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Fuzzy generalized closed sets via fuzzy grills

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Abstract. In this paper we introduce a kind of generalized fuzzy closed sets in terms of fuzzy grills, termed $\mathcal{G}_{gf}$-closed sets, and discuss their basic properties. We also define $\mathcal{G}_{gf}$-continuity and characterize it by $\mathcal{G}_{gf}$-closed sets. We further define $\mathcal{G}_{gf}$-regularity and $\mathcal{G}_{gf}$-normality and study their behaviours by means of $\mathcal{G}_{gf}$-closed sets.

1. Introduction and preliminaries

It is by now well known that the concept of fuzzy sets, as invented by Zadeh [19], has enormous applications to various fields of present-day science and technology. Now-a-days much research is being conducted on the generalizations of the notion of fuzzy sets. The idea of fuzzy topology was given by Chang [5] and some other mathematicians [15, 18]. After the initiation of grill in general topology by Choquet [7], Azad [2] introduced the concept of fuzzy grill to study fuzzy proximities. Since then a good many research papers have appeared which have encompassed different areas of fuzzy topology by use of fuzzy grill.

The idea of generalized fuzzy closed sets and generalization of fuzzy continuous functions for fuzzy topological spaces and their applications were first given by Balasubramanian and Sundaram [3]. Since then many modifications have been introduced and investigated by others.

In the present paper, we also introduce the concept of a kind of generalized fuzzy closed sets, termed $\mathcal{G}_{gf}$-closed sets, in a fuzzy topological space via fuzzy grill, and study it to obtain some information and properties of such sets. It is seen that the class of such sets contains the class of gf-closed sets introduced in [3]. In Section 3, we consider a generalization of fuzzy continuous functions and that too in terms of fuzzy grill and its associated fuzzy topology. In the last section, we take up certain applications of $\mathcal{G}_{gf}$-closed sets and the aforesaid ideas. To that end we consider some modified versions of fuzzy regular and fuzzy normal spaces, effected in terms of grills, and try for some preservation theorems.

Throughout this paper, by an fts X, we mean a fuzzy topological space (X, τ), as initiated by Chang [5]. For two fuzzy sets [19] A, B in X, i.e., $A, B \in I^X$ ($I = [0, 1]$), we write $A \leq B$ if $A(x) \leq B(x)$ for each $x \in X$.

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A fuzzy singleton or a fuzzy point [15] with support x and value α $(0 < \alpha \leq 1)$, is denoted by x_α. A fuzzy point x_α is said to be q-coincident with a fuzzy set A in X, denoted by $x_\alpha qA$, if there exists $x \in X$ such that $A(x) + B(x) > 1$, whereas the notation AqB means that A is q-coincident[15] with B. i.e., AqB implies $A(x)+B(x) > 1$ for some $x \in X$. The negations of these statements are denoted by $A \not\leq B$ and $A\bar{q}B$ respectively. The fuzzy sets in X taking on respectively the constant values 0 and 1 are denoted by 0_X and 1_X respectively. A fuzzy set A is non-null if $A \neq 0_X$. For $A, B \in I^X$, A is called a q-nbd of B [15] if for some fuzzy open set U in X, $BqU \leq A$; if, in addition, A itself is fuzzy open then it is called an open-q-nbd of B. The collection of all open q-nbds of any fuzzy point x_α is denoted by $\mathcal{Q}(x_\alpha)$. For a fuzzy set A in an fts X, the fuzzy complement, fuzzy interior and fuzzy closure of A in X are written as $(1 - A)$ [or sometimes as $(1_X - A)$], $intA$ and clA, respectively.

A non-void collection $\mathcal{G}$ of fuzzy sets in an fts (X, τ) is called a fuzzy grill [2] on X if (i) $0_X \notin \mathcal{G}$ (ii) $A \in \mathcal{G}$, $B \in I^X$ and $A \leq B \Rightarrow B \in \mathcal{G}$ and (iii) $A, B \in I^X$ and $A \bigcup B \in \mathcal{G} \Rightarrow A \in \mathcal{G}$ or $B \in \mathcal{G}$. Let (X, τ) be an fts and $\mathcal{G}$ be a fuzzy grill on X. An operator $\phi_\mathcal{G} : I^X \to I^X$, associated with the fuzzy grill $\mathcal{G}$ and the fuzzy topology τ, is defined [14] as follows:
For any $A \in I^X$, $\phi_\mathcal{G}(A)$, simply denoted by $\phi(A)$, is the union of all fuzzy points x_α of X such that if $U \in Q(x_\alpha)$, then $A * U \in \mathcal{G}$, where $A * B$ is the Lukasiewicz conjunction [12] on the powerset I^X, defined by $A * B = \max(0, A + B - 1_X)$, for $A, B \in I^X$, where $(A * B)(x) = A(x) + B(x) - 1$ if $A(x) + B(x) > 1$ and $(A * B)(x) = 0$ otherwise.

We need as follows some results about the operator 'ϕ', as pre-requisites.

Result 1.1. ([14]) Let (X, τ) be an fts, and A, B $\in I^X$.
(i) If $\mathcal{G}$ is any fuzzy grill on X, then $A \leq B \Rightarrow \phi_\mathcal{G}(A) \leq \phi_\mathcal{G}(B)$. i.e., $\phi_\mathcal{G}$ is an increasing function.
(ii) If $\mathcal{G}_1$ and $\mathcal{G}_2$ are two fuzzy grills on X with $\mathcal{G}_1 \subseteq \mathcal{G}_2$, then $\phi_{\mathcal{G}_1}(A) \leq \phi_{\mathcal{G}_2}(A)$.
(iii) For any fuzzy grill $\mathcal{G}$ on X, if $A \notin \mathcal{G}$ then $\phi_\mathcal{G}(A) = 0_X \notin \mathcal{G}$.
(iv) $\phi(A \cup B) = \phi(A) \cup \phi(B)$.
(v) $\phi(\phi(A)) \leq \phi(A) = cl(\phi(A)) \leq cl(A)$.
(vi) $\phi(A \cup G) = \phi(A)$, for every $G \notin \mathcal{G}$.

Definition 1.2. ([14]) A map $\psi : I^X \to I^X$ is defined by $\psi(A) = A \cup \phi(A)$ for all fuzzy set A in X.

Then we have

Theorem 1.3. ([14]) *In an fts (X, τ), corresponding to a fuzzy grill $\mathcal{G}$, there exists a unique fuzzy topology $\tau_\mathcal{G}$ (say) on X given by $\tau_\mathcal{G} = \{U \in I^X / \psi(1_X - U) = 1_X - U\}$, where for any $A \in I^X$, $\psi(A) = A \cup \phi(A) = \tau_\mathcal{G}$-$cl(A)$.*

Theorem 1.4. ([14]) *In an fts (X, τ),*
(i) if $\mathcal{G}_1$ and $\mathcal{G}_2$ be two fuzzy grills with $\mathcal{G}_1 \subseteq \mathcal{G}_2$, then $\tau_{\mathcal{G}_2} \subseteq \tau_{\mathcal{G}_1}$.
(ii) if $\mathcal{G}$ be a fuzzy grill and $B \notin \mathcal{G}$, then B is closed in $(X, \tau_\mathcal{G})$.
(iii) for any fuzzy set A and any fuzzy grill $\mathcal{G}$ on X, $\phi(A)$ is $\tau_\mathcal{G}$-closed.

Theorem 1.5. ([14]) *For a fuzzy topological space (X, τ) together with a fuzzy grill $\mathcal{G}$, $\tau \subseteq \tau_\mathcal{G}$.*

Theorem 1.6. ([13]) *For any two fuzzy open sets A and B in an fts X, $A \bar{q} B \Rightarrow clA \bar{q} B$ and $A \bar{q} clB$.*

From now on by a fuzzy $\mathcal{G}$-space we mean an fts (X, τ) with a fuzzy grill $\mathcal{G}$ and denote it by $(X, \tau, \mathcal{G})$ (or sometimes simply by X).

2. $\mathcal{G}_{gf}$-closed and $\mathcal{G}_{gf}$-open sets in fuzzy $\mathcal{G}$-spaces

The concepts of generalized closed and open sets in fuzzy setting were introduced in [3] in the following way:

Definition 2.1. ([3]) A fuzzy set A in an fts (X, τ) is called *generalized fuzzy closed* (in short, *gf-closed*) if $cl(A) \leq U$ whenever $A \leq U$ and U is fuzzy open. A fuzzy set A is called *generalized fuzzy open* (in short, *gf-open*) if its fuzzy complement $(1 - A)$ is gf-closed.

We now initiate below the notion of a class of generalized closed fuzzy sets, larger than the class of gf-closed sets, in terms of the operator 'ϕ'.

Definition 2.2. A fuzzy set A in a fuzzy $\mathcal{G}$-space $(X,\tau,\mathcal{G})$ is called a *generalized fuzzy closed set* with respect to a fuzzy grill $\mathcal{G}$ ($\mathcal{G}_{gf}$-closed, for short) if $\phi(A) \le U$ whenever $A \le U$ and U is fuzzy open with respect to τ. A is called *generalized fuzzy open* with respect to a fuzzy grill $\mathcal{G}$ ($\mathcal{G}_{gf}$-open, for short) if its complement $(1-A)$ is $\mathcal{G}_{gf}$-closed.

As every fuzzy closed (respectively fuzzy open) set is gf-closed(respectively gf-open) [3], we have the following:

Observation 2.3. For a fuzzy $\mathcal{G}$-space $(X,\tau,\mathcal{G})$, we observe as follows:
(i) For any fuzzy set A in X, since $\phi(\phi(A)) \le \phi(A)$ [by Result 1.1(v)], $\phi(A)$ is $\mathcal{G}_{gf}$-closed.
(ii) If $A \notin \mathcal{G}$ then $\phi(A) = 0_X$ [by Result 1.1(iii)]. Thus every non-member of $\mathcal{G}$ is $\mathcal{G}_{gf}$-closed in $(X,\tau,\mathcal{G})$.
(iii) By the definition of $\tau_{\mathcal{G}}$ and Definition 2.2, we see that a fuzzy set is $\mathcal{G}_{gf}$-closed if and only if ($A \le U \in \tau \Rightarrow \tau_{\mathcal{G}}\text{-}cl(A) = A \vee \phi(A) \le U$).
(iv) Every fuzzy closed (resp. fuzzy open) as well as every gf-closed [resp. gf-open] set is $\mathcal{G}_{gf}$-closed(resp. $\mathcal{G}_{gf}$-open)[this is clear from Result 1.1(v)].

Example 2.4. Let $\mathcal{G} = I^X \setminus \{0_X\}$. Then for any non-null fuzzy set A in X, we have $A \le \phi(A)$. Indeed, $x_\lambda \le A$ and $U \in \mathcal{Q}(x_\lambda) \Rightarrow \lambda + U(x) > 1$ and $\lambda \le A(x) \Rightarrow A(x) + U(x) - 1 > 0$, i.e., $A + U - 1 \ne 0_X \Rightarrow A + U - 1 \in \mathcal{G} \Rightarrow x_\lambda \le \phi(A)$. Then by Result 1.1(v) and Theorem 1.3, we have $\phi(A) = cl(A)$ and $\psi(A) = \tau_{\mathcal{G}}\text{-}clA = A \vee \phi(A) = clA$, for any fuzzy set A in X. So the class of $\mathcal{G}_{gf}$-closed sets coincides with that of gf-closed sets.

But in general a $\mathcal{G}_{gf}$-closed set may not be a gf-closed set which we show by the following example:

Example 2.5. Let $(X,\tau,\mathcal{G})$ be a fuzzy $\mathcal{G}$-space where X is any infinite set, $\tau = \{U \in I^X : 0 \le U(x) < \frac{1}{2},\ x \in X\}$ together with 1_X and $\mathcal{G} = \{G \in I^X : 0.7 < G(x) \le 1,\ x \in X\}$. Let x_0 be a chosen point of X and A be the fuzzy set in X given by $A(x_0) = 0.2$ and $A(x) = 0$ for $x \in X \setminus \{x_0\}$. Then A is not gf-closed, since for any $U(\ne 1_X) \in \tau$ with $A \le U$, we see that $cl(A) \nleq U$.
But A is $\mathcal{G}_{gf}$-closed. In fact, if $A \le 1_X$, then $\phi(A) \le 1_X$. So consider the case when $A \le V(\ne 1_X) \in \tau$. Here for any fuzzy point x_λ in X and for any $W \in \mathcal{Q}(x_\lambda) \subseteq \tau$, $A + W - 1_X \notin \mathcal{G}$. Thus $\phi(A) = 0_X \le V$ so that A is $\mathcal{G}_{gf}$-closed.

Theorem 2.6. *If A_1 and A_2 are two $\mathcal{G}_{gf}$-closed sets in a fuzzy $\mathcal{G}$-space $(X,\tau,\mathcal{G})$, then $A_1 \vee A_2$ is also a $\mathcal{G}_{gf}$-closed set.*

Proof. Let A_1 and A_2 be two $\mathcal{G}_{gf}$-closed sets in a fuzzy $\mathcal{G}$-space $(X,\tau,\mathcal{G})$. Suppose $A_1 \vee A_2 \le U$ for some fuzzy open set U. Then $A_1 \le U$ and $A_2 \le U$. By hypothesis, $\phi(A_1) \le U$ and $\phi(A_2) \le U$ and hence $\phi(A_1 \vee A_2) = \phi(A_1) \vee \phi(A_2) \le U$[by (iv) of Result 1.1]. □

But the intersection of two $\mathcal{G}_{gf}$-closed sets may not be $\mathcal{G}_{gf}$-closed as shown by the following example:

Example 2.7. Let $X = \{a,b\}$ and $\tau = \{0_X, 1_X, P\}$, where $P(a) = 0.6$ and $P(b) = 0.8$, be a fuzzy topology on X. Let $\mathcal{G}$ be a fuzzy grill on X consisting of those fuzzy sets G on X, such that $0.1 \le G(x) \le 1$ for all $x \in X$. Let A and B be two fuzzy sets in X defined as $A(a) = 0.8, A(b) = 0.4$ and $B(a) = 0.5, B(b) = 0.9$. Then A and B are both $\mathcal{G}_{gf}$-closed sets [indeed, the only fuzzy open set containing A or B is 1_X]. But $A \wedge B$ is not a $\mathcal{G}_{gf}$-closed set, Here $A \wedge B \le P$, but $\phi(A \wedge B) \nleq P$. Indeed, $a_{0.7} \nleq P$ but for any $U \in \mathcal{Q}(a_{0.7})$, $(A \wedge B) + U - 1 \in \mathcal{G}$, so that $a_{0.7} \le \phi(A \wedge B)$.

We next investigate as to under which condition a $\mathcal{G}_{gf}$-closed set is gf-closed. For this we need the following:

Definition 2.8. A fuzzy set A in a fuzzy $\mathcal{G}$-space $(X,\tau,\mathcal{G})$ is said to be $\tau_{\mathcal{G}}$-dense-in-itself if $A \le \phi(A)$.

Theorem 2.9. *If A is $\tau_{\mathcal{G}}$-dense-in-itself and a $\mathcal{G}_{gf}$-closed set in a fuzzy $\mathcal{G}$-space $(X, \tau, \mathcal{G})$, then A is gf-closed.*

Proof. Let A be $\tau_{\mathcal{G}}$-dense-in-itself and $\mathcal{G}_{gf}$-closed in $(X, \tau, \mathcal{G})$, and U be any fuzzy open set in X such that $A \leq U$. Then $\tau_{\mathcal{G}}$- $cl(A) = A \cup \phi(A) \leq U$ [as $A \leq \phi(A)$ and A is $\mathcal{G}_{gf}$-closed]. Since A is $\tau_{\mathcal{G}}$-dense-in-itself, then by (v) of Result 1.1, $cl(A) = cl\phi(A) = \phi(A) \leq U$ and hence A is gf-closed. □

Theorem 2.10. *For any $\mathcal{G}_{gf}$-closed set A in a fuzzy $\mathcal{G}$-space $(X, \tau, \mathcal{G})$,*
(i) if $A \leq B \leq \phi(A)$, where $B \in I^X$, then B is also $\mathcal{G}_{gf}$-closed.
(ii) if $A \leq B \leq \tau_{\mathcal{G}}$-$cl(A)$ [$B \in I^X$], then B is also $\mathcal{G}_{gf}$-closed.
(iii) $\tau_{\mathcal{G}}$-closure of every $\mathcal{G}_{gf}$-closed set is $\mathcal{G}_{gf}$-closed

Proof. (i) Suppose $B \leq U$ for some fuzzy open set U. Then $A \leq U$ and since A is $\mathcal{G}_{gf}$-closed, $\phi(A) \leq U$. Now $B \leq \phi(A) \Rightarrow \phi(B) \leq \phi(\phi(A)) \leq \phi(A) \leq U$ [by Result 1.1(i) and (v)] $\Rightarrow B$ is also $\mathcal{G}_{gf}$-closed.
(ii) Let $B \leq V$ for some $V \in \tau$. Then $A \leq V$ and hence $\phi(A) \leq V$. Since $B \leq \tau_{\mathcal{G}}$-$cl(A) = A \vee \phi(A)$, we have $\phi(B) \leq \phi(A \vee \phi(A)) = \phi(A)) \vee \phi(\phi(A)) \leq \phi(A) \leq V \Rightarrow B$ is $\mathcal{G}_{gf}$-closed.
(iii) For any $\mathcal{G}_{gf}$-closed set A, $\tau_{\mathcal{G}}$-$cl(A) \leq U \in \tau \Rightarrow \phi(A) \leq U \Rightarrow \phi(\tau_{\mathcal{G}}\text{-}cl(A)) = \phi(A \vee \phi(A)) = \phi(A) \vee \phi(\phi(A)) \leq \phi(A) \leq U$, which implies that $\tau_{\mathcal{G}}$-$cl(A)$ is $\mathcal{G}_{gf}$-closed. □

Corollary 2.11. *Let A and B be two fuzzy sets in a fuzzy $\mathcal{G}$-space $(X, \tau, \mathcal{G})$ such that $A \leq B \leq \phi(A)$. If A is $\mathcal{G}_{gf}$-closed then A and B are gf-closed.*

Proof. Since $A \leq B \leq \phi(A)$ and A is $\mathcal{G}_{gf}$-closed, then by Theorem 2.10, B is $\mathcal{G}_{gf}$-closed. Also by Result 1.1 and $A \leq B \leq \phi(A)$, we get $\phi(A) \leq \phi(B) \leq \phi(\phi(A)) \leq \phi(A)$ i.e., $\phi(A) = \phi(B)$. Then A and B are $\tau_{\mathcal{G}}$-dense-in-itself and by Theorem 2.9, A and B are gf-closed. □

Theorem 2.12. *Let $(X, \tau, \mathcal{G})$ be a fuzzy $\mathcal{G}$-space. Then the following are equivalent:*
(a) Every fuzzy set in X is $\mathcal{G}_{gf}$-closed.
(b) Every fuzzy open set is $\tau_{\mathcal{G}}$-closed.

Proof. $(a) \Rightarrow (b)$: Suppose every fuzzy set in $(X, \tau, \mathcal{G})$ is $\mathcal{G}_{gf}$-closed. Let U be any fuzzy open set in X. Then by hypothesis, U is $\mathcal{G}_{gf}$-closed and $U \leq U \Rightarrow \phi(U) \leq U$ and hence U is $\tau_{\mathcal{G}}$-closed.
$(b) \Rightarrow (a)$: Suppose every fuzzy open set is $\tau_{\mathcal{G}}$-closed. Let A be any fuzzy set in X such that $A \leq U$, where U is fuzzy open. Then $\phi(A) \leq \phi(U) \leq U$ (as U is $\tau_{\mathcal{G}}$-closed). So A is $\mathcal{G}_{gf}$-closed. □

Theorem 2.13. *Let $(X, \tau, \mathcal{G})$ be a fuzzy $\mathcal{G}$-space , where $\mathcal{G}$ is so taken that $\tau \cap \mathcal{G} = \{1_X\}$. Then each $A(\neq 1_X) \in \tau$ is a $\mathcal{G}_{gf}$-closed set.*

Proof. In fact, $A(\neq 1_X)$ is fuzzy open $\Rightarrow A \notin \mathcal{G} \Rightarrow A$ is $\tau_{\mathcal{G}}$-closed[by Theorem 1.4(ii)]$\Rightarrow \phi(A) \leq A$. Then for any fuzzy open set U with $A \leq U$ we have $\phi(A) \leq U$. Hence A is $\mathcal{G}_{gf}$-closed. □

We shall now prove some properties of $\mathcal{G}_{gf}$-open sets in a fuzzy $\mathcal{G}$-space $(X, \tau, \mathcal{G})$.

Theorem 2.14. *Let $(X, \tau, \mathcal{G})$ be a fuzzy $\mathcal{G}$-space. A fuzzy set A in X is $\mathcal{G}_{gf}$-open if and only if $F \leq \tau_{\mathcal{G}}$-$int(A)$, whenever F is fuzzy closed and $F \leq A$.*

Proof. Straightforward. □

Theorem 2.15. *Let $(X, \tau, \mathcal{G})$ be a fuzzy $\mathcal{G}$-space. If A and B are two $\mathcal{G}_{gf}$-open sets with $A \wedge cl(B) = B \wedge cl(A) = 0_X$, then $A \vee B$ is $\mathcal{G}_{gf}$-open.*

Proof. Let F be a fuzzy closed set such that $F \leq A \vee B$. Since $B \wedge cl(A) = 0_X$, then $F \wedge cl(A) \leq A$ and $F \wedge cl(A)$ is fuzzy closed. Since A is $\mathcal{G}_{gf}$-open, by Theorem 2.14, $F \wedge cl(A) \leq \tau_{\mathcal{G}}$-$int(A)$.
Similarly we get $F \wedge cl(B) \leq \tau_{\mathcal{G}}$-$int(B)$.
Now we have, $F = F \wedge (A \vee B) \leq (F \wedge cl(A)) \vee (F \wedge cl(B)) \leq \tau_{\mathcal{G}}\text{-}int(A) \vee \tau_{\mathcal{G}}\text{-}int(B) \leq \tau_{\mathcal{G}}\text{-}int(A \vee B)$. Thus $F \leq \tau_{\mathcal{G}}$-$int(A \vee B)$ and then by Theorem 2.14, $(A \vee B)$ is $\mathcal{G}_{gf}$-open. □

Theorem 2.16. *In a fuzzy $\mathcal{G}$-space $(X, \tau, \mathcal{G})$ if $\tau_{\mathcal{G}}$-int$(A) \leq B \leq A$ and A is $\mathcal{G}_{gf}$-open then B is $\mathcal{G}_{gf}$-open.*

Proof. If $\tau_{\mathcal{G}}$-$int(A) \leq B \leq A$ then $(1 - A) \leq (1 - B) \leq 1 - \tau_{\mathcal{G}}$-$int(A) = \tau_{\mathcal{G}}$-$cl(1 - A)$. Since $(1 - A)$ is $\mathcal{G}_{gf}$-closed and $(1 - A) \leq (1 - B) \leq \tau_{\mathcal{G}}$-$cl(1 - A) = (1 - A) \vee \phi(1 - A)$, then by Theorem 2.10(ii), $(1 - B)$ is also $\mathcal{G}_{gf}$-closed and hence B is $\mathcal{G}_{gf}$-open. □

Definition 2.17. A function $f : (X, \tau, \mathcal{G}_1) \to (Y, \sigma, \mathcal{G}_2)$ is called *fuzzy $(\tau_{\mathcal{G}_1}$-$\sigma_{\mathcal{G}_2})$-closed* if the image of every fuzzy $\tau_{\mathcal{G}_1}$-closed set in X is $\sigma_{\mathcal{G}_2}$-closed in Y.

In what follows, whenever $\mathcal{G}_1$ (resp. $\mathcal{G}_2$) is a fuzzy grill on an fts (X, τ) (resp. (Y, σ)) and $A \subseteq X$ (resp. $\subseteq Y$)is generalized fuzzy closed or open with respect to $\mathcal{G}_1$ (resp. $\mathcal{G}_2$), we shall simply write, to shorten the notation, that 'A is $\mathcal{G}_{gf}$-closed or $\mathcal{G}_{gf}$-open in X (resp.Y)' and hope that the context will not allow any confusion to arise.

Theorem 2.18. *If A be a $\mathcal{G}_{gf}$-closed set in X and $f : (X, \tau, \mathcal{G}_1) \to (Y, \sigma, \mathcal{G}_2)$ is fuzzy continuous and fuzzy $(\tau_{\mathcal{G}_1}$-$\sigma_{\mathcal{G}_2})$-closed, then $f(A)$ is $\mathcal{G}_{gf}$-closed in Y.*

Proof. Let B be a fuzzy open set in Y such that $f(A) \leq B$. Then $A \leq f^{-1}(B)$ and $f^{-1}(B)$ is fuzzy open in X. Since A is $\mathcal{G}_{gf}$-closed, $\phi(A) \leq f^{-1}(B)$. Thus $A \vee \phi(A) \leq f^{-1}(B) \Rightarrow \tau_{\mathcal{G}_1}$-$cl(A) \leq f^{-1}(B) \Rightarrow f(\tau_{\mathcal{G}_1}$-$cl(A)) \leq B$. Since f is fuzzy $\tau_{\mathcal{G}_1}$-closed, $f(\tau_{\mathcal{G}_1}$-$cl(A))$ is $\sigma_{\mathcal{G}_2}$-closed and $\sigma_{\mathcal{G}_2}$-$cl(f(A)) \leq f(\tau_{\mathcal{G}_1}$-$cl(A) \leq B$. Then $\phi(f(A)) \leq B$ and so $f(A)$ is $\mathcal{G}_{gf}$-closed in Y. □

However the image of a $\mathcal{G}_{gf}$-open set need not be $\mathcal{G}_{gf}$-open under a fuzzy continuous map as is shown below:

Example 2.19. Let $X = \{a, b\}$, $Y = \{c, d\}$ be two fuzzy sets. Also let $\tau = \{0_X, 1_X, P, Q\}$, where $P(a) = 0.5$, $P(b) = 0.2$, $Q(a) = 0.6$, $Q(b) = 0.8$ and $\sigma = \{0_X, 1_X, R\}$ where $R(c) = 0.8$, $R(d) = 0.6$, be two fuzzy topologies on X and Y respectively. Let $\mathcal{G}_1 = \{G \in I^X / 0.2 \leq G(x) \leq 1; x \in X\}$ and $\mathcal{G}_2 = \{G \in I^X / 0.1 \leq G(x) \leq 1; x \in X\}$ be two fuzzy grills on X and Y respectively. Let $f : (X, \tau, \mathcal{G}_1) \to (Y, \sigma, \mathcal{G}_2)$ be a function such that $f(a) = d$ and $f(b) = c$. Then we see that f is continuous since $f^{-1}(R)(a) = Rf(a) = R(d) = 0.6 = Q(a)$ and $f^{-1}(R)(b) = Rf(b) = R(c) = 0.8 = Q(b)$. Now P, being fuzzy open in X, is also $\mathcal{G}_{gf}$-open in X. Also $f(P)(c) = \sup_{z \in f^{-1}(c)} P(z) = P(b) = 0.2$ and $f(P)(d) = P(a) = 0.5$ imply that $(1 - f(P)) \leq R$. But $\phi(1 - f(P)) \nleq R$. Indeed, $c_{0.9} \nleq R$ but for any $U \in \mathcal{Q}(c_{0.9}) \in \sigma$, we get $(1 - f(P)) + U - 1_Y \in \mathcal{G}_2$. So $c_{0.9} \leq \phi(1 - f(P))$. Thus $(1 - f(P))$ is not $\mathcal{G}_{gf}$-closed in Y and consequently $f(P)$ is not $\mathcal{G}_{gf}$-open in Y.

3. Generalized fuzzy continuous functions for fuzzy $\mathcal{G}$-spaces

According to our proposed scheme, in this section we introduce a sort of generalized fuzzy continuous function, termed $\mathcal{G}_{gf}$-continuous function, in terms of some fuzzy grill $\mathcal{G}$.

Definition 3.1. ([3]) A function $f : X \to Y$ is called *generalized fuzzy continuous* (gf-continuous, for short) if the inverse image of every fuzzy closed set in Y is gf-closed in X.

Definition 3.2. A function $f : X \to Y$ is called *generalized fuzzy continuous with respect to some fuzzy grill $\mathcal{G}$* on X ($\mathcal{G}_{gf}$-continuous, for short) if the inverse image of every fuzzy closed set in Y is $\mathcal{G}_{gf}$-closed in X.

Corollary 3.3. *A function $f : (X, \tau, \mathcal{G}) \to (Y, \sigma)$ is $\mathcal{G}_{gf}$-continuous if and only if the inverse image of each fuzzy open set in Y is $\mathcal{G}_{gf}$-open in X.*

Remark 3.4. Every fuzzy continuous function is gf-continuous [3] and every gf-continuous function is $\mathcal{G}_{gf}$-continuous. But the converses are false. The first one was done by Balasubramanian and Sundaram [3]. Here we give two examples of which the first one shows that a function may be $\mathcal{G}_{gf}$-continuous without being fuzzy continuous.

Example 3.5. Let $X = Y = \{a, b, c\}$; $\tau = \{0_X, 1_X, A\}$ and $\sigma = \{0_X, 1_X, B\}$, where $A(a) = 0.2$, $A(b) = 0.4$, $A(c) = 0.6$ and $B(a) = 0.3$, $B(b) = 0.5$, $B(c) = 0.7$, be two fuzzy topologies on X and Y respectively. Let $f : (X, \tau) \to (Y, \sigma)$ be the identity function. Then f is not fuzzy continuous since $f^{-1}(B) \notin \tau$. But for any fuzzy grill $\mathcal{G}$ on (X, τ), f is $\mathcal{G}_{gf}$-continuous, since for $f^{-1}(1 - B)$ the only τ-open fuzzy set containing $f^{-1}(1 - B)$ is 1_X and then obviously $\phi(f^{-1}(1 - B)) \le 1_X$. Hence f is $\mathcal{G}_{gf}$-continuous on X.

We next show that a $\mathcal{G}_{gf}$-continuous function need not be gf-continuous.

Example 3.6. Let X be any set consisting of more than one point and $'a'$ be any fixed element of X. Let $\tau = \{0_X, 1_X, A\}$ and $\tau' = \{0_X, 1_X, B\}$, where $A(a) = 0.6$, $A(x) = 0$ for $x \in X \setminus \{a\}$, and $B(a) = 0.5$, $B(x) = 1$ for $x \in X \setminus \{a\}$. Then (X, τ) and (X, τ') are both fuzzy topological spaces. Let $\mathcal{G} = \{G \in I^X : 0.6 < G(x) \le 1;\ x \in X\}$ be a fuzzy grill on X. Consider the identity function $f : (X, \tau) \to (X, \tau')$. Then f is not gf-continuous on (X, τ) since for $B \in \tau'$, $f^{-1}(1 - B)$ is not gf-closed in (X, τ). Indeed, $f^{-1}(1 - B)(a) = (1 - B)f(a) = 1 - B(a) = 0.5$ and $f^{-1}(1 - B)(x) = 1 - B(x) = 0$ for any $x \in X \setminus \{a\}$. Thus $f^{-1}(1 - B) \le A$, but $cl(f^{-1}(1 - B)) = 1_X \nleq A$. However, f is $\mathcal{G}_{gf}$-continuous, because for any fuzzy point x_λ in X and for any $U \in Q(x_\lambda) \subseteq \tau$, $f^{-1}(1 - B) + U - 1 \notin \mathcal{G}$ and hence $\phi(f^{-1}(1 - B)) = 0_X \le A$.

We now show that the inverse image of a $\mathcal{G}_{gf}$-closed set under a $\mathcal{G}_{gf}$-continuous map need not be $\mathcal{G}_{gf}$-closed.

Example 3.7. Let $X = \{a, b\}$. $\tau = \{0_X, 1_X, U\}$ and $\sigma = \{0_X, 1_X, V\}$, where $U(a) = 0.7$, $U(b) = 0.8$ and $V(a) = 0.5$, $V(b) = 0.1$. Then τ and σ are two fuzzy topologies on X. Consider the fuzzy grill $\mathcal{G} = \{G \in I^X / 0.1 \le G(x) \le 1; x \in X\}$ on X. Let $f : (X, \tau, \mathcal{G}) \to (X, \sigma, \mathcal{G})$ be the identity function. Then f is $\mathcal{G}_{gf}$-continuous since $f^{-1}(1 - V)$ is $\mathcal{G}_{gf}$-closed in X. In fact, the only fuzzy open set containing $f^{-1}(1 - V)$ is 1_X and also $\phi(f^{-1}(1 - V)) \le 1_X$. Now let us take a fuzzy set A in $(X, \tau, \mathcal{G})$ such that $A(a) = 0.6$ and $A(b) = 0.4$. Then A is $\mathcal{G}_{gf}$-closed in $(X, \sigma, \mathcal{G})$. Also, $f^{-1}(A)(a) = Af(a) = A(a) = 0.6$ and $f^{-1}(A)(b) = Af(b) = A(b) = 0.4$. Now $f^{-1}(A) \le U$. But $\phi(f^{-1}(A)) \nleq U$ as $a_{0.8} \nleq U$, although for any $P \in Q(a_{0.8}) \subseteq \tau$, $f^{-1}(A) + P - 1 \in \mathcal{G}$ so that $a_{0.8} \le \phi(f^{-1}(A))$. Thus $f^{-1}(A)$ is not $\mathcal{G}_{gf}$-closed in X.

4. Some applications of $\mathcal{G}_{gf}$-closed sets

In this section we introduce fuzzy generalized regular and normal spaces via $\mathcal{G}_{gf}$-closed sets. Before going to that we require the following preliminaries:

Definition 4.1. ([11]) An fts X is said to be *fuzzy regular* if every fuzzy open set V can be expressed as union of fuzzy open sets U_α's such that $cl(U_\alpha) \le V$, for each α.

Definition 4.2. ([16]) An fts X is called *fuzzy strongly regular* if for any fuzzy open set A in X and each fuzzy point $x_\alpha \le A$, there exists a fuzzy open set B in X such that $x_\alpha \le B \le cl(B) \le A$.

Theorem 4.3. ([16]) *A fuzzy strongly regular space is fuzzy regular.*

Definition 4.4. A fuzzy $\mathcal{G}$-space $(X, \tau, \mathcal{G})$ is said to be $\mathcal{G}_{gf}$-regular if for every $\mathcal{G}_{gf}$-closed set F and each fuzzy point x_λ in X such that $x_\lambda \, \overline{q} \, F$, there exist fuzzy open sets U and V in X for which $x_\lambda \le U$, $F \le V$ and $U \, \overline{q} \, V$.

Theorem 4.5. *In a fuzzy $\mathcal{G}$-space $(X, \tau, \mathcal{G})$, the following are equivalent:*
(a) $(X, \tau, \mathcal{G})$ is $\mathcal{G}_{gf}$-regular.
(b) For each fuzzy point x_λ in X and each $\mathcal{G}_{gf}$-closed set F such that $x_\lambda \, \overline{q} \, F$, there exist two fuzzy open sets U and V such that $x_\lambda \le U$, $\tau_{\mathcal{G}}$-$clF \le V$ and $U \, \overline{q} \, V$.
(c) For each $\mathcal{G}_{gf}$-closed set F in X and each fuzzy point x_λ with $x_\lambda \, \overline{q} \, F$, there exist two fuzzy open sets U and V such that $x_\lambda \le U$, $F \le V$ and $clU \, \overline{q} \, clV$.
(d) For each fuzzy point x_λ in X and each $\mathcal{G}_{gf}$-open set U containing x_λ, there exists a fuzzy open set V such that $x_\lambda \le V \le clV \le U$.

Proof. $(a) \Rightarrow (b)$: Let x_λ be any fuzzy point and F be any $\mathcal{G}_{gf}$-closed set in X such that $x_\lambda \, \overline{q} \, F$. By (a), there exist fuzzy open sets U and V such that $x_\lambda \leq U$, $F \leq V$ and $U \, \overline{q} \, V$. Now F being $\mathcal{G}_{gf}$-closed, $\phi(F) \leq V$ i.e., $\tau_{\mathcal{G}}\text{-}clF \leq V$.
$(b) \Rightarrow (a)$: Obvious.
$(b) \Rightarrow (c)$: Let x_λ be any fuzzy point and F be any $\mathcal{G}_{gf}$-closed set in X such that $x_\lambda \, \overline{q} \, F$. Then by (b), there exist $W, V \in \tau$ such that $x_\lambda \leq W$, $\tau_{\mathcal{G}}\text{-}clF \leq V$ and $W \, \overline{q} \, V$. Thus $W \, \overline{q} \, clV$, and hence $x_\lambda \, \overline{q} \, clV$.
Now clV, being a fuzzy closed set, is $\mathcal{G}_{gf}$-closed [by Observation 2.3 (iv)] and $x_\lambda \, \overline{q} \, clV$. Then again by (b), there exist two fuzzy open sets G and H in X such that $x_\lambda \leq G$, $\tau_{\mathcal{G}}\text{-}cl(clV) \leq H$ and $G \, \overline{q} \, H$.i.e., $x_\lambda \leq G$, $clV \leq H$ and $G \, \overline{q} \, H$. Hence $clG \, \overline{q} \, H$ [by Theorem 1.6].
Let us put $U = W \wedge G$. Then U and V are fuzzy open sets in X such that $x_\lambda \leq U$, $F \leq V$ and $clU \, \overline{q} \, clV$. Indeed, $cl(U) = cl(W \wedge G) \leq clG$ and $clG \, \overline{q} \, H \Rightarrow clU \, \overline{q} \, H \Rightarrow clU \, \overline{q} \, clV$ [since $clV \leq H$].
$(c) \Rightarrow (d)$: Let y_β be any fuzzy point and U be any $\mathcal{G}_{gf}$-open set in X such that $y_\beta \leq U$. Then $y_\beta \overline{q} \, (1-U)$ and $(1-U)$ is $\mathcal{G}_{gf}$-closed. Then by (c), there exist $V, W \in \tau$ such that $y_\beta \leq V$, $(1-U) \leq W$ and $clV \, \overline{q} \, clW$ $\Rightarrow y_\beta \leq V$, $(1-W) \leq U$ and $clV \, \overline{q} \, W \Rightarrow y_\beta \leq V$, $(1-W) \leq U$ and $clV \leq (1-W) \Rightarrow y_\beta \leq V \leq clV \leq 1-W \leq U$ $\Rightarrow y_\beta \leq V \leq clV \leq U$.
$(d) \Rightarrow (a)$: Let x_λ be any fuzzy point and F be any $\mathcal{G}_{gf}$-closed set in X such that $x_\lambda \, \overline{q} \, F$. Then $x_\lambda \leq 1-F$ and $1-F$ is a $\mathcal{G}_{gf}$-open set in X. By assumption, there exists a $V \in \tau$ such that $x_\lambda \leq V \leq clV \leq 1-F$ which gives $F \leq 1-clV$. Thus we have two fuzzy open sets V and $(1-clV)$ such that $x_\lambda \leq V$ $F \leq (1-clV)$ and $V \, \overline{q}$ $(1-clV)$. □

Theorem 4.6. *Every $\mathcal{G}_{gf}$-regular space is fuzzy strongly regular and hence fuzzy regular.*

Proof. Let x_λ be any fuzzy point in X and A be any fuzzy open set in X such that $x_\lambda \leq A$. Since every fuzzy open set is $\mathcal{G}_{gf}$-open, then by Theorem 4.5(d), there exists a $U \in \tau$ such that $x_\lambda \leq U \leq clU \leq A$. Thus X is fuzzy strongly regular and then by Theorem 4.3, X is fuzzy regular. □

But in general, the reverse implications in the above theorem are not true. The counterexample to show that a fuzzy regular space need not be fuzzy strongly regular is given in [16]. We only show that a fuzzy strongly regular space may not be $\mathcal{G}_{gf}$-regular.

Example 4.7. Let $X = \{a, b\}$ and $\tau = \{0_X, 1_X, P, Q\}$, where $P(a) = 0.3$, $P(b) = 0.5$ and $Q(a) = 0.7$, $Q(b) = 0.5$. Then (X, τ) is an fts. Here we see that P and Q are both fuzzy open and fuzzy closed. Thus for any $x_\lambda \leq P$, we have the fuzzy open set P such that $x_\lambda \leq P \leq clP = P$. Similarly for any $y_\beta \leq Q$, we get $y_\beta \leq Q \leq clQ = Q$. Thus X is fuzzy strongly regular. But we claim that X is not $\mathcal{G}_{gf}$-regular. Indeed, let us take a fuzzy set F in X where $F(a) = 0.6$, $F(b) = 0.3$. Then $F \leq Q$ and $\phi(F) \leq clF$ [by Result 1.1(v)] $= Q \Rightarrow F$ is $\mathcal{G}_{gf}$-closed. Now $b_{0.6} \, \overline{q} \, F$ [since $0.6 + F(b) < 1$], but we cannot find two fuzzy open sets U and V such that $b_{0.6} \leq U$, $F \leq V$ and $U \, \overline{q} \, V$.

Definition 4.8. ([10]) An fts X is called *fuzzy normal* if for any fuzzy closed set F and a fuzzy open set U in X such that $F \leq U$ there exists a fuzzy open set V such that $F \leq V \leq cl(V) \leq U$.

Theorem 4.9. ([16]) *An fts X is fuzzy normal if and only if for two fuzzy closed sets F and G with $F \, \overline{q} \, G$, there exist fuzzy open sets U and V such that $F \leq U$, $G \leq V$ and $U \, \overline{q} \, V$.*

Definition 4.10. A fuzzy $\mathcal{G}$-space $(X, \tau, \mathcal{G})$ is said to be $\mathcal{G}_{gf}$-normal if for any two $\mathcal{G}_{gf}$-closed sets F_1 and F_2 in X such that $F_1 \, \overline{q} \, F_2$, there exist fuzzy open sets U and V in X for which $F_1 \leq U$, $F_2 \leq V$ and $U \, \overline{q} \, V$.

Evidently every $\mathcal{G}_{gf}$-normal space is fuzzy normal.

Theorem 4.11. *In a fuzzy $\mathcal{G}$-space $(X, \tau, \mathcal{G})$, the following are equivalent:*
(a) $(X, \tau, \mathcal{G})$ is $\mathcal{G}_{gf}$-normal.
(b) For any two $\mathcal{G}_{gf}$-closed sets F_1 and F_2 in X such that $F_1 \, \overline{q} \, F_2$, there exist two fuzzy open sets U and V such that $F_1 \leq U$, $F_2 \leq V$ and $clU \, \overline{q} \, clV$.
(c) For each $\mathcal{G}_{gf}$-closed set F and and for each $\mathcal{G}_{gf}$-open set G containing F, there exists a fuzzy open set V such that $F \leq V \leq clV \leq G$.

Proof. $(a) \Rightarrow (b)$: Let $F_1 \,\overline{q}\, F_2$, where F_1 and F_2 are two $\mathcal{G}_{gf}$-closed sets in X. Then by (a), there exist two fuzzy open sets W and V such that $F_1 \le W$, $F_2 \le V$ and $W \,\overline{q}\, V$. Then $W \,\overline{q}\, clV$ and thus $F_1 \,\overline{q}\, clV$. Now clV, being fuzzy closed, is $\mathcal{G}_{gf}$-closed also. Thus F_1 and clU are two $\mathcal{G}_{gf}$-closed sets in X such that $F_1 \,\overline{q}\, clV$. Again by $\mathcal{G}_{gf}$-normality of X, there exist two fuzzy open sets G and H such that $F_1 \le G$, $clV \le H$ and $G \,\overline{q}\, H$. Hence $clG \,\overline{q}\, H$[by Theorem 1.6]. Let us put $U = W \wedge G$. Then U and V are two fuzzy open sets in X such that $F_1 \le U$, $F_2 \le V$ and $clU \,\overline{q}\, clV$.
$(b) \Rightarrow (c)$: Let F be any $\mathcal{G}_{gf}$-closed set and G be any $\mathcal{G}_{gf}$-open set in X such that $F \le G$. Thus $F \,\overline{q}\, (1 - G)$ and $(1 - G)$ is $\mathcal{G}_{gf}$-closed. Hence by (b), there exist fuzzy open sets U and V such that $F \le U$, $1 - G \le V$ and $clU \,\overline{q}\, clV$. Now $1 - G \le V \le clV \Rightarrow 1 - clV \le G$ and $clU \,\overline{q}\, clV \Rightarrow clU \le 1 - clV \le G$. Thus U is the required fuzzy open set in X for which $F \le U \le clU \le G$.
$(c) \Rightarrow (a)$: Let F_1 and F_2 be two $\mathcal{G}_{gf}$-closed sets in X with $F_1 \,\overline{q}\, F_2$. Then $F_1 \le (1 - F_2)$ and $(1 - F_2)$ is $\mathcal{G}_{gf}$-open. By (c), there exists a fuzzy open set U in X such that $F_1 \le U \le clU \le 1 - F_2$, i.e., $F_1 \le U$, $F_2 \le (1 - clU)$ and $U \,\overline{q}\, (1 - clU)$. □

Proposition 4.12. *For any two $\mathcal{G}_{gf}$-closed sets F_1 and F_2 in a $\mathcal{G}_{gf}$-normal space $(X, \tau, \mathcal{G})$, $F_1 \,\overline{q}\, F_2$ if and only if $\tau_{\mathcal{G}}$-$clF_1 \,\overline{q}\, \tau_{\mathcal{G}}$-$clF_2$.*

Proof. First let $F_1 \,\overline{q}\, F_2$. By $\mathcal{G}_{gf}$-normality of X, there exist fuzzy open sets U and V in X for which $F_1 \le U$, $F_2 \le V$ and $U \,\overline{q}\, V$. Since F_1 and F_2 are $\mathcal{G}_{gf}$-closed sets, $\phi(F_1) \le U$, $\phi(F_2) \le V \Rightarrow \tau_{\mathcal{G}}$-$clF_1 \le U$ and $\tau_{\mathcal{G}}$-$clF_2 \le V$ and thus $\tau_{\mathcal{G}}$-$clF_1 \,\overline{q}\, \tau_{\mathcal{G}}$-$clF_2$ [since $U \,\overline{q}\, V$].
The converse part is obvious. □

Definition 4.13. A function $f : X \to Y$ is called *$\mathcal{G}_{gf}$-irresolute* if the inverse image of every $\mathcal{G}_{gf}$-closed set in Y is $\mathcal{G}_{gf}$-closed in X, or equivalently the inverse image of every $\mathcal{G}_{gf}$-open set in Y is $\mathcal{G}_{gf}$-open in X.

Theorem 4.14. *Let $f : X \to Y$ be a fuzzy open, $\mathcal{G}_{gf}$-irresolute surjection.*
(a) If X is $\mathcal{G}_{gf}$-regular then Y is also $\mathcal{G}_{gf}$-regular.
(b) If X is $\mathcal{G}_{gf}$-normal then Y is also $\mathcal{G}_{gf}$-normal.

Proof. (a) Let F be any $\mathcal{G}_{gf}$-closed fuzzy set in Y and y_λ be any fuzzy point in Y such that $y_\lambda \,\overline{q}\, F$. Then $f^{-1}(y_\lambda) \,\overline{q}\, f^{-1}(F)$. Since f is $\mathcal{G}_{gf}$-irresolute, $f^{-1}(F)$ is $\mathcal{G}_{gf}$-closed in X. Select some $x \in f^{-1}(y)$ (as f is onto) and consider the fuzzy point x_λ in X. Then $x_\lambda \,\overline{q}\, f^{-1}(F)$. Since X is $\mathcal{G}_{gf}$-regular, there exist fuzzy open sets U and V such that $x_\lambda \le U$, $f^{-1}(F) \le V$ and $U \,\overline{q}\, V$. Since f is fuzzy open and surjective, $f(U)$ and $f(V)$ are fuzzy open in Y such that $y_\lambda \le f(U)$, $F \le f(V)$ and $f(U) \,\overline{q}\, f(V)$. Thus Y is $\mathcal{G}_{gf}$-regular.
(b) Let F_1 and F_2 be two $\mathcal{G}_{gf}$-closed sets in Y such that $F_1 \,\overline{q}\, F_2$. Since f is $\mathcal{G}_{gf}$-irresolute, $f^{-1}(F_1)$ and $f^{-1}(F_2)$ are $\mathcal{G}_{gf}$-closed sets in X and also $f^{-1}(F_1) \,\overline{q}\, f^{-1}(F_2)$. Since X is $\mathcal{G}_{gf}$-normal, there exist fuzzy open sets U and V in X for which $f^{-1}(F_1) \le U$, $f^{-1}(F_2) \le V$ and $U \,\overline{q}\, V$.
Now f is surjective, so $F_1 \le f(U)$, $F_2 \le f(V)$ and $f(U) \,\overline{q}\, f(V)$ and since f is fuzzy open, $f(U)$ and $f(V)$ are fuzzy open sets in Y. Thus Y is $\mathcal{G}_{gf}$-normal. □

Theorem 4.15. *Let $(X, \tau, \mathcal{G}_1)$ and $(X, \sigma, \mathcal{G}_2)$ be two fuzzy $\mathcal{G}$-spaces. If $f : X \to Y$ is a fuzzy continuous, fuzzy $(\tau_{\mathcal{G}_1}$-$\sigma_{\mathcal{G}_2})$-closed surjective map, then*
(a) Y is $\mathcal{G}_{gf}$-regular $\Rightarrow$ X is $\mathcal{G}_{gf}$-regular.
(b) Y is $\mathcal{G}_{gf}$-normal $\Rightarrow$ X is also $\mathcal{G}_{gf}$-normal.

Proof. (a) Let F be any $\mathcal{G}_{gf}$-closed fuzzy set in X and x_λ be any fuzzy point in X such that $x_\lambda \,\overline{q}\, F$. Since f is fuzzy continuous and fuzzy $(\tau_{\mathcal{G}_1}$-$\sigma_{\mathcal{G}_2})$-closed then by Theorem 2.18, $f(F)$ is $\mathcal{G}_{gf}$-closed in Y. Now f being surjective, $f(x_\lambda) \,\overline{q}\, f(F)$. Also since Y is $\mathcal{G}_{gf}$-regular, there exist fuzzy open sets U and V such that $f(x_\lambda) \le U$, $f(F) \le V$ and $U \,\overline{q}\, V$. Thus we get $x_\lambda \le f^{-1}(U)$, $F \le f^{-1}(V)$ and $f^{-1}(U) \,\overline{q}\, f^{-1}(V)$. By fuzzy continuity of f, $f^{-1}(U)$ and $f^{-1}(V)$ are both fuzzy open. So X is $\mathcal{G}_{gf}$-regular.
(b) Let F_1 and F_2 be two $\mathcal{G}_{gf}$-closed sets in X such that $F_1 \,\overline{q}\, F_2$. Then by Theorem 2.18, $f(F_1)$ and $f(F_2)$ are $\mathcal{G}_{gf}$-closed in Y. Then $f(F_1) \,\overline{q}\, f(F_2)$ [f being surjective]. By $\mathcal{G}_{gf}$-normality of X, there exist fuzzy open sets U and V in X for which $f(F_1) \le U$, $f(F_2) \le V$ and $U \,\overline{q}\, V$, i.e., $F_1 \le f^{-1}(U)$, $F_2 \le f^{-1}(V)$ and $f^{-1}(U) \,\overline{q}\, f^{-1}(V)$. Now by fuzzy continuity of f, $f^{-1}(U)$ and $f^{-1}(V)$ are both fuzzy open in X. Hence X is $\mathcal{G}_{gf}$-normal. □

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