[21] E. Melin, Extension of continuous functions in digital spaces with the Khalimsky topology, Topology and Its Applications 153(1)(2005) 52–65.
[22] B.M. Munshi, D.S. Bassan, Super-continuous mappings, Indian Jour. Pure Appl. Math. 13 (1982) 229–236.
[23] T. Noiri, On δ-continuous functions, Journal of the Korean Mathematical Society 16 (1980) 161–166.
[24] A. Rosenfeld, Arcs and curves in digital pictures, Jour. of the ACM 20 (1973) 81–87.
[25] M.K. Singal, A.R. Singal, Almost continuous mappings, Yokohama Math. J. 16 (1968) 63–73.