

The optimal convergence factor of the gradient based iterative algorithm for linear matrix equations

Xiang Wang^a, Dan Liao^a

^a*Department of Mathematics, Nanchang University, Nanchang 330031, China*

Abstract. A hierarchical gradient based iterative algorithm of [L. Xie et al., *Computers and Mathematics with Applications* 58 (2009) 1441-1448] has been presented for finding the numerical solution for general linear matrix equations, and the convergent factor has been discussed by numerical experiments. However, they pointed out that how to choose a best convergence factor is still a project to be studied. In this paper, we discussed the optimal convergent factor for the gradient based iterative algorithm and obtained the optimal convergent factor. Moreover, the theoretical results of this paper can be extended to other methods of gradient-type based. Results of numerical experiments are consistent with the theoretical findings.

1. Introduction

Numerical methods for solving matrix equations become interesting as soon as they play an important role in various fields, such as neural network [18], model reduction [14] and image processing [1] etc. Recently, iterative approaches for solving matrix equations and recursive identification for parameter estimation have received much attention, e.g., ([6]-[10], [11], [15]-[16], [20]-[22]). In [17], Xie et al. presented an efficient gradient based iterative algorithms for solving a class of matrix equations by applying the hierarchical identification principle ([2]-[4]) and the convergence properties of the method are investigated. Because the convergent rate relies on the convergent factor μ and the larger the convergent factor μ , the faster the convergent rate of algorithm. However, when the convergent factor μ is too large, the algorithm will diverge. Therefore, there exists a best convergent factor μ . In [17], the authors pointed out that how to choose a best convergent factor μ is still a project to be studied. In this paper, we derived the optimal convergent factor. Results of numerical experiments verify the theoretical findings.

The paper is organized as follows: In Section 2, we introduce the gradient based iterative algorithm for a class of general linear matrix equations proposed by Xie et al. in [17]. The main results are presented in Section 3. Finally, we present three numerical experiments to verify the theoretical finding.

2010 *Mathematics Subject Classification.* Primary 15A24; Secondary 65F20, 65F22, 65K10

Keywords. Linear matrix equations, optimal convergent factor, gradient based iterative algorithm

Received: 11 April 2011; Accepted: 14 August 2011

Communicated by Dragana Cvetković-Ilić

This work is supported by NNSF of China No. 11101204, NSF of Jiangxi, China No. 20114BAB201004, Science Funds of The Education Department of Jiangxi Province No. GJJ12011, and the Scientific Research Foundation of Graduate School of Nanchang University with No. YC2011-S007

Email address: wangxiang49@ncu.edu.cn (Xiang Wang)

2. The gradient based iterative algorithm for a class of general linear matrix equations

Considering the following general linear matrix equations of form

$$\sum_{i=1}^p A_i X B_i + \sum_{i=1}^q C_i X^T D_i = F, \tag{1}$$

where $A_i \in R^{r \times m}, B_i \in R^{n \times s}, C_i \in R^{r \times n}$ and $D_i \in R^{m \times s}$ and $F = [f_1, f_2, \dots, f_s] \in R^{r \times s}$ are given constant matrix, $X \in R^{m \times n}$ is the unknown matrix to be solved.

Let us introduce some notations which are used in [17]. The symbol I or I_n stands for an identity matrix of appropriate sizes or size $n \times n$. For two matrices M and $N, M \otimes N$ is their Kronecker product; for an $m \times n$ matrix $X = [x_1, x_2, \dots, x_n] \in R^{m \times n}, x_k \in R^m, \text{col}[X]$ is an mn -dimensional vector formed by the columns of X , i.e. $\text{col}[X] = [x_1^T, x_2^T, \dots, x_n^T]^T \in R^{mn}$.

Referring to Al Zhou and Kilicman’s work [19], the $mn \times mn$ square matrix is defined by

$$P_{mn} = \sum_{i=1}^m \sum_{j=1}^n E_{ij} \otimes E_{ij}^T, \tag{2}$$

where $E_{ij} = e_i e_j^T$ called an elementary matrix of order $m \times n$, and $e_i (e_j)$ is a column vector with a unity in the i th(j th) positions and zeros elsewhere of order $m \times 1 (n \times 1)$. According to the definition above, we have $P_{mn} \text{col}[X^T] = \text{col}[X], P_{nm} P_{mn} = I_{mn}, P_{mn}^T = P_{mn}^{-1} = P_{nm}$.

In [17], the authors presented the following lemma.

Lemma 2.1. *Let $S = \sum_{i=1}^p B_i^T \otimes A_i + \sum_{i=1}^q (D_i^T \otimes C_i) P_{nm}$, then (1) has a unique solution if and only if $\text{rank}[S, \text{col}[F]] = \text{rank}[S] = \min(\text{i.e., } S \text{ has a full column rank})$. In this case, the unique solution is given by $\text{col}[X] = (S^T S)^{-1} S^T \text{col}[F]$, and the corresponding homogeneous matrix equation in (1) with $F = 0$ has a unique solution $X = 0$.*

By Lemma 2.1, we can obtain the solution of (1) but it requires excessive computer memory. Therefore, the iterative methods are preferred.

In [17], the authors presented the following gradient based iterative algorithm for solving (1) by applying the hierarchical identification principle.

Algorithm 1: The gradient based iterative algorithm for Eq. (1)

1. $X(k) = \frac{1}{p+q} [\sum_{j=1}^p X_j(k) + \sum_{l=1}^q X_{p+l}(k)],$
2. $X_j(k) = X(k-1) + \mu A_j^T [F - \sum_{i=1}^p A_i X(k-1) B_i - \sum_{i=1}^q C_i X^T(k-1) D_i] B_j^T,$
3. $X_{p+l}(k) = X(k-1) + \mu D_l [F - \sum_{i=1}^p A_i X(k-1) B_i - \sum_{i=1}^q C_i X^T(k-1) D_i]^T C_l.$

The algorithm above has been proved to be convergent under certain conditions.

Lemma 2.2. ([17]) *If the equation in (1) has a unique solution X and the convergent factor μ satisfies the following condition*

$$0 < \mu < 2 \left(\sum_{j=1}^p \lambda_{\max}[A_j A_j^T] \lambda_{\max}[B_j^T B_j] + \sum_{l=1}^q \lambda_{\max}[C_l C_l^T] \lambda_{\max}[D_l^T D_l] \right)^{-1},$$

then the iterative solution $X(k)$ given by Algorithm 1 converges to X , i.e., $\lim_{k \rightarrow \infty} X(k) = X$; or, the error $X(k) - X$ converges to zero for any initial value $X(0)$.

Denoted by $\widetilde{X}(k) = X(k) - X$, we can get the error equation of Eq. (1) (see [17]) as

$$\text{col}[\widetilde{X}(k)] = (I_{mn} - \frac{\mu}{p+q}\Phi)\text{col}[\widetilde{X}(k-1)], \tag{3}$$

where

$$\Phi \equiv \sum_{j=1}^p \sum_{i=1}^p B_j B_i^T \otimes A_j^T A_i + \sum_{l=1}^q \sum_{i=1}^q C_l^T C_i \otimes D_l D_i^T + \sum_{j=1}^p \sum_{i=1}^q (B_j D_i^T \otimes A_j^T C_i) P_{nm} + \sum_{l=1}^q \sum_{i=1}^p (C_l^T A_i \otimes D_l B_i^T) P_{nm}.$$

According to Eq. (3), we can easily get the following lemma.

Lemma 2.3. *If the equation in (1) has a unique solution X , then Algorithm 1 converges for any initial value $X(0)$ if and only if the spectrum radius of the matrix $I_{mn} - \frac{\mu}{p+q}\Phi$ is less than one, that is, $\rho(I_{mn} - \frac{\mu}{p+q}\Phi) < 1$; If the matrix Φ is positive definite, then Algorithm 1 converges if and only if the convergent factor satisfies the following condition*

$$0 < \mu < \frac{2(p+q)}{\lambda_{\max}(\Phi)}.$$

In other words, the closer the spectrum radius of $I_{mn} - \frac{\mu}{p+q}\Phi$ is to 0, the faster the error $\widetilde{X}(k)$ converges to zero. However, in [17] the authors didn't discuss that how to choose a best convergent factor μ , and pointed out that which is a project to be studied in the future.

In the next section, we will discuss this problem and derive the optimal convergent factor formula.

3. The selection of optimal convergent factor

Firstly, we present the following lemma, which is Theorem 3.4 in [19].

Lemma 3.1. *Let P_{mn} be the $nm \times nm$ matrix defined by (2), $A \in R^{n \times m}$, $B \in R^{m \times n}$, then we have $P_{mn}(A \otimes B)P_{mn} = B \otimes A$.*

According to Lemma 3.1, it is trivial to prove that the matrix Φ is a symmetric matrix. The following lemma plays an important role in determining the optimal convergent factor μ , and the proof is similar with Lemma 5 of [12].

Lemma 3.2. *Let $b, a \in R$, $b > a$ and $\mu > 0$, then we have*

- (a) *If $b > a > 0$, then $\min_{0 < \mu < 2/b} \{\max\{|1 - \mu a|, |1 - \mu b|\}\} = \frac{b-a}{b+a}$, and the minimizer can be reached at the point $\mu = \frac{2}{a+b}$;*
- (b) *If $b > 0 > a$, then $\min_{0 < \mu} \{\max\{|1 - \mu a|, |1 - \mu b|\}\} > 1$, that is, for any $\mu > 0$ we have $\max\{|1 - \mu a|, |1 - \mu b|\} > 1$;*
- (c) *If $a < b < 0$, then $\min_{0 < \mu} \{\max\{|1 - \mu a|, |1 - \mu b|\}\} > 1$, and for any $\mu > 0$ we have $\max\{|1 - \mu a|, |1 - \mu b|\} > 1$.*

Proof. (a) If $b > a > 0$, then we have

$$\max\{|1 - \mu a|, |1 - \mu b|\} = \begin{cases} 1 - \mu a, & \mu \leq \frac{2}{a+b}, \\ \mu b - 1, & \mu \geq \frac{2}{a+b}, \end{cases}$$

which implies $1 - \mu a \geq 1 - \frac{2}{a+b}a = \frac{b-a}{b+a}$ and $\mu b - 1 \geq \frac{2}{a+b}b = \frac{b-a}{b+a}$, i.e., $\min_{0 < \mu < 2/b} \{\max\{|1 - \mu a|, |1 - \mu b|\}\} = \frac{b-a}{b+a}$, and the minimizer can be reached at the point $\mu = \frac{2}{a+b}$. The proof of (b) and (c) is trivial. \square

The following lemma is an immediately subsequence of Lemma 3.2.

Lemma 3.3. *If Algorithm 1 converges to the solution of Eq. (1) for any initial value $X(0)$, then the matrix Φ must be positive definite and so is symmetric positive definite.*

Now we can present the main result of this paper.

Theorem 3.4. *If the matrix Φ is negative definite and indefinite, then Algorithm 1 will diverge for some initial values, otherwise if $0 < \mu < \frac{2(p+q)}{\lambda_{\max}(\Phi)}$ it will converge and in this case the optimal convergent factor should be*

$$\mu_{\text{optimal}} = \frac{2(p+q)}{\lambda_{\min}(\Phi) + \lambda_{\max}(\Phi)}. \quad (4)$$

Proof. The first part is trivial by Lemma 3.1. According to (3), we can see that the optimal convergent factor μ should be chosen to minimize the spectrum radius of the matrix Φ . As Φ is symmetric positive definite, the spectrum radius can be obtained by the following formula

$$\max\left\{\left|1 - \frac{\mu}{p+q}\lambda_{\min}(\Phi)\right|, \left|1 - \frac{\mu}{p+q}\lambda_{\max}(\Phi)\right|\right\},$$

which is less than one when $0 < \mu < \frac{2(p+q)}{\lambda_{\max}(\Phi)}$, and so Algorithm 1 will converge. Then by Lemma 3.1, the optimal convergent factor can be achieved by (4). \square

Remark 1. F. Ding et al. presented the gradient based algorithm for Sylvester matrix equations in [5] and Q. Niu et al. presented the relaxed gradient based algorithm for the same matrix equations in [13]. The authors only discussed the convergent factor μ by numerical experiments, and pointed out that how to choose the best convergent factor in [6] and the best relaxed factor in [13] is very difficult and is a subject to be studied in the future. In fact, by similar idea, we can get the optimal convergent factor of [6] and the optimal relaxed factor of [13] easily. As well, we can show Ding's algorithm in [6] and Niu's Algorithm in [13] are completely equivalent (both numerical and mathematical) if take $\mu_d = \omega(1 - \omega)\mu_n$, where μ_d and μ_n are the convergent factors of Ding's algorithm and Niu's algorithm respectively.

Remark 2. According to Theorem 3.4, to achieve good convergence, eigenvalues estimates are required in order to obtain the optimal or a near-optimal μ , and this may cause difficulties. In addition, when $\lambda_{\max}(\Phi)$ is very large, the curve $\rho(I_{mn} - \frac{\mu}{p+q}\Phi)$ can be extremely sensitive near the optimal value of μ . These observations are common to many iterative approaches that depend on an acceleration parameter.

4. Numerical examples

This section gives two examples which are the same as those in [17] to verify the theoretical findings.

Example 1. ([17]) Suppose that $AX + X^T B = F$, where

$$A = \begin{pmatrix} 1 & 1 \\ 2 & -1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}, \quad F = \begin{pmatrix} 8 & 8 \\ 5 & 2 \end{pmatrix}.$$

The exact solution of the matrix equations above can be obtained as follows $X = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$. The matrix

$$P_{22} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad \Phi = \begin{bmatrix} 9 & 1 & 1 & 1 \\ 1 & 6 & 0 & 0 \\ 1 & 0 & 3 & 2 \\ 1 & 0 & 2 & 2 \end{bmatrix}.$$

By the Theorem 3.4, the optimal convergent factor is 0.3961 and $\frac{2(1+1)}{\lambda_{\max}(\Phi)} \approx 0.414$. Take $X(0) = 10^{-6}1_{2 \times 2}$ and $\mu = 0.3961, 0.0961, 0.1961, 0.2961, 0.412$. Applying Algorithm 1 to compute $X(k)$, the iterative errors $\delta := \|X(k) - X\|_F / \|X\|_F$ versus k are shown in Fig. 1 by Matlab command `semilogy`.

According to Fig. 1 and Fig. 2, it is clear that the larger the convergence factor μ , the faster the convergent rate and when the convergent factor μ is taken to be 0.3961, the convergent rate is the fastest. However,

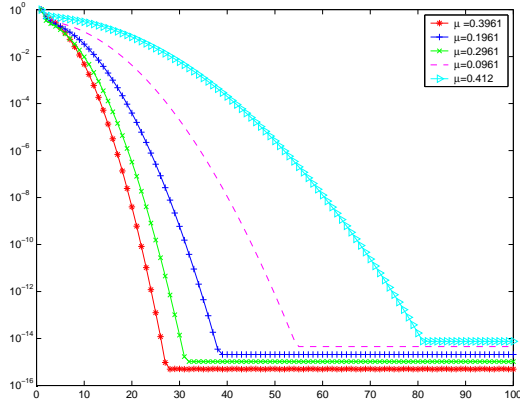


Fig. 1. The comparison of convergent rate for different μ .

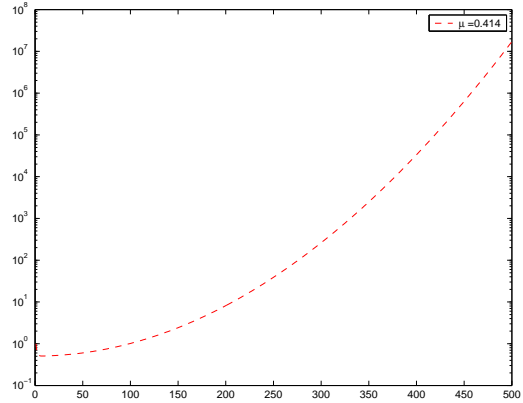


Fig. 2. The convergent curve for $\mu=0.414$.

when the convergent factor μ is greater than 0.3961, the convergent rate will slow and when μ is greater than 0.414, Algorithm 1 will diverge (see Fig. 2), which verifies the theoretical findings.

Also, if we take the termination condition of Algorithm 1 to be the relative error $\delta \leq 10^{-5}$, we can plot the figure of the iterative steps k versus the convergent factor μ (see Fig. 3).

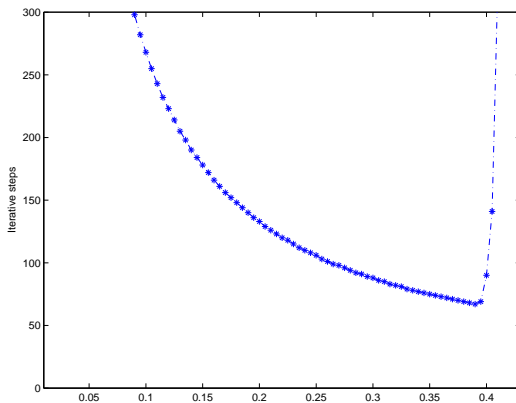


Fig. 3. The iterative steps k versus the convergent factor μ .

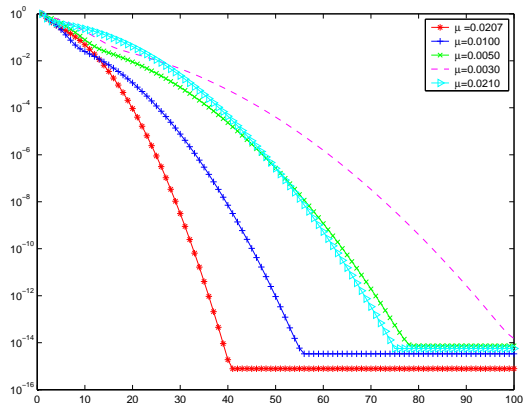


Fig. 4. The comparison of convergent rate for different μ .

Example 2. ([17]) Suppose that $A_1XB_1 + A_2XB_2 + C_1X^TD_1 + C_2X^TD_2 = F$, where

$$A_1 = \begin{pmatrix} 1 & 0 \\ 2 & -1 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0 & 1 \\ 3 & -1 \end{pmatrix}, \quad B_1 = \begin{pmatrix} 2 & -1 \\ 1 & 1 \end{pmatrix}, \quad B_2 = \begin{pmatrix} 3 & -1 \\ 2 & 1 \end{pmatrix},$$

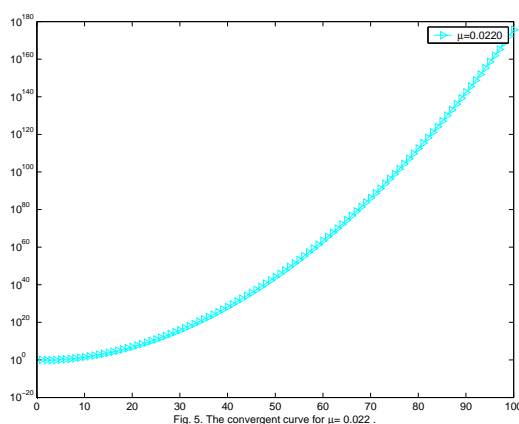
$$C_1 = \begin{pmatrix} 1 & 2 \\ -1 & 2 \end{pmatrix}, \quad C_2 = \begin{pmatrix} -1 & 3 \\ -1 & 2 \end{pmatrix}, \quad D_1 = \begin{pmatrix} 2 & -1 \\ 1 & 2 \end{pmatrix}, \quad D_2 = \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}, \quad F = \begin{pmatrix} 35 & 9 \\ 20 & 7 \end{pmatrix}.$$

The exact solution of the above linear matrix equations is

$$X = \begin{pmatrix} 1 & 2 \\ 3 & 1 \end{pmatrix}.$$

The matrix

$$P_{22} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad \Phi = \begin{bmatrix} 143 & -38 & 133 & -52 \\ -38 & 51 & -28 & 25 \\ 133 & -28 & 289 & -14 \\ -52 & 25 & -14 & 39 \end{bmatrix}.$$



By the Theorem 3.4, the optimal convergent factor is equal to 0.0207 and $\frac{2(2+2)}{\lambda_{\max}(\Phi)} \approx 0.0211$. Take $X(0) = 10^{-6}1_{2 \times 2}$ and $\mu = 0.0207, 0.01, 0.00500, 0.0030, 0.0210$. Applying Algorithm 1 to compute $X(k)$, the iterative errors $\delta := \|X(k) - X\|_F / \|X\|_F$ versus k are shown in Fig. 4 by Matlab command semilogy.

Obviously, when the convergent factor $\mu = 0.0207$, the convergent rate is fastest, and when $\mu \geq 0.0220$ Algorithm 1 will diverge (see Fig. 5), which are consistent with the theoretical analysis.

Acknowledgements

The authors wish to thank the anonymous referee and Professor Dragana Cvetković-Ilić for valuable suggestions that have significantly improved the presentation of this paper.

References

- [1] D. Calvetti, L. Reichel, Application of ADI iterative methods to the restoration of noisy images, *SIAM J. Matrix Anal. Appl.* 17 (1996) 165-186.
- [2] F. Ding, T.-W. Chen, Hierarchical gradient based identification of multivariable discrete-time systems, *Automatica* 41(2) (2005) 315-325.
- [3] F. Ding, T.-W. Chen, Hierarchical identification of lifted state-space models for general dual-rate systems, *IEEE Trans. Circuits. Syst.* 52 (2005) 1179-1187.
- [4] F. Ding, T.-W. Chen, Hierarchical least squares identification methods for multivariable systems, *IEEE Trans. Autom. Control* 50 (2005) 397-402.
- [5] F. Ding, T.-W. Chen, Gradient based iterative algorithms for solving a class of matrix equations, *IEEE Trans. Autom. Control* 50 (2005) 1216-1221.
- [6] F. Ding, T.-W. Chen, On iterative solutions of general coupled matrix equations, *SIAM Journal on Control and Optimization* 44 (2006) 2269-2284.
- [7] F. Ding, T.-W. Chen, Iterative least squares solutions of coupled Sylvester matrix equations, *Systems & Control Letters* 54 (2005) 95-107.
- [8] F. Ding, T.-W. Chen, Performance analysis of multi-innovation gradient type identification methods, *Automatica* 43(1) (2007) 1-14.
- [9] F. Ding, P.-X. Liu, Auxiliary model based multi-innovation extended stochastic gradient parameter estimation with colored measurement noises, *Signal Processing* 89(10) (2009) 1883-1890.
- [10] F. Ding, L. Qiu, T.-W. Chen, Reconstruction of continuous-time systems from their non-uniformly sampled discrete-time systems, *Automatica* 45(2) (2009) 324-332.
- [11] A. Kılıçman, Z. Al Zhou, Vector least-squares solutions for coupled singular matrix equations, *Journal of Computational and Applied Mathematics* 206(2) (2007) 1051-1069.
- [12] Z.-Y. Li, Y. Wang, B. Zhou, G.-R. Duan, Least squares solution with the minimum-norm to general matrix equations via iteration, *Applied Mathematics and Computation* 215 (2010) 3547-3562.
- [13] Q. Niu, X. Wang, L.-Z. Lu, A relaxed gradient based algorithm for solving Sylvester equations, *Asian Journal of Control* 13(3) (2011) 461-464.
- [14] G. Obinata, B.D.O. Anderson, *Model reduction for control system design*, Communications and Control Engineering Series, Springer-Verlag, London, UK, 2001.

- [15] X. Wang, W.-H. Wu, A finite iterative algorithm for solving the generalized (P, Q) -reflexive solution of the linear systems of matrix equations, *Mathematical and Computer Modelling* 54(2011) 2117-2131.
- [16] X. Wang, L. Dai, D. Liao, A modified gradient based algorithm for solving Sylvester equations, *Applied Mathematics and Computation* 218 (2012) 5620-5628.
- [17] L. Xie, J. Ding, F. Ding, Gradient based iterative solutions for general linear matrix equations, *Computers & Mathematics with Applications* 58 (2009) 1441-1448.
- [18] Y.-N. Zhang, D.-C. Jiang, J. Wang, A recurrent neural network for solving Sylvester equation with time-varying coefficients, *IEEE Trans. On Neural Networks* 13(5) (2002) 1053-1063.
- [19] Z. Al Zhoul, A. Kilicman, Some new connections between matrix products for partitioned and non-partioned matrices, *Computers & Mathematics with Applications* 54(6) (2007) 763-784.
- [20] B. Zhou, G.-R. Duan, A new solution to the generalized Sylvester matrix equation $AV - EVF = BW$, *Systems & Control Letters* 55(3) (2006) 193-198.
- [21] B. Zhou, G.-R. Duan, An explicit solution to the matrix equation $AX - XF = BY$, *Linear Algebra and its Applications* 402(1-3) (2005) 345-366.
- [22] B. Zhou, J. Lam, G.-R. Duan, Gradient-based maximal convergence rate iterative method for solving linear matrix equations, *International Journal of Computer Mathematics* 87(3) (2010) 515-527.