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Portfolio problems based on jump-diffusion models

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Abstract. Zhou and Li [49] by virtue of stochastic linear-quadratic control theory studied the optimal portfolio problems with the asset price process satisfying a diffusion stochastic differential equation, and proposed the celebrated LQ framework and the efficient frontier for the given portfolio problem. In this paper, we consider the optimal portfolio problems based on the asset price process satisfying a jump-diffusion stochastic differential equation. Similarly, we also arrive at the efficient frontier of the optimal portfolio selection problem. The conclusions obtained here can be regarded as a natural generalization of the work by Zhou and Li [49].

1. Introduction

Since Markowitz [25, 26], many researchers studied the portfolio theory, W. Sharpe [42], J. Lintner [23] and J. Mossin [32] respectively put forward the famous capital asset pricing model (CAPM); Fama [11] and Samuelson [39] proposed the efficient market theory; Merton [30] derived the capital gains rate is a lognormal distribution of the capital asset pricing model in the case of continuous-time transactions; Ross [38] developed breakthroughly the capital asset pricing model, and put forward the arbitrage pricing theory (APT), it laid a solid foundations for the development of modern portfolio theory and the growth of financial markets, and so on. On the other hand, many scholars consider the applications with the improved mean-variance models such as CVaR, spectral risk measures, variance hedge, dynamic portfolio with transaction costs, etc., and obtain some celebrated works, one can refer to [2, 6, 7, 9, 15, 18, 22, 24, 34, 41, 45, 46] for details.

After the pioneering work of Markowitz, the single-period mean-variance model was quickly extended to the multi-period portfolio selection problems. However, comparing to the single-period model, the analytical solution to multi-period models had not given until the work was posed by Merton [29]. The study of multi-period portfolio problems has been dominated by the maximizing the expected utility of the final wealth, namely maximize $E[U(x(T))]$, where U is a utility function. Logarithmic, exponential, quadratic utility functions have been extensively studied before. The shortcoming of these findings is the lack of long-term consideration of the optimal investment strategy. On the one hand, it is difficult to determine the investor's utility functions. On the other hand, the transaction information of risk and expected return has not been clear, many investors depended on intuition to make decisions. In this sense,

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Markowitz's mean-variance approach has not been fully utilized in the dynamic, multi-period investment decisions.

The research initially on the portfolio, was only limited to single-period portfolio problems. In the actual investment environment, the distribution of asset returns often changes with different stages. The article of Mossin [33] published in 1968 was the first to consider multi-stage portfolio problems, he extended the single-stage model of Markowitz to multi-stage situation with the dynamic programming methods. In 1972, Merton [29] gave the standard form of the analytical solution of Markowitz mean-variance model. Since then, Merton [27, 28] and Samuleson [40] constructed a reference frame of multi-stage model. Chen, Jen and Zionts [4] improved Smith's model and extended to the multi-stage situation. Hakansson [17] gave the analysis of multi-stage mean-variance. Mossin [32], Ross [38] studied the mean-variance hedging problem, the conclusion of [32] is obtained based on the following assumptions: all the factors (including interest rates, volatility, etc.) are identified, constant and does not changes over time. In [12], the authors considered and obtained the solution to the variance minimization problem by embedding the constraint equations into the target functions. In 2000, Zhou and Li [49] used stochastic linear quadratic control theory to study the problem of mean-variance optimization, and they obtained the best analytical solution of the optimal investment strategy and the explicit representation of effective frontier of the mean-variance portfolio selection problem. This article is unusual in that the original problem being changed into a stochastic optimal linear quadratic problem, and solved it using linear quadratic theory. The solution of the original problem can be obtained by dealing with the transformed problem. Its contribution is to link the standard portfolio choice problem with stochastic control model, and to provide a general framework to deal with more complex situations.

It is generally believed that the stock price follows geometric Brownian motion, Zhou and Li's research is based on the process of financial asset prices being a continuous process with respect to time. However, in financial practice, the process of financial asset prices is not necessarily a continuous process. For example, subjected to significant information (such as sudden major events, policy changes, etc.), a jump of ups and downs of the stock price will occur. The time and impact of such information occurring is random, therefore, one can describe it by the Poisson process. In this paper, similar to the work by Zhou and Li [49], we study the mean variance portfolio problem based on the asset prices satisfying the jump-diffusion stochastic differential equation, and also arrive at the efficient frontier.

The organization of this paper is as follows. The first two sections briefly introduce some necessary notations and terminologies. Section 3 proposes a model based on a stochastic jump-diffusion differential equation framework. The authors get an analytic solution of the optimal control problem by using the stochastic linear-quadratic method. Similarly, we also arrive at the efficient frontier. The results posed here can be regarded as a natural generalization of Zhou and Li [49].

2. Preliminaries

2.1. Mean-variance models

The portfolio based on the mean-variance that was originally proposed by Markowitz is the core of modern portfolio theory. It uses probability theory and optimization techniques model to get the investment behavior under uncertainty. Investment income is described by the mean income, and investment risk is described by income variance. Assume that the investor chooses n kinds of risk assets to do invest. For convenience, we denote by the following

$R \doteqdot (r_1, r_2, \cdots, r_n)^T$ is the expected return vector of risky assets;

$\sum \doteqdot (\sigma_{ij})_{n\times n}$ is the risk covariance matrix of asset returns;

$x = (x_1, x_2, \cdots, x_n)^T$ is the vector with investment ratio x_i in i-th risk asset;

$l = (1, 1, \cdots, 1)^T$ is the vector with all components being 1;

We call $r(x) = R^T x$, $\sigma^2(x) = x^T \sum x$ as the portfolio expected return, the risk, respectively.

Then, Markowitz's mean-variance model can be expressed in two mathematical models: The first model is as follows

$$\text{Min } x^T \sum x$$
$$\text{s.t.} \begin{cases} R^T x = \mu \\ l^T x = 1 \\ Ax \leq B \end{cases}$$

where μ is the level of expected return given by investor, $A_{m\times n}$ is a constant matrix, B is an n-dimensional constant vector

The second model is as follows

$$\text{Max } R^T x$$
$$\text{s.t.} \begin{cases} x^T \sum x = \sigma^2 \\ l^T x = 1 \\ Ax \leq B \end{cases}$$

where $Ax \leq B$ is the additional constraints that investors or the investment market for investment behaviors, for example, not allowing short selling, restricting the number of investments and so on.

2.2. Efficient frontier

A portfolio is efficient when it meets the following conditions: the first is for a given level of the expectations income, it has minimum risk; the second is for a given level of risk, it has the greatest expected return. All efficient portfolios form the efficient frontier, or namely effective boundary [26].

2.3. Optimal controls

The optimal control is an important part of modern control theory, the key issue of it is how to select control $u(t)$ to make the control system be optimal. The optimal control theory is usually solved after abstracted into mathematical problem, the formulation and related concepts are [47]:

(1) The formulation of optimal control

Given the state equation of the controlled system, a initial state, and a target set, one will find an admissible control that make the system start from the initial state at the initial time t_0 and transfer the state to the target set at the termination time t_f, and minimize the performance index.

(2) The performance index

It is a performance index function that evaluate the control effect or the quality is good or bad.

(3) The allow control

The point set provided by control constraints is called the control domain, a control function that is defined on the closed interval $[t_0, t_f]$, and valued in the same control domain is called allow control.

(4) The target set

The final state constraints, in general, is used to represent the requirements in the final state (state at t_f), the final state constraints provides a time-varying set or a time-invariant set of the state space, the set of states meet the constraints is called the target set.

3. The portfolio based on jump-diffusions

3.1. Assumptions and notations

3.1.1. Assumptions

This study is based on changes of asset prices and obey such a hypothesis: the jump-diffusion model. In the actual financial markets, the uncertainty of asset price is consist of two parts: The first change is price's "normal" fluctuations, such as temporary imbalance between supply and demand, changes of the economic outlook and so on, this change can be described by Brown motion $W(t)$ on the probability space $(\Omega^W, \mathcal{F}^W, \mathbb{P}^W)$, it has a continuous sample path; The second change is the abnormal "vibration" of price, it is due to the arrival of important information, that have a significant impact on stock prices. Generally, this information is about specific companies and industries and have little effect on the entire market, it is "non-systematic" risk. The change can be described by the "jump" that is affected importantly by the response information, the jumping can be represented by the Poisson process $N(t)$ with intensity $\lambda(t)$ on the probability space $(\Omega^N, \mathcal{F}^N, \mathbb{P}^N)$, where $W(t), N(t)$ are independent on $(\Omega, \mathcal{F}, \mathbb{P})$.

3.1.2. Notations

We mainly adopt the notations and terminologies from [49] here. For convenience, we also write down some necessary notations and terminologies in this subsection as follows:
A^T is the transpose of any vector or matrix A;
A_j is the j-th-antry of any vector A;
$|A| \doteq \sqrt{\sum_{i,j} a_{ij}^2}$ for any matrix or vector $A = (a_{ij})$;
$\mathbb{S}^n$ is the space of all $n \times n$ symmetric matrices;
$\mathbb{S}_+^n$ is a subspace of all non-negative definite matrices in $\mathbb{S}^n$;
$\hat{\mathbb{S}}_+^n$ denotes the subspace of all positive definite matrices in $\mathbb{S}^n$;
$C([0, T], X)$ is a Banach space associated with X-valued continuous functions defined on $[0, T]$ endowed with the maximum norm $\|\cdot\|$ for a given Hilbert space X;
$L^2([0, T]; X)$ is the L^2-integrable function endowed with the norm as $\left(\int_0^T \|f(t)\|_X^2 dt\right)^{\frac{1}{2}}$;
$L_{\mathcal{F}}^2(0, T; \mathbb{R}^m)$ is a collection of measurable random process $f(t)$ adapted to the field-flow $\{\mathcal{F}\}_{t\geq 0}$ and $f(t)$ satisfies $E\int_0^T |f(t)|^2 dt < +\infty$;
$(\Omega, \mathcal{F}, \mathbb{P}, \{\mathcal{F}_t\}_{t\geq 0})$ is the full probability space with field-flow field $\{\mathcal{F}_t\}_{t\geq 0}$;
$W(t) = (W_1(t), W_2(t), \cdots, W_n(t))^T$ is the standard Brown motion defined on complete probability space.

3.2. Model descriptions

Suppose that there are $m + 1$ kinds of securities on the market, where there is one risk-free security, its price process $P_0(t)$ satisfies

$$\begin{cases} dP_0(t) = r(t)P_0(t)dt, t \in [0, T] \\ P_0(0) = P_0 > 0 \end{cases} \tag{1}$$

where $r(t)$ is the risk-free interest rate. On the other hand, the price process of m kinds of risky securities $P_1(t), P_2(t), \cdots, P_m(t)$ satisfy the following jump-diffusion stochastic differential equations

$$\begin{cases} dP_i(t) = P_i(t)\Big[b_i(t)dt + \sum\limits_{j=1}^m \sigma_{ij}(t)dW_j(t) + \sum\limits_{k=1}^m \varphi_{ik}(t)dN_k(t)\Big], t \in [0, T] \\ P_i(0) = P_i \end{cases} \tag{2}$$

where $b_i(t) > 0$ are the appreciation return rate; $\sigma_i(t) = (\sigma_{i1}(t), \cdots, \sigma_{im}(t)) : [0, T] \to \mathbb{R}^m$ is the volatility, $\varphi_i(t) = (\varphi_{i1}(t), \cdots, \varphi_{im}(t)) : [0, T] \to \mathbb{R}^m$ is the jump amplitude.

Assume that the investor has total assets $x(t)$ at t, and the share of the *ith* asset $(i = 0, 1, \cdots, m)$ at the time t is $N_i(t)$, then one gets

$$x(t) = \sum_{i=0}^m N_i(t)P_i(t), t \geq 0 \tag{3}$$

Assume that the transactions are continuous, the transaction costs and consumptions are not considered here, then there holds

$$\begin{cases} dx(t) = \sum\limits_{i=0}^m N_i(t)dP_i(t) = \{r(t)x(t) + \sum\limits_{i=1}^m [b_i(t) - r(t)]u_i(t)\}dt + \sum\limits_{j=1}^m \sum\limits_{i=1}^m \sigma_{ij}(t)u_i(t)dW_j(t) + \sum\limits_{k=1}^m \sum\limits_{i=1}^m \varphi_{ik}(t)u_i(t)dN_k(t) \\ x(0) = x_0 > 0 \end{cases} \tag{4}$$

where $u_i(t) = N_i(t)P_i(t), i = 0, 1, 2, \cdots, m$, $u(t) = (u_1(t), \cdots, u_m(t))^T$ is said to be a portfolio.

Denote by $J_1(u(\cdot)) = -E[x(T)], J_2(u(\cdot)) = Var[x(T)]$, then we can propose the following

Definition 3.1. A portfolio $u(\cdot)$ is said to be *admissible* if $u(\cdot) \in L^2_{\mathcal{F}}(0,T;\mathbb{R}^m)$.

Definition 3.2. For an admissible portfolio $\bar{u}(\cdot)$, if there doesn't exist any admissible portfolio $u(\cdot)$ such that $J_1(u(\cdot)) \leq J_1(\bar{u}(\cdot)), J_2(u(\cdot)) \leq J_2(\bar{u}(\cdot))$, where there is at least one inequality holds strictly, then we say $\bar{u}(\cdot)$ is an effective investment portfolio, $(J_1(\bar{u}(\cdot)), J_2(\bar{u}(\cdot))) \in \mathbb{R}^2$ is called an efficient point, and all efficient points constitutes the efficient frontier.

Consider the following optimal control problem $P(\mu)$:

$$\begin{aligned} &\text{Min} J_1(u(\cdot)) + \mu J_2(u(\cdot)) = -E[x(T)] + \mu Var[x(T)], \mu > 0 \\ &\text{s.t.} \begin{cases} u(\cdot) \in L^2_{\mathcal{F}}(0,T;\mathbb{R}^m) \\ (x(\cdot), u(\cdot)) \text{ meets (4)} \end{cases} \end{aligned} \tag{5}$$

Definition 3.3. $U_{P(\mu)} = \{u(\cdot) | u(\cdot) \text{ is an optimal control in } P(\mu)\}$

3.3. The equivalent transformation of $P(\mu)$

We will first convert $P(\mu)$ to a standard stochastic linear quadratic problem $P(\mu,\lambda)$ as follows

$$\begin{aligned} &\text{Min} J(u(\cdot);\mu,\lambda) = E\{\mu x(T)^2 - \lambda x(T)\}, \mu > 0, -\infty < \lambda < +\infty \\ &\text{s.t.} \begin{cases} u(\cdot) \in L^2_{\mathcal{F}}(0,T;\mathbb{R}^m) \\ (x(\cdot), u(\cdot)) \text{ meets (4)} \end{cases} \end{aligned} \tag{6}$$

Definition 3.4. $U_{P(\mu,\lambda)} = \{u(\cdot) | u(\cdot) \text{ is an optimal control in } P(\mu,\lambda)\}$

3.4. The stochastic linear quadratic control

Considering the following linear stochastic differential equations:

$$\begin{cases} dx(t) = \Big(A(t)x(t) + B(t)u(t) + f(t)\Big)dt + \sum\limits_{j=1}^{m} D_j(t)u(t)dW_j(t) \\ x(0) = x_0 \in \mathbb{R}^n \end{cases} \tag{7}$$

where x_0 is the initial state, $W(t)$ is a given m-dimensional Brown motion on a given filtered probability space $(\Omega, \mathcal{F}, \mathbb{P}, \{\mathcal{F}_t\}_{t\geq 0})$, $u(\cdot) \in L^2_{\mathcal{F}}(0,T;\mathbb{R}^m)$ is a control. For each $u(\cdot) \in L^2_{\mathcal{F}}(0,T;\mathbb{R}^m)$, the corresponding quadratic objective functional is

$$J(u(\cdot)) = E\{\int_0^T \frac{1}{2}(x^T(t)Q(t)x(t) + u^T(t)R(t)u(t))dt + \frac{1}{2}x^T(T)Hx(T)\} \tag{8}$$

where $R(t) \in C([0,T];\mathbb{S}^m), H \in \mathbb{S}^n_+, Q \in C([0,T];\mathbb{S}^n_+)$.

By using the classical optimal control theory [47], we get the following Riccati equations

$$\begin{cases} \dot{P}(t) = -P(t)A(t) - A^T(t)P(t) - Q(t) + P(t)B(t)\Big(R(t) + \sum\limits_{j=1}^{m} D_j^T(t)P(t)D_j(t)\Big)^{-1}B^T(t)P(t) \\ P(T) = H \\ K(t) \doteqdot R(t) + \sum\limits_{j=1}^{m} D_j^T(t)P(t)D_j(t) > 0, \forall t \in [0,T] \end{cases} \tag{9}$$

$$\begin{cases} \dot{g}(t) = -A^T(t)g(t) + P(t)B(t)\Big(R(t) + \sum\limits_{j=1}^{m} D_j^T(t)P(t)D_j(t)\Big)^{-1}B^T(t)g(t) - P(t)f(t) \\ g(T) = 0 \end{cases} \tag{10}$$

where $P(t)$ is a positive semi-definite matrix, $B, D_j \in C([0,T];\mathbb{R}^{n\times m}), A \in C([0,T];\mathbb{R}^n)$.

Theorem 3.5. ([49]) *If $P \in C([0,T]; S^n_+)$, $g \in C([0,T]; \mathbb{R}^n)$ are the solutions of (9) and (10), respectively, then the problem (7) and (8) have an optimal control*

$$u^*(t,x) = -(R(t) + \sum_{j=1}^{m} D_j^T(t)P(t)D_j(t))^{-1}B^T(t)(P(t)x + g(t)) \tag{11}$$

and the optimal value function is

$$J^* = \frac{1}{2}\int_0^T \Big(2f^T(t)g(t) - g(t)B(t)[R(t) + \sum_{j=1}^{m} D_j^T(t)P(t)D_j(t)]^{-1}B^T(t)g(t)\Big)dt + \frac{1}{2}x_0^T P(0)x_0 + x_0 g(0) \tag{12}$$

3.5. The solution of equivalent problems

Considering the dynamic equation of assets

$$\begin{cases} dx(t) = [A(t)x(t) + B(t)u(t) + f(t)]dt + \sum\limits_{j=1}^{m} D_j(t)u(t)dW_j(t) + \sum\limits_{j=1}^{m} \Phi_k(t)u(t)dN_k(t) \\ x(0) = x_0 \end{cases} \tag{13}$$

For any $u(\cdot) \in L^2_{\mathcal{F}}(0,T;\mathbb{R}^m)$, there holds the associated cost functional

$$J(u(\cdot)) = E\{\int_0^T \frac{1}{2}(x^T(t)Q(t)x(t) + u^T(t)R(t)u(t))dt + \frac{1}{2}x^T(T)H(T)x(T)\} \tag{14}$$

For convenience, we give a generalized Itô formula as a Lemma 3.6 as follows

Lemma 3.6. ([31](Generalized Ito Formula)) *Assume that $X(t)$ is a d-dimensional semi-martingale satisfying*

$$X_i(t) = X_i(0) + \int_0^t f_i(s)dW_s + \int_0^t g_i(s)dM_s$$

where W is a Brown motion, M is a martingale related to the Poisson process N with intensity λ, then there is

$$\begin{aligned} F(t,X(t)) - F(0,X(0)) = & \int_0^t \frac{\partial F}{\partial s}(s,X(s))ds + \sum_{i=1}^{d}\int_0^t \frac{\partial F}{\partial x_i}(s,X(s))f_i(s)dW(s) + \frac{1}{2}\sum_{i,j}\int_0^t \frac{\partial^2 F}{\partial x_i \partial x_j} f_i(s)f_j(s)ds \\ + & \int_0^t [F(s,X_{s-} + g(s)) - F(s,X_{s-})]dM(s) + \int_0^t \{F(s,X_s + g(s)) - F(s,X_s) \\ - & \sum_i \frac{\partial F}{\partial x_i}(s,X_s)g_i(s)\}\lambda(s)ds \end{aligned}$$

From Lemma 3.6, we know that the Poisson counting process N (the jump of assets) with intensity λ can be decomposed into two parts: an non-stochastic part and a stochastic part. In other words, by using Lemma 3.6, we arrive at

$$dx(t) = [A(t)x(t) + B(t)u(t) + f(t) + \sum_{k=1}^{m}\Phi_k(t)u(t)\lambda_k(t)]dt + \sum_{j=1}^{m} D_j(t)u(t)dW_j(t) + \sum_{k=1}^{m}\Phi_k(t)u(t)dN_k(t) \tag{15}$$

where $\lambda_k(t)$ is the no-stochastic jump part, $\Phi_k(t)u(t)dN_k(t)$ is the stochastic-jump part.

For the problem Min$J(u(\cdot);\mu,\lambda) = E\{\mu x(T)^2 - \lambda x(T)\}$, it can be simplified as a standard stochastic linear quadratic objective function

$$E\{\mu x(T)^2 - \lambda x(T)\} = E\{\mu[x(T) - \frac{\lambda}{2\mu}]^2 - \frac{\lambda^2}{4\mu}\}.$$

Let $\gamma = \frac{\lambda}{2\mu}$, $y(t) = x(t) - \gamma$, then we know that

$$E\{\mu x(T)^2 - \lambda x(T)\} = E\{\mu y(T)^2 - \frac{\lambda^2}{4\mu}\}.$$

This implies that the problem $P(\mu, \lambda)$ is equivalent to minimizing $E[\frac{1}{2}\mu y(T)^2]$ subject to

$$\begin{cases} dy(t) = [A(t)y(t) + B(t)u(t) + f(t) + \sum_{k=1}^{m} \Phi_k(t)u(t)\lambda_k(t)]dt + \sum_{j=1}^{m} D_j(t)u(t)dW_j(t) + \sum_{k=1}^{m} \Phi_k(t)u(t)dN_k(t) \\ y(0) = x_0 - \gamma \end{cases} \quad (16)$$

By using a direct and complex computation and by virtue of the proof similar to the argument posed by [49], we can write down the optimal control for (11) as follows

$$\bar{u}(t, y) = (\bar{u}_1(t, y), \cdots, \bar{u}_m(t, y)) = -(\sigma(t)\sigma^T(t) + \phi(t)\phi^T(t))^{-1}(B(t) + \sum_{k=1}^{m} \Phi_k(t)\lambda_k(t))^T(y + \frac{g(t)}{f(t)}) \quad (17)$$

In fact, roughly speaking, by introducing the stochastic Riccati equation

$$\begin{cases} \dot{P}(t) = -P(t)A(t) - A^T(t)P(t) - Q(t) + P(t)(B(t) + \sum_{k=1}^{m} \Phi_k(t)\lambda_k(t))\Big(R(t) \\ \qquad + \sum_{j=1}^{m} D_j^T(t)P(t)D_j(t) + \sum_{k=1}^{m} \Phi_k^T(t)P(t)\Phi_k(t)\Big)^{-1}(B(t) + \sum_{k=1}^{m} \Phi_k(t)\lambda_k(t))^T P(t) \\ P(T) = H \\ K(t) = R(t) + \sum_{j=1}^{m} D_j^T(t)P(t)D_j(t) + \sum_{k=1}^{m} \Phi_k^T(t)P(t)\Phi_k(t) > 0, \forall t \in [0, 1] \end{cases} \quad (18)$$

along with an equation

$$\begin{cases} \dot{g}(t) = -A^T(t)g(t) + P(t)(B(t) + \sum_{k=1}^{m} \Phi_k(t)\lambda_k(t))\Big(R(t) + \sum_{j=1}^{m} D^T(t)P(t)D_j(t) \\ \qquad + \sum_{k=1}^{m} \Phi_k^T(t)P(t)\Phi_k(t)\Big)(B(t) + \sum_{k=1}^{m} \Phi_k(t)\lambda_k(t))^T g(t) - P(t)f(t) \\ g(T) = 0 \end{cases} \quad (19)$$

In order to obtain the optimal feedback control to (18), one can simplify Equation (18) by letting

$$\rho(t) = (B(t) + \sum_{k=1}^{m} \Phi_k(t)\lambda_k(t))\Big(\sum_{j=1}^{m} D_j^T(t)D_j(t) + \sum_{k=1}^{m} \Phi_k^T(t)\Phi_k(t)\Big)^{-1}(B(t) + \sum_{k=1}^{m} \Phi_k(t)\lambda_k(t))^T$$

and $(Q(t), R(t)) = (0, 0)$, $H = \mu$, $\sigma(t) = (D_i(t), \cdots, D_m(t))$, $\phi(t) = (\Phi_1(t), \cdots, \Phi_m(t))$, then one gets the simplified equation as follows

$$\begin{cases} \dot{P}(t) = -2r(t)P(t) + \rho(t)P(t) \\ P(T) = \mu \\ P(t)[\sigma^T(t)\sigma(t) + \phi^T(t)\phi(t)] > 0 \end{cases} \quad (20)$$

along with an equation

$$\begin{cases} \dot{g}(t) = (\rho(t) - r(t))g(t) - \gamma r(t)P(t) \\ g(T) = 0 \end{cases} \quad (21)$$

It is easy to see that there is a solution to Equation (20) as $P(t) = \mu e^{-\int_t^T(\rho(s)-2r(s))ds}$. Then, we get the optimal control (17) as

$$\bar{u}(t, y) \;=\; (\bar{u}_1(t, y), \cdots, \bar{u}_m(t, y)) = -[\sigma(t)\sigma^T(t) + \phi(t)\phi^T(t)]^{-1}\Big(B(t) + \sum_{k=1}^{d} \Phi_k(t)\lambda_k(t)\Big)^T(y + \frac{g(t)}{P(t)}) \quad (22)$$

Simplifying (22) with a similar argument posed in [49], then there holds the optimal control

$$\bar{u}(t,x) = (\bar{u}_1(t,x),\cdots,\bar{u}_m(t,x)) = [\sigma(t)\sigma^T(t)+\phi(t)\phi^T(t)]^{-1}(B(t)+\phi(t)\lambda(t))^T(\gamma e^{-\int_t^T r(s)ds}-x) \tag{23}$$

This optimal control generates the formula (5.12) in [49].

4. Efficient frontier

In this subsection we will give out the efficient frontier for the jump-diffusion model, which is similar to that posed by Zhou and Li in [49]. First, by a series of computation similar to [49], we have the following

$$\begin{aligned}
dx(t) &= \{r(t)x(t)+\sum_{i=1}^m[b_i(t)-r(t)]u_i(t)\}dt+\sum_{j=1}^m\sum_{i=1}^m\sigma_{ij}(t)u_i(t)dW_j(t)+\sum_{k=1}^m\sum_{i=1}^m\phi_{ik}(t)u_i(t)dN_k(t)\\
&= \{r(t)x(t)+B(t)[\sigma(t)\sigma(t)^T+\phi(t)\phi(t)^T]^{-1}(B(t)+\phi(t)\lambda(t)^T)(\gamma e^{-\int_t^T r(s)ds}-x)\\
&+ \phi(t)\lambda(t)[\sigma(t)\sigma(t)^T+\phi(t)\phi(t)^T]^{-1}(B(t)+\phi(t)\lambda(t)^T)(\gamma e^{-\int_t^T r(s)ds}-x)\}dt\\
&+ \sigma(t)[\sigma(t)\sigma(t)^T+\phi(t)\phi(t)^T]^{-1}(B(t)+\phi(t)\lambda(t)^T)(\gamma e^{-\int_t^T r(s)ds}-x)dW(t)\\
&+ \phi(t)[\sigma(t)\sigma(t)^T+\phi(t)\phi(t)^T]^{-1}(B(t)+\phi(t)\lambda(t)^T)(\gamma e^{-\int_t^T r(s)ds}-x)dN(t)\\
&\doteq \{(r(t)-\rho(t))x(t)+\gamma e^{-\int_t^T r(s)ds}\rho(t)\}dt\\
&+ (B(t)+\phi(t)\lambda(t))(\sigma(t)\sigma(t)^T+\phi(t)\phi(t)^T)^{-1}\sigma(t)(\gamma e^{-\int_t^T r(s)ds}-x(t))dW(t)\\
&+ (B(t)+\phi(t)\lambda(t))(\sigma(t)\sigma(t)^T+\phi(t)\phi(t)^T)^{-1}\phi(t)(\gamma e^{-\int_t^T r(s)ds}-x(t))dN(t)\\
x(0) &= x_0
\end{aligned} \tag{24}$$

where $\rho(t)=(B(t)+\phi(t)\lambda(t))[\sigma(t)\sigma^T(t)+\phi(t)\phi^T(t)]^{-1}(B(t)+\phi(t)\lambda(t))^T$. By Îto formula, we have

$$\begin{aligned}
dx^2(t) &= \{(2r(t)-\rho(t))x^2(t)+\gamma^2e^{-2\int_t^T r(s)ds}\rho(t)\}dt+2x(t)(B(t)+\phi(t)\lambda(t))[\sigma(t)\sigma^T(t)+\phi(t)\phi^T(t)]^{-1}\\
&\quad (\sigma(t)+\phi(t))[\gamma e^{-\int_t^T r(s)ds}-x(t)](dW(t)+dN(t))\\
x^2(0) &= x_0^2
\end{aligned} \tag{25}$$

¿From these statements, we know that

$$\begin{cases} dEx(t)=\{(r(t)-\rho(t))Ex(t)+\gamma e^{-\int_t^T r(s)ds}\rho(t)\}dt\\ Ex(0)=x_0\end{cases} \tag{26}$$

$$\begin{cases} dEx^2(t)=\{(2r(t)-\rho(t))Ex^2(t)+\gamma^2e^{-2\int_t^T r(s)ds}\rho(t)\}dt\\ Ex^2(0)=x_0^2\end{cases} \tag{27}$$

Similar to the work by [49], for the final wealth, we get the variance

$$\text{Var}\bar{x}(T)=\frac{1-\beta}{\beta}[(\beta\bar{\gamma}+\alpha x_0)^2-2\frac{\alpha\beta x_0\bar{\gamma}}{1-\beta}+\frac{\beta(\delta-\alpha^2)}{1-\beta}x_0^2] \tag{28}$$

where $\alpha=e^{\int_0^T(r(t)-\rho(t))dt}$, $\beta=1-e^{-\int_0^T\rho(t)dt}$, $\delta=e^{\int_0^T(2r(t)-\rho(t))dt}$, and $\gamma\doteq\frac{\lambda}{2H}$.

The expect, under the optimal control $\bar{u}(t)$, of the final wealth is

$$E\bar{x}(T)=x_0e^{\int_0^T r(t)dt}+\sqrt{\frac{1-e^{-\int_0^T\rho(t)dt}}{e^{-\int_0^T\rho(t)dt}}}\sigma_{\bar{x}(T)} \tag{29}$$

where $\rho(t)$ is defined as above.

Example 4.1. Suppose that there is a market risk-free security, the average annual return rate is $r = 5\%$, there also is a stock with the average annual return rate is $b = 15\%$ and the standard fluctuation rate is $\sigma = 16\%$. Assuming the magnitude of each jump is 1, that is $\phi = 1$, the intensity of jumps is 1, that is $\lambda = 1$. Set $T = 1$ (one year) and $\rho = \frac{(b-r+1)^2}{\sigma^2+\phi^2} = 1.179797$.

By a direct computation, we get the security market line

$$E\bar{x}(1) = x_0 e^{\int_0^1 r(t)dt} + \sqrt{\frac{1 - e^{-\int_0^1 \rho(t)dt}}{e^{-\int_0^1 \rho(t)dt}}}\sigma_{\bar{x}(1)} = x_0 e^{0.05} + 1.501238\sigma_{\bar{x}(1)}$$

Assume now there is an investor who has initial assets $x_0 = 100$, the expected return rate is 18% after one year, then we can calculate how much risk that the investor bear under the expected revenue.

By the security market line, when $x_0 = 100$ and $E\bar{x}(1) = 118$, then we arrive at

$$\sigma_{\bar{x}(1)} = \frac{E\bar{x}(1) - x_0 e^{0.05}}{1.501238} = \frac{118 - 100e^{0.05}}{1.501238} = 8.574852$$

The result shows that the standard deviation of investor's target is 8.574852%.

Next, we calculate the investor's portfolio as follows

$$\gamma = \frac{E\bar{x}(T) - \alpha x_0}{\beta} = \frac{118 - 100e^{-1.022}}{1 - e^{-1.072}} = 123.7118$$

Therefore, the investor's asset invested in risky securities is

$$\bar{u}(t, x(t)) = [\sigma(t)\sigma^T(t) + \phi(t)\phi^T(t)]^{-1}(B(t) + \phi(t)\lambda(t))^T(\gamma e^{-\int_t^T r(s)ds} - x) = 1.072(123.7118e^{0.05(t-1)} - x(t))$$

In the initial time $t = 0, \bar{u}(0, x_0) = 18.96076$, this implies that one can buy the risky security with his initial assets without borrowing additional money.

Excluding the occurrence of jumps, that is, we use the general diffusion model to describe fluctuations of the market price, at the same time there is a risk-free security, the average annual return rate is $r = 5\%$, there is an stock, its average annual return rate is $b = 15\%$ and the standard fluctuation rate is $\sigma = 16\%$, set $T = 1$, then there holds

$$\rho_1(t) = \frac{(b(t) - r(t))^2}{\sigma^2(t)} = \frac{(0.15 - 0.05)^2}{0.16^2} = 0.3906$$

$$E\bar{x}_1(1) = x_0 e^{\int_0^1 r(t)dt} + \sqrt{\frac{1 - e^{-\int_0^1 \rho_1(t)dt}}{e^{-\int_0^1 \rho_1(t)dt}}}\sigma_{\bar{x}_1(1)} = x_0 e^{0.05} + \sqrt{\frac{1 - e^{-0.3906}}{e^{-0.3906}}}\sigma_{\bar{x}_1(1)} = x_0 e^{0.05} + 0.6912\sigma_{\bar{x}_1(1)}$$

Suppose that an investor has a initial asset $x_0 = 100$, the expected return rate is 18% after one year, let's calculate how much risk at this expected level. By $E\bar{x}_1(1) = x_0 e^{0.05} + 0.6912\sigma_{\bar{x}_1(1)}$, there is

$$\sigma_{\bar{x}_1(1)} = \frac{E\bar{x}_1(1) - x_0 e^{0.05}}{0.6912} = 18.6240$$

This shows that the expected standard deviation of the investor is 18.6240%. Now, we look at his portfolio choice. Since

$$\gamma = \frac{118 - 100e^{0.05-0.3906}}{1 - e^{-0.3906}} = 144.9604$$

then we get

$$\bar{u}_1(t, x) = \frac{(b(t) - r(t))^2(\gamma e^{-\int_t^T r(s)ds} - x)}{\sigma_1^2(t)} = \frac{0.01(144.9604e^{-0.05} - x)}{0.16^2} = 0.3906(144.9604e^{0.05(t-1)} - x)$$

Considering the case $t = 0$, then one gets $\bar{u}_1(0, x) = 14.80$, this result shows that the investor should loan 14.80 (yuan) at the initial time with his own initial principal 100 (yuan) all to invest in risky securities. Compare to the results under the two models, showing that when the asset price volatility in the market contains jumps, the investor will bear a larger bit of risks under the same expected revenue.

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