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Geodesic mappings of equiaffine and anti-equiaffine general affine connection spaces preserving torsion

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Abstract. In this paper we consider equitorsion geodesic mappings of equiaffine spaces of the θ-kind, $\theta \in \{0, ..., 5\}$. In the case when $\theta \in \{0, 5\}$, vector ψ_i, which determines that mapping, is gradient, which doesn't hold in general case. We found the condition when the vector ψ_i is gradient in the case of $\theta \in \{1, ..., 4\}$. Some invariant geometric objects of such mappings are found. Anti-equiaffine spaces of θ-kind, $\theta \in \{0, ..., 5\}$ are introduced and discussed. Equitorsion geodesic mappings of such spaces are described and some invariants are found.

1. Motivation

The study of the theory of geodesic mappings of Riemannian spaces, affine connection spaces and their generalizations has been an active field over the past several decades. Many new and interesting results appeared in the papers of N. S. Sinyukov [26], J. Mikeš [7]-[10], G. S. Hall and D. P. Lonie [4]-[6], M. Prvanović [21, 22], S. Minčić [18, 19, 27, 29], Z. Radulović [25], N. Pušić [23, 24], etc. The investigation of geodesic mappings theory for special spaces is an important and active research topic.

The beginning of the study of general (non-symmetric) affine connection spaces is especially related to the works of A. Einstein [1, 2] on Unified Field Theory (UFT). Einstein was not satisfied with his General Theory of Relativity (GTR, 1916), and from 1923 to the end of his life (1955), he worked on different variants of UFT. This theory had the aim to unite the gravitation theory, to which is related GTR, and the theory of electromagnetism.

At the second step, L. P. Eisenhart [3] investigated the properties of generalized Riemannian spaces enabled with non-symmetric metrics. He put the problem to define the class of linear connections which are compatible with the symmetric part of a non-symmetric metric. Further developments extended the problem of generalization to noncommutative quantum gravity and applications in modern cosmology.

Equiaffine spaces are determined by the symmetry of the Ricci tensor. These spaces with symmetric affine connection in which the volume of an N-dimensional parallelepiped is invariant under parallel transport, play an important role in the theory of geodesic mappings. Geodesic mappings of equiaffine spaces with symmetric affine connection were studied in [10]-[14]. For instance, in [11], it is proved that a

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manifold with symmetric affine connection is locally projectively equivalent to an equiaffine manifold. In [14] more general theorem is given: All manifold with affine connection are projectively equiaffine. In [10] fundamental equations of geodesic mappings of equiaffine spaces into (pseudo-) Riemannian spaces were found.

Four kinds of covariant derivatives [15], five independent curvature tensors [17] and five Ricci tensors exist in the spaces with non-symmetric affine connection. Also, there exists one more Ricci tensor, which we take from the adjoint symmetric space. If one of them is symmetric, it doesn't mean that all other Ricci tensors are symmetric, too. In this way, we can define six types of equiaffine spaces with non-symmetric affine connection. We consider geodesic mappings of such spaces. Also, if we suppose anti-symmetry of any Ricci tensor, we get anti-equiaffine space. Geodesic mapping of anti-equiaffine spaces are especially interesting.

Equitorsion mappings are special geodesic mappings of non-symmetric affine connection spaces. These mappings are introduced by M. Stanković at [18] and they represent geodesic mappings which preserve torsion of the spaces.

The paper is organized in the following way. In Section 2, some used notations and preliminaries are introduced. In Section 3, equiaffine general affine connection spaces of the θ-kind, $\theta \in \{0, ..., 5\}$ are introduced and equitorsion geodesic mappings of such spaces are considered. It is proved that in the case when $\theta \in \{0, 5\}$, vector ψ_i, which determines that mapping, is gradient. For the rest of the cases vector ψ_i is not a gradient. Also, we found the condition when the vector ψ_i is gradient in the case of $\theta \in \{1, .., 4\}$. Some invariant geometric objects of such mapping are found. In Section 4, anti-equiaffine general affine connection spaces of θ-kind, $\theta \in \{0, ..., 5\}$ are introduced and discussed. Equitorsion geodesic mappings of such spaces is described and some invariants were found. Specially, it is proved that the fifth curvature tensor is invariant of equitorsion geodesic mappings of anti-equiaffine generalized Riemannian spaces of θ-kind, $\theta \in \{1, ..., 4\}$. In Section 5, we give a concluding remark of the paper.

2. Notations and preliminaries

A *generalized Riemannian space* $\mathbb{GR}_N$ in the sense of Eisenhart's definition [3] is a differentiable N-dimensional manifold, equipped with a non-symmetric basic tensor g_{ij}.

The *general (non-symmetric) affine connection space* $\mathbb{GA}_N$ is a differentiable N-dimensional manifold where introduced magnitudes are L^i_{jk} instead of basic tensor g_{ij} which transform themselves by the low:

$$L^{i'}_{j'k'}(x') = L^i_{jk}(x) x^{i'}_i x^j_{j'} x^k_{k'} + x^{i'}_i x^i_{j'k'},$$

($x^i_{i'} = \frac{\partial x^i}{\partial x^{i'}}, x^{i'}_i = \frac{\partial x^{i'}}{\partial x^i}, x^i_{j'k'} = \frac{\partial^2 x^i}{\partial x^{j'} \partial x^{k'}}$, (x^i) and $(x^{i'})$ are two kinds of local coordinates) where is

$$L^i_{jk} \neq L^i_{kj} \tag{2.1}$$

in the general case. The magnitudes L^i_{jk} are coefficients of the non-symmetric affine connection.

The space $\mathbb{GR}_N$ is special case of the space $\mathbb{GA}_N$. Connection coefficients of the space $\mathbb{GR}_N$ are generalized Cristoffel's symbols of the second kind Γ^i_{jk}. Generally, $\Gamma^i_{jk} \neq \Gamma^i_{kj}$.

Based on (2.1), one can define the symmetric part of L^i_{jk} :

$$L^i_{\underline{jk}} = \frac{1}{2}(L^i_{jk} + L^i_{kj}) = \frac{1}{2} L^i_{(jk)}$$

and anti-symmetric part $L^i_{\underset{\vee}{jk}} = \frac{1}{2}(L^i_{jk} - L^i_{kj}) = \frac{1}{2} L^i_{[jk]}$.

The magnitude $L^i_{\underset{\vee}{jk}}$ is *torsion tensor*. Obviously,

$$L^i_{jk} = L^i_{\underline{jk}} + L^i_{\underset{\vee}{jk}}. \tag{2.2}$$

The magnitudes $L^i_{\underline{jk}}$ can be considered as the coefficients of the symmetric affine connection space $\mathbb{A}_N$.

Using the non-symmetric connection in the space $\mathbb{GA}_N$ one can define four kinds of covariant derivatives [15, 16]. For example, for a tensor a^i_j, we have

$$\begin{aligned} a^i_{j\underset{1}{|}m} &= a^i_{j,m} + L^i_{pm}a^p_j - L^p_{jm}a^i_p, \qquad & a^i_{j\underset{2}{|}m} &= a^i_{j,m} + L^i_{mp}a^p_j - L^p_{mj}a^i_p,\\ a^i_{j\underset{3}{|}m} &= a^i_{j,m} + L^i_{pm}a^p_j - L^p_{mj}a^i_p, \qquad & a^i_{j\underset{4}{|}m} &= a^i_{j,m} + L^i_{mp}a^p_j - L^p_{jm}a^i_p. \end{aligned} \tag{2.3}$$

In the case of the space $\mathbb{GA}_N$ we have five independent curvature tensors [17]:

$$\begin{aligned} \underset{1}{R}^i_{jmn} &= L^i_{jm,n} - L^i_{jn,m} + L^p_{jm}L^i_{pn} - L^p_{jn}L^i_{pm},\\ \underset{2}{R}^i_{jmn} &= L^i_{mj,n} - L^i_{nj,m} + L^p_{mj}L^i_{np} - L^p_{nj}L^i_{mp},\\ \underset{3}{R}^i_{jmn} &= L^i_{jm,n} - L^i_{nj,m} + L^p_{jm}L^i_{np} - L^p_{nj}L^i_{pm} + L^p_{nm}(L^i_{pj} - L^i_{jp}),\\ \underset{4}{R}^i_{jmn} &= L^i_{jm,n} - L^i_{nj,m} + L^p_{jm}L^i_{np} - L^p_{nj}L^i_{pm} + L^p_{mn}(L^i_{pj} - L^i_{jp}),\\ \underset{5}{R}^i_{jmn} &= \frac{1}{2}(L^i_{jm,n} + L^i_{mj,n} - L^i_{jn,m} - L^i_{nj,m} + L^p_{jm}L^i_{pn} + L^p_{mj}L^i_{nm} - L^p_{jn}L^i_{pm} - L^p_{nj}L^i_{pm}). \end{aligned} \tag{2.4}$$

These curvature tensors produce Ricci tensors of θ-kind, i.e. $\underset{\theta}{R}^\alpha_{jm\alpha} = \underset{\theta}{R}_{jm}$, $\theta \in \{1, \ldots, 5\}$.

Riemanian curvature tensor formed by symmetric part of connection L^i_{jk} exists in the $\mathbb{GA}_N$:

$$R^i_{jmn} = L^i_{\underline{jm},n} - L^i_{\underline{jn},m} + L^p_{\underline{jm}}L^i_{\underline{pn}} - L^p_{\underline{jn}}L^i_{\underline{pm}}. \tag{2.5}$$

The Ricci tensor is given by $R^\alpha_{ij\alpha} = R_{ij}$.

Corollary 2.1. *In the case when $L^i_{\underset{\vee}{jk}} = 0$, all tensors given by (2.4) reduce to one curvature tensor (2.5).*

Let $\mathbb{GA}_N$ and $\mathbb{G}\overline{\mathbb{A}}_N$ be two general affine connection spaces. A diffeomorphism $f : \mathbb{GA}_N \to \mathbb{G}\overline{\mathbb{A}}_N$ is called *geodesic mapping* of $\mathbb{GA}_N$ into $\mathbb{G}\overline{\mathbb{A}}_N$ if f maps any geodesic curve in $\mathbb{GA}_N$ into a geodesic curve in $\mathbb{G}\overline{\mathbb{A}}_N$ [18].

In the corresponding points $M(x)$ and $\overline{M}(x)$ we can put

$$\overline{L}^i_{jk}(x) = L^i_{jk}(x) + P^i_{jk}(x), \quad (i, j, k = 1, ..., N) \tag{2.6}$$

where $P^i_{jk}(x)$ is the *deformation tensor* of the connection L of $\mathbb{GA}_N$ according to a mapping f.

A necessary and sufficient condition that the mapping f can be geodesic [18] is that the deformation tensor P^i_{jk} has the form

$$P^i_{jk} = \delta^i_j\psi_k + \delta^i_k\psi_j + \xi^i_{jk}, \tag{2.7}$$

$$\psi_i = \frac{1}{1+N}P^p_{\underline{ip}} = \frac{1}{1+N}(\overline{L}^p_{\underline{ip}} - L^p_{\underline{ip}}), \tag{2.8}$$

$$\xi^i_{jk} = P^i_{\underset{\vee}{jk}} = \overline{L}^i_{\underset{\vee}{jk}} - L^i_{\underset{\vee}{jk}}. \tag{2.9}$$

Definition 2.1. ([18]) A geodesic mapping $f : \mathbb{GA}_N \to \mathbb{G}\overline{\mathbb{A}}_N$ is *equitorsion* if the torsion tensors of the spaces $\mathbb{GA}_N$ and $\mathbb{G}\overline{\mathbb{A}}_N$ are equal in the corresponding points.

According to (2.9), it means that

$$\overline{L}^i_{\underset{\vee}{jk}} - L^i_{\underset{\vee}{jk}} = \xi^i_{jk} = 0. \tag{2.10}$$

3. Equitorsion geodesic mapping of equiaffine spaces

Definition 3.1. The general affine connection space $\mathbb{GA}_N$ is *equiaffine of the θ-kind*, $\theta \in \{1, ..., 5\}$, if the Ricci tensor of the θ-kind is symmetric, i.e. $\underset{\theta}{R}_{jm} = \underset{\theta}{R}_{mj}$. The space $\mathbb{GA}_N$ is *zero-equiaffine* if the Ricci tensor is symmetric i.e. $R_{ij} = R_{ji}$.

Let us consider an equitorsion geodesic mapping of two equiaffine spaces of the θ-kind, $\theta \in \{0, 1, .., 5\}$. By virtue of the geodesic mapping $f : \mathbb{GA}_N \to \mathbb{G}\overline{\mathbb{A}}_N$ we obtain tensors $\underset{\theta}{\overline{R}}$, $(\theta = 1, ..., 5)$, where for example

$$\underset{1}{\overline{R}}{}^i_{jmn} = \overline{L}^i_{jm,n} - \overline{L}^i_{jn,m} + \overline{L}^p_{jm}\overline{L}^i_{pn} - \overline{L}^p_{jn}\overline{L}^i_{pm}.$$

We demonstrate here how one can represent the tensor $\underset{1}{R}$, given by equation (2.4), using (2.2):

$$\begin{aligned}
\underset{1}{R}{}^i_{jmn} &= L^i_{\underline{jm},n} + L^i_{\underset{\vee}{jm},n} - L^i_{\underline{jn},m} - L^i_{\underset{\vee}{jn},m} + L^p_{\underline{jm}}L^i_{\underline{pn}} + L^p_{\underset{\vee}{jm}}L^i_{\underline{pn}} + L^p_{\underline{jm}}L^i_{\underset{\vee}{pn}} + L^p_{\underset{\vee}{jm}}L^i_{\underset{\vee}{pn}} \\
&\quad - L^p_{\underline{jn}}L^i_{\underline{pm}} - L^p_{\underset{\vee}{jn}}L^i_{\underline{pm}} - L^p_{\underline{jn}}L^i_{\underset{\vee}{pm}} - L^p_{\underset{\vee}{jn}}L^i_{\underset{\vee}{pm}} \\
&= R^i_{jmn} + L^i_{jm;n} - L^i_{jn;m} + L^p_{jm}L^i_{pn} - L^p_{jn}L^i_{pm},
\end{aligned} \tag{3.1}$$

where (;) denotes covariant derivative with respect to the symmetric connection $L^i_{\underline{jk}}$ and R^i_{jmn} is Riemannian curvature tensor given by (2.5).

Following this procedure, we obtain:

$$\begin{aligned}
\underset{2}{R}{}^i_{jmn} &= R^i_{jmn} - L^i_{\underset{\vee}{jm};n} + L^i_{\underset{\vee}{jn};m} + L^p_{\underset{\vee}{jm}}L^i_{\underset{\vee}{pn}} - L^p_{\underset{\vee}{jn}}L^i_{\underset{\vee}{pm}}, \\
\underset{3}{R}{}^i_{jmn} &= R^i_{jmn} + L^i_{\underset{\vee}{jm};n} + L^i_{\underset{\vee}{jn};m} - L^p_{\underset{\vee}{jm}}L^i_{\underset{\vee}{pn}} + L^p_{\underset{\vee}{jn}}L^i_{\underset{\vee}{pm}} - 2L^p_{\underset{\vee}{mn}}L^i_{\underset{\vee}{pj}}, \\
\underset{4}{R}{}^i_{jmn} &= R^i_{jmn} + L^i_{\underset{\vee}{jm};n} + L^i_{\underset{\vee}{jn};m} - L^p_{\underset{\vee}{jm}}L^i_{\underset{\vee}{pn}} + L^p_{\underset{\vee}{jn}}L^i_{\underset{\vee}{pm}} + 2L^p_{\underset{\vee}{mn}}L^i_{\underset{\vee}{pj}}, \\
\underset{5}{R}{}^i_{jmn} &= R^i_{jmn} + L^p_{\underset{\vee}{jm}}L^i_{\underset{\vee}{pn}} + L^p_{\underset{\vee}{jn}}L^i_{\underset{\vee}{pm}}.
\end{aligned} \tag{3.2}$$

Contracting by indices i and n in (3.1) and (3.2) we get five Ricci tensors:

$$\begin{aligned}
\underset{1}{R}_{jm} &= R_{jm} + L^p_{\underset{\vee}{jm};p} - L^p_{\underset{\vee}{jp};m} + L^p_{\underset{\vee}{jm}}L^q_{\underset{\vee}{pq}} - L^p_{\underset{\vee}{jq}}L^q_{\underset{\vee}{pm}}, \\
\underset{2}{R}_{jm} &= R_{jm} - L^p_{\underset{\vee}{jm};p} + L^p_{\underset{\vee}{jp};m} + L^p_{\underset{\vee}{jm}}L^q_{\underset{\vee}{pq}} - L^p_{\underset{\vee}{jq}}L^q_{\underset{\vee}{pm}}, \\
\underset{3}{R}_{jm} &= R_{jm} + L^p_{\underset{\vee}{jm};p} + L^p_{\underset{\vee}{jp};m} - L^p_{\underset{\vee}{jm}}L^q_{\underset{\vee}{pq}} + L^p_{\underset{\vee}{jq}}L^q_{\underset{\vee}{pm}} - 2L^p_{\underset{\vee}{mq}}L^q_{\underset{\vee}{pj}}, \\
\underset{4}{R}_{jm} &= R_{jm} + L^p_{\underset{\vee}{jm};p} + L^p_{\underset{\vee}{jp};m} - L^p_{\underset{\vee}{jm}}L^q_{\underset{\vee}{pq}} + L^p_{\underset{\vee}{jq}}L^q_{\underset{\vee}{pm}} + 2L^p_{\underset{\vee}{mq}}L^q_{\underset{\vee}{pj}}, \\
\underset{5}{R}_{jm} &= R_{jm} + L^p_{\underset{\vee}{jm}}L^q_{\underset{\vee}{pq}} + L^p_{\underset{\vee}{jq}}L^q_{\underset{\vee}{pm}}.
\end{aligned} \tag{3.3}$$

By alternating without division with respect to the indices j and m in (3.3) we get

$$\begin{aligned}
\underset{1}{R}_{[jm]} &= R_{[jm]} + 2L^p_{\underset{\vee}{jm};p} - L^p_{\underset{\vee}{jp};m} + L^p_{\underset{\vee}{mp};j} + 2L^p_{\underset{\vee}{jm}}L^q_{\underset{\vee}{pq}}, \\
\underset{2}{R}_{[jm]} &= R_{[jm]} - 2L^p_{\underset{\vee}{jm};p} + L^p_{\underset{\vee}{jp};m} - L^p_{\underset{\vee}{mp};j} + 2L^p_{\underset{\vee}{jm}}L^q_{\underset{\vee}{pq}}, \\
\underset{3}{R}_{[jm]} &= R_{[jm]} + 2L^p_{\underset{\vee}{jm};p} + L^p_{\underset{\vee}{jp};m} - L^p_{\underset{\vee}{mp};j} - 2L^p_{\underset{\vee}{jm}}L^q_{\underset{\vee}{pq}}, \\
\underset{4}{R}_{[jm]} &= R_{[jm]} + 2L^p_{\underset{\vee}{jm};p} + L^p_{\underset{\vee}{jp};m} - L^p_{\underset{\vee}{mp};j} - 2L^p_{\underset{\vee}{jm}}L^q_{\underset{\vee}{pq}}, \\
\underset{5}{R}_{[jm]} &= R_{[jm]} + 2L^p_{\underset{\vee}{jm}}L^q_{\underset{\vee}{pq}},
\end{aligned} \tag{3.4}$$

where [ ] denotes anti-symmetrization without division.

Remark 3.1. In the space $\mathbb{GA}_N$ is valid $\underset{3}{R}_{[jm]} = \underset{4}{R}_{[jm]}$, where $\underset{3}{R}_{jm}$ and $\underset{4}{R}_{jm}$ are Rici tensors of the third and the fourth kind, respectively.

In the case of geodesic mapping of two generalized Riemannian spaces, vector ψ_i given by formula (2.8), which determines the geodesic mapping, is gradient [18]. Also, this fact is valid for geodesic mapping of two spaces of symmetric affine connection i.e. the next theorem is valid:

Theorem 3.1. ([12]) *If the mapping $f : \mathbb{A}_N \to \overline{\mathbb{A}}_N$ is geodesic mapping of two equiaffine spaces, then the vector ψ_i is gradient.*

In the general case of geodesic mappings of two general affine connection spaces it is not valid. In that case the next theorems are valid:

Theorem 3.2. *If the mapping $f : \mathbb{GA}_N \to \mathbb{G}\overline{\mathbb{A}}_N$ is equitorsion geodesic mapping of two equiaffine spaces of the θ-kind, $\theta \in \{0, 5\}$, then the vector ψ_i is gradient.*

Proof. Using (2.6), (2.7), (2.10) and (2.4) we get the relation between the fifth curvature tensors of the spaces $\mathbb{GA}_N$ and $\mathbb{G}\overline{\mathbb{A}}_N$ under equitorsion geodesic mapping:

$$\underset{5}{\overline{R}}{}^i_{jmn} = \underset{5}{R}{}^i_{jmn} + \frac{1}{2}\delta^i_j(\underset{3}{\psi}_{mn} - \underset{4}{\psi}_{nm} + \underset{4}{\psi}_{mn} - \underset{3}{\psi}_{nm}) + \frac{1}{2}\delta^i_m(\underset{3}{\psi}_{jn} + \underset{4}{\psi}_{jn}) - \frac{1}{2}\delta^i_n(\underset{4}{\psi}_{jm} + \underset{3}{\psi}_{jm}), \tag{3.5}$$

where we denoted

$$\underset{\alpha}{\psi}_{mn} = \psi_{m\underset{\alpha}{|}n} - \psi_m\psi_n, \ \alpha \in \{3, 4\}. \tag{3.6}$$

Contracting by indices i and n in (3.5), we get

$$\underset{5}{\overline{R}}_{jm} = \underset{5}{R}_{jm} + \frac{1}{2}(\underset{3}{\psi}_{mj} - \underset{4}{\psi}_{jm} + \underset{4}{\psi}_{mj} - \underset{3}{\psi}_{jm}) + \frac{1}{2}(\underset{3}{\psi}_{jm} + \underset{4}{\psi}_{jm}) - \frac{1}{2}N(\underset{4}{\psi}_{jm} + \underset{3}{\psi}_{jm}). \tag{3.7}$$

Alternating by indices j and m without division in (3.7), we get

$$\underset{5}{\overline{R}}_{[jm]} = \underset{5}{R}_{[jm]} - \frac{N+1}{2}(\underset{3}{\psi}_{[jm]} + \underset{4}{\psi}_{[jm]}). \tag{3.8}$$

According to (2.3) and (3.6) we have

$$\underset{3}{\psi}_{[jm]} + \underset{4}{\psi}_{[jm]} = 2(\psi_{j,m} - \psi_{m,j}). \tag{3.9}$$

Replacing (3.9) in (3.8) one obtains

$$\underset{5}{\overline{R}}_{[jm]} = \underset{5}{R}_{[jm]} - (N+1)(\psi_{j,m} - \psi_{m,j}). \tag{3.10}$$

If we suppose that the tensors $\underset{5}{\overline{R}}_{jm}$ and $\underset{5}{R}_{jm}$ are symmetric, it is valid $\psi_{j,m} = \psi_{m,j}$, i.e. ψ_i is gradient.

Otherwise, replacing (3.4) into (3.10) we get

$$\overline{R}_{[jm]} + 2\overline{L}{}^p_{\underset{\vee}{jm}}\overline{L}{}^q_{\underset{\vee}{pq}} = R_{[jm]} + 2L^p_{\underset{\vee}{jm}}L^q_{\underset{\vee}{pq}} - (1+N)(\psi_{j,m} - \psi_{m,j}).$$

As the spaces $\mathbb{GA}_N$ and $\mathbb{G}\overline{\mathbb{A}}_N$ have the same torsion tensors under the equitorsion mapping, we have

$$\overline{R}_{[jm]} = R_{[jm]} - (1+N)(\psi_{j,m} - \psi_{m,j}). \tag{3.11}$$

If the tensors $\overline{R}_{jm}$ and R_{jm} are symmetric, then ψ_i is gradient. □

From the equalities (3.10), (3.11) we have the following corollaries:

Corollary 3.1. *Let $f:\mathbb{G}\mathbb{A}_N \to \mathbb{G}\overline{\mathbb{A}}_N$ be equitorsion geodesic mapping. The tensor $R_{[jm]}$ is invariant of this mapping if and only if $\underset{5}{R}_{[jm]}$ is invariant of this mapping.*

Corollary 3.2. *If $f:\mathbb{G}\mathbb{A}_N \to \mathbb{G}\overline{\mathbb{A}}_N$ is equitorsion geodesic mapping of two equiaffine spaces of the zero kind, then the tensor $\underset{5}{R}_{[jm]}$, is an invariant of this mapping. And inversely, if $f:\mathbb{G}\mathbb{A}_N \to \mathbb{G}\overline{\mathbb{A}}_N$ is equitorsion geodesic mapping of two equiaffine spaces of fifth kind, then the tensor $R_{[jm]}$, is an invariant of this mapping.*

Under equitorsion geodesic mappings of two equiaffine spaces of θ-kind $\theta \in \{1,\ldots,4\}$, vector ψ_i isn't gradient in general case, but it holds

Theorem 3.3. *Let $f:\mathbb{G}\mathbb{A}_N \to \mathbb{G}\overline{\mathbb{A}}_N$ be equitorsion geodesic mapping of two equiaffine spaces of the θ-kind, $\theta\in\{1,\ldots,4\}$. Vector ψ_i is gradient if and only if*

$$\underset{\vee}{L}{}^p_{mj}\psi_p = \frac{1}{N-1}(\underset{\vee}{L}{}^\alpha_{m\alpha}\psi_j - \underset{\vee}{L}{}^\alpha_{j\alpha}\psi_m). \tag{3.12}$$

Proof. Let us suppose that the spaces $\mathbb{G}\mathbb{A}_N$ and $\mathbb{G}\overline{\mathbb{A}}_N$ are equiaffine of the first kind. Let us start from the first curvature tensor. Using (2.6), (2.7), (2.10) and (2.4) we get the relation between the first curvature tensors of the spaces $\mathbb{G}\mathbb{A}_N$ and $\mathbb{G}\overline{\mathbb{A}}_N$ under equitorsion geodesic mapping

$$\underset{1}{\overline{R}}{}^i_{jmn} = \underset{1}{R}{}^i_{jmn} + \delta^i_j(\underset{1}{\psi}_{mn} - \underset{1}{\psi}_{nm}) + \delta^i_m\underset{1}{\psi}_{jn} - \delta^i_n\underset{1}{\psi}_{jm} + 2\underset{\vee}{L}{}^i_{mn}\psi_j + 2\underset{\vee}{L}{}^\alpha_{mn}\psi_\alpha\delta^i_j. \tag{3.13}$$

Contracting by indices i and n in (3.13), we get

$$\underset{1}{\overline{R}}_{jm} = \underset{1}{R}_{jm} + (\underset{1}{\psi}_{mj} - \underset{1}{\psi}_{jm}) + \underset{1}{\psi}_{jm} - N\underset{1}{\psi}_{jm} + 2\underset{\vee}{L}{}^\alpha_{m\alpha}\psi_j + 2\underset{\vee}{L}{}^\alpha_{mj}\psi_\alpha, \tag{3.14}$$

where we denoted

$$\underset{1}{\psi}_{mn} = \psi_{m\underset{1}{|}n} - \psi_m\psi_n. \tag{3.15}$$

Alternating by indices j and m without division in (3.14), we get

$$\underset{1}{\overline{R}}_{[jm]} = \underset{1}{R}_{[jm]} - (1+N)\underset{1}{\psi}_{[jm]} + 2\underset{\vee}{L}{}^\alpha_{m\alpha}\psi_j - 2\underset{\vee}{L}{}^\alpha_{j\alpha}\psi_m + 4\underset{\vee}{L}{}^\alpha_{mj}\psi_\alpha.$$

As $\underset{1}{\overline{R}}_{[jm]} = 0$, $\underset{1}{R}_{[jm]} = 0$, we get

$$(1+N)\underset{1}{\psi}_{[jm]} = 2\underset{\vee}{L}{}^\alpha_{m\alpha}\psi_j - 2\underset{\vee}{L}{}^\alpha_{j\alpha}\psi_m + 4\underset{\vee}{L}{}^\alpha_{mj}\psi_\alpha. \tag{3.16}$$

According to (3.15) and (2.3), we have

$$\underset{1}{\psi}_{[jm]} = \psi_{j,m} - \psi_{m,j} + 2\underset{\vee}{L}{}^p_{mj}\psi_p. \tag{3.17}$$

Replacing (3.17) into (3.16), one obtains $\psi_{j,m} - \psi_{m,j} = \frac{2}{1+N}\Big((1-N)\underset{\vee}{L}{}^p_{mj}\psi_p + \underset{\vee}{L}{}^\alpha_{m\alpha}\psi_j - \underset{\vee}{L}{}^\alpha_{j\alpha}\psi_m\Big)$. From here we can see that ψ_i is gradient if and only if the equation (3.12) holds.

On the same way, it is easy to prove that for the other curvature tensors, equation (3.12) is necessary and sufficient condition to ψ_i be gradient. □

The Corollaries 3.1 and 3.2 don't hold for the other Ricci tensors in general case, but they do in the case of special equitorsion geodesic mappings. Namely,

Theorem 3.4. *Let $f : \mathbb{GA}_N \to \mathbb{G}\overline{\mathbb{A}}_N$ be equitorsion geodesic mapping of two general affine connection spaces which satisfies*

$$\underset{\vee}{L^p_{mj}}\psi_p = \frac{1}{N-1}(\underset{\vee}{L^\alpha_{m\alpha}}\psi_j - \underset{\vee}{L^\alpha_{j\alpha}}\psi_m). \tag{3.18}$$

Then the next conditions are equivalent:

$$a)\quad \overline{R}_{[jm]} = R_{[jm]},\quad b)\quad \underset{1}{\overline{R}}_{[jm]} = \underset{1}{R}_{[jm]},\quad c)\quad \underset{2}{\overline{R}}_{[jm]} = \underset{2}{R}_{[jm]},$$
$$d)\quad \underset{3}{\overline{R}}_{[jm]} = \underset{3}{R}_{[jm]},\quad e)\quad \underset{4}{\overline{R}}_{[jm]} = \underset{4}{R}_{[jm]},\quad f)\quad \underset{5}{\overline{R}}_{[jm]} = \underset{5}{R}_{[jm]}.$$

Proof. Let us prove the equivalence $(a) \Leftrightarrow (b)$, i.e. $\overline{R}_{[jm]} = R_{[jm]} \Leftrightarrow \underset{1}{\overline{R}}_{[jm]} = \underset{1}{R}_{[jm]}$. Let $(;)$ denote covariant derivative with respect to the symmetric connection $L^i_{\underline{jk}}$ and $(\bar{;})$ covariant derivative with respect to the $\overline{L}^i_{\underline{jk}}$. According to (2.3) and (2.6), one obtains

$$\begin{aligned}
\underset{\vee}{\overline{L}^i_{jm\bar{;}k}} &= \underset{\vee}{\overline{L}^i_{jm,k}} + \overline{L}^i_{\underline{pk}}\underset{\vee}{\overline{L}^p_{jm}} - \overline{L}^p_{\underline{jk}}\underset{\vee}{\overline{L}^i_{pm}} - \overline{L}^p_{\underline{mk}}\underset{\vee}{\overline{L}^i_{jp}}\\
&= \underset{\vee}{L^i_{jm,k}} + (L^i_{\underline{pk}} + P^i_{pk})\underset{\vee}{L^p_{jm}} - (L^p_{\underline{jk}} + P^p_{jk})\underset{\vee}{L^i_{pm}} - (L^p_{\underline{mk}} + P^p_{mk})\underset{\vee}{L^i_{jp}}\\
&= \underset{\vee}{L^i_{jm;k}} + P^i_{pk}\underset{\vee}{L^p_{jm}} - P^p_{jk}\underset{\vee}{L^i_{pm}} - P^p_{mk}\underset{\vee}{L^i_{jp}},
\end{aligned}$$

and from (3.4), we get

$$\begin{aligned}
\underset{1}{\overline{R}}_{[jm]} &= \overline{R}_{[jm]} + 2\underset{\vee}{\overline{L}^p_{jm\bar{;}p}} - \underset{\vee}{\overline{L}^p_{jp\bar{;}m}} + \underset{\vee}{\overline{L}^p_{mp\bar{;}j}} + 2\underset{\vee}{\overline{L}^p_{jm}}\underset{\vee}{\overline{L}^q_{pq}}\\
&= \overline{R}_{[jm]} + 2(\underset{\vee}{L^p_{jm;p}} + P^p_{pq}\underset{\vee}{L^q_{jm}} - P^q_{jp}\underset{\vee}{L^p_{qm}} - P^q_{mp}\underset{\vee}{L^p_{jq}}) - (\underset{\vee}{L^p_{jp;k}} + P^p_{qm}\underset{\vee}{L^q_{jp}} - P^q_{jm}\underset{\vee}{L^p_{qp}} - P^q_{pm}\underset{\vee}{L^p_{jq}})\\
&+ (\underset{\vee}{L^p_{mp;j}} + P^p_{qj}\underset{\vee}{L^q_{mp}} - P^q_{mj}\underset{\vee}{L^p_{qp}} - P^q_{pj}\underset{\vee}{L^p_{mq}}) + 2\underset{\vee}{L^p_{jm}}\underset{\vee}{L^q_{pq}}\\
&= \overline{R}_{[jm]} + 2\underset{\vee}{L^p_{jm;p}} - \underset{\vee}{L^p_{jp;m}} + \underset{\vee}{L^p_{mp;j}} + 2\underset{\vee}{L^p_{jm}}\underset{\vee}{L^q_{pq}} + 2(P^p_{pq}\underset{\vee}{L^q_{jm}} - P^q_{jp}\underset{\vee}{L^p_{qm}} - P^q_{mp}\underset{\vee}{L^p_{jq}}),
\end{aligned}$$

Now, according to (2.7), (2.10) and (3.18), one obtains

$$\begin{aligned}
P^p_{pq}\underset{\vee}{L^q_{jm}} - P^q_{jp}\underset{\vee}{L^p_{qm}} - P^q_{mp}\underset{\vee}{L^p_{jq}} &= (\delta^p_q\psi_p + \delta^p_p\psi_q)\underset{\vee}{L^q_{jm}} - (\delta^q_j\psi_p + \delta^q_p\psi_j)\underset{\vee}{L^p_{qm}} - (\delta^q_m\psi_p + \delta^q_p\psi_m)\underset{\vee}{L^p_{jq}}\\
&= \underset{\vee}{L^p_{jm}}\psi_p + N\underset{\vee}{L^p_{jm}}\psi_p - \underset{\vee}{L^p_{jm}}\psi_p - \underset{\vee}{L^p_{pm}}\psi_j - \underset{\vee}{L^p_{jm}}\psi_p - \underset{\vee}{L^q_{jq}}\psi_m\\
&= (N-1)\underset{\vee}{L^p_{jm}}\psi_p + \underset{\vee}{L^p_{mp}}\psi_j - \underset{\vee}{L^q_{jq}}\psi_m = 0,
\end{aligned}$$

Therefore, $\underset{1}{\overline{R}}_{[jm]} = \overline{R}_{[jm]} + 2\underset{\vee}{L^p_{jm;p}} - \underset{\vee}{L^p_{jp;m}} + \underset{\vee}{L^p_{mp;j}} + 2\underset{\vee}{L^p_{jm}}\underset{\vee}{L^q_{pq}}$. It is easy to see that $\overline{R}_{[jm]} = R_{[jm]} \Leftrightarrow \underset{1}{\overline{R}}_{[jm]} = \underset{1}{R}_{[jm]}$ holds. On the same way we get the other equivalences. □

Immediately follows

Corollary 3.3. *Let $f : \mathbb{GA}_N \to \mathbb{G}\overline{\mathbb{A}}_N$ be equitorsion geodesic mapping of two equiaffine spaces of the θ-kind, $\theta \in \{0, ..., 5\}$, which satisfies the condition (3.18), then $R_{[jm]}, \underset{1}{R}_{[jm]}, \underset{2}{R}_{[jm]}, \underset{3}{R}_{[jm]}, \underset{4}{R}_{[jm]}, \underset{5}{R}_{[jm]}$ are invariants of this mapping.*

Specially, in the generalized Riemannian space $\mathbb{GR_N}$ it is valid that $\underset{\vee}{\Gamma}{}^p_{ip} = \underset{\vee}{\Gamma}{}^p_{pi} = 0$ (see [3]), and Ricci tensor is always symmetric, so the equations (3.4) are reduced to

$$\underset{1}{R}_{[jm]} = 2\underset{\vee}{\Gamma}{}^p_{jm;p},\ \underset{2}{R}_{[jm]} = -2\underset{\vee}{\Gamma}{}^p_{jm;p},\ \underset{3}{R}_{[jm]} = 2\underset{\vee}{\Gamma}{}^p_{jm;p},\ \underset{4}{R}_{[jm]} = 2\underset{\vee}{\Gamma}{}^p_{jm;p},\ \underset{5}{R}_{[jm]} = 0.$$

Corollary 3.4. *In the space $\mathbb{GR}_N$ is valid*

$$\underset{1}{R}_{[jm]} = -\underset{2}{R}_{[jm]} = \underset{3}{R}_{[jm]} = \underset{4}{R}_{[jm]},\ \underset{5}{R}_{[jm]} = 0,$$

i.e. the fifth Ricci tensor $\underset{5}{R}_{jm}$ is always symmetric.

Also, ψ_i is always gradient, and if the spaces $\mathbb{GR}_N$ and $\mathbb{G}\overline{\mathbb{R}}_N$ are equiaffine of the θ-kind, $\theta \in \{1, 2, 3, 4\}$, then the other Ricci tensors are symmetric, too. The equality (3.12) holds and it is reduced to $\underset{\vee}{\Gamma}{}^\alpha_{mj}\psi_\alpha = 0$.

The simplified connections between curvature tensors in the case of equitorsion geodesic mapping of the generalized Riemannian spaces which are equiaffine of θ-kind, $\theta \in \{0, ..., 5\}$ are:

$$\begin{aligned}
\underset{1}{\overline{R}}{}^i_{jmn} &= \underset{1}{R}{}^i_{jmn} + \delta^i_m \underset{1}{\psi}_{jn} - \delta^i_n \underset{1}{\psi}_{jm} + 2\underset{\vee}{L}{}^i_{mn}\psi_j,\\
\underset{2}{\overline{R}}{}^i_{jmn} &= \underset{2}{R}{}^i_{jmn} + \delta^i_m \underset{2}{\psi}_{jn} - \delta^i_n \underset{2}{\psi}_{jm} + 2\underset{\vee}{L}{}^i_{nm}\psi_j,\\
\underset{3}{\overline{R}}{}^i_{jmn} &= \underset{3}{R}{}^i_{jmn} + \delta^i_m \underset{2}{\psi}_{jn} - \delta^i_n \underset{1}{\psi}_{jm} + 2\underset{\vee}{L}{}^i_{mj}\psi_n + 2\underset{\vee}{L}{}^i_{nj}\psi_m,\\
\underset{4}{\overline{R}}{}^i_{jmn} &= \underset{4}{R}{}^i_{jmn} + \delta^i_m \underset{2}{\psi}_{jn} - \delta^i_n \underset{1}{\psi}_{jm} + 2\underset{\vee}{L}{}^i_{mj}\psi_n + 2\underset{\vee}{L}{}^i_{nj}\psi_m,\\
\underset{5}{\overline{R}}{}^i_{jmn} &= \underset{5}{R}{}^i_{jmn} + \delta^i_m \underset{4}{\psi}_{\underline{jn}} - \delta^i_n \underset{4}{\psi}_{\underline{jm}},
\end{aligned}$$

where $\underset{\alpha}{\psi}_{jm} = \psi_{j\underset{\alpha}{|}m} - \psi_j\psi_m$, $\alpha \in \{1, ..., 4\}$.

4. Equitorsion geodesic mappings of anti-equiaffine spaces

Definition 4.1. The general affine connection space $\mathbb{GA}_N$ is *anti-equiaffine of the θ-kind*, $\theta \in \{1, ..., 5\}$, if the Ricci tensor of the θ-kind is anti-symmetric, i.e $\underset{\theta}{R}_{jm} = -\underset{\theta}{R}_{mj}$. The space $\mathbb{GA}_N$ is *zero-anti-equiaffine* if the Ricci tensor is anti-symmetric, i.e. $R_{ij} = -R_{ji}$.

In the sequel we will describe equitorsion geodesic mappings of two anti-equiaffine spaces of the θ-kind, $\theta \in \{0, 1, .., 5\}$.

By symmetrization with respect to the indices j and m without division in (3.3) we get

$$\begin{aligned}
\underset{1}{R}_{(jm)} &= R_{(jm)} - \underset{\vee}{L}{}^p_{jp;m} - \underset{\vee}{L}{}^p_{mp;j} - 2\underset{\vee}{L}{}^p_{jq}\underset{\vee}{L}{}^q_{pm},\\
\underset{2}{R}_{(jm)} &= R_{(jm)} + \underset{\vee}{L}{}^p_{jp;m} + \underset{\vee}{L}{}^p_{mp;j} - 2\underset{\vee}{L}{}^p_{jq}\underset{\vee}{L}{}^q_{pm},\\
\underset{3}{R}_{(jm)} &= R_{(jm)} + \underset{\vee}{L}{}^p_{jp;m} + \underset{\vee}{L}{}^p_{mp;j} - 2\underset{\vee}{L}{}^p_{jq}\underset{\vee}{L}{}^q_{pm},\\
\underset{4}{R}_{(jm)} &= R_{(jm)} + \underset{\vee}{L}{}^p_{jp;m} + \underset{\vee}{L}{}^p_{mp;j} + 6\underset{\vee}{L}{}^p_{jq}\underset{\vee}{L}{}^q_{pm},\\
\underset{5}{R}_{(jm)} &= R_{(jm)} + 2\underset{\vee}{L}{}^p_{jq}\underset{\vee}{L}{}^q_{pm}.
\end{aligned} \tag{4.1}$$

where ( ) denotes symmetrization without division.

Remark 4.1. In the space $\mathbb{GA}_N$ is valid $\underset{2}{R}_{(jm)} = \underset{3}{R}_{(jm)}$.

Let us denote $\psi_{jm} = \psi_{j;m} - \psi_j\psi_m$, where ψ_j is given at (2.8) and (;) is covariant differentiation with respect to the symmetric part of connection L^i_{jk}.

Theorem 4.1. *Let $f : \mathbb{GA}_N \to \mathbb{G}\overline{\mathbb{A}}_N$ be equitorsion geodesic mapping of two anti-equiaffine spaces of θ-kind, $\theta \in \{0, 5\}$, then ψ_{ij} is anti-symmetric.*

Proof. By symmetrization of the equation (3.7) with respect to the indices j and m, we get

$$\underset{5}{\overline{R}}_{(jm)} = \underset{5}{R}_{(jm)} + \frac{1-N}{2}\Big(\underset{3}{\psi}_{(jm)} + \underset{4}{\psi}_{(jm)}\Big),$$

where

$$\begin{aligned}\underset{3}{\psi}_{(jm)} + \underset{4}{\psi}_{(jm)} &= \psi_{j,m} - L^p_{mj}\psi_p + \psi_{m,j} - L^p_{jm}\psi_p + \psi_{j,m} - L^p_{jm}\psi_p + \psi_{m,j} - L^p_{mj}\psi_p - 4\psi_m\psi_j \\ &= 2(\psi_{j,m} - L^p_{\underline{jm}}\psi_p + \psi_{m,j} - L^p_{\underline{mj}}\psi_p - 2\psi_m\psi_j) \\ &= 2(\psi_{jm} + \psi_{mj}) = 2\psi_{(jm)}.\end{aligned}$$

Finally,

$$\underset{5}{\overline{R}}_{(jm)} = \underset{5}{R}_{(jm)} + (1-N)\psi_{(jm)}. \tag{4.2}$$

Suppose that the spaces $\mathbb{GA}_N$ and $\mathbb{G}\overline{\mathbb{A}}_N$ are anti-equiaffine of the fifth kind. Then $\psi_{(jm)} = 0$, i.e. ψ_{jm} is anti-symmetric. Otherwise, using (4.1) and the fact that the corresponding torsions are equal under equitorsion mapping, we have

$$\overline{R}_{(jm)} = R_{(jm)} + (1-N)\psi_{(jm)}. \tag{4.3}$$

Suppose that $\mathbb{GA}_N$ and $\mathbb{G}\overline{\mathbb{A}}_N$ are anti-equiaffine spaces of the zero kind, then $\psi_{(jm)} = 0$, and ψ_{jm} is anti-symmetric. □

Obviously, under equitorsion geodesic mapping of two anti-equiaffine spaces of θ-kind, $\theta \in \{0, 5\}$, $\psi_{ij} = 0$ if and only if ψ_i is gradient.

From the equations (4.2) and (4.3), we have

Corollary 4.1. *Let $f : \mathbb{GA}_N \to \mathbb{G}\overline{\mathbb{A}}_N$ be equitorsion geodesic mapping. A necessary and sufficient that the tensor $R_{(jm)}$ is invariant of this mapping is that the tensor $\underset{5}{R}_{(jm)}$ is invariant of this mapping.*

Corollary 4.2. *If $f : \mathbb{GA}_N \to \mathbb{G}\overline{\mathbb{A}}_N$ is equitorsion geodesic mapping of two anti-equiaffine spaces of the zero kind, then the tensor $\underset{5}{R}_{(jm)}$, is an invariant of this mapping. And inversely, if $f : \mathbb{GA}_N \to \mathbb{G}\overline{\mathbb{A}}_N$ is equitorsion geodesic mapping of two anti-equiaffine spaces of the fifth kind, then the tensor $R_{(jm)}$ is an invariant of this mapping.*

Theorem 4.2. *Let $f : \mathbb{GA}_N \to \mathbb{G}\overline{\mathbb{A}}_N$ be equitorsion geodesic mapping of two anti-equiaffine spaces of the θ-kind, $\theta \in \{1, ..., 4\}$. The magnitude ψ_{ij} is antisymmetric if and only if*

$$L^\alpha_{\underset{\vee}{m\alpha}}\psi_j + L^\alpha_{\underset{\vee}{j\alpha}}\psi_m = 0. \tag{4.4}$$

Proof. Assume that the spaces $\mathbb{GA}_N$ and $\mathbb{G}\overline{\mathbb{A}}_N$ are anti-equiaffine of the first kind. The relation between the first curvature tensors of the spaces $\mathbb{GA}_N$ and $\mathbb{G}\overline{\mathbb{A}}_N$ under equitorsion geodesic mapping is

$$\underset{1}{\overline{R}}{}^i_{jmn} = \underset{1}{R}{}^i_{jmn} + \delta^i_j(\underset{1}{\psi}_{mn} - \underset{1}{\psi}_{nm}) + \delta^i_m \underset{1}{\psi}_{jn} - \delta^i_n \underset{1}{\psi}_{jm} + 2L^i_{\underset{\vee}{mn}}\psi_j + 2L^\alpha_{\underset{\vee}{mn}}\psi_\alpha\delta^i_j. \tag{4.5}$$

By symmetrization with respect to the indices j and m without division in (4.5) we get

$$\underset{1}{\overline{R}}_{(jm)} = \underset{1}{R}_{(jm)} + (1-N)\underset{1}{\psi}_{(jm)} + 2L^\alpha_{\underset{\vee}{m\alpha}}\psi_j + 2L^\alpha_{\underset{\vee}{j\alpha}}\psi_m.$$

As $\underset{1}{\overline{R}}_{(jm)} = 0$, $\underset{1}{R}_{(jm)} = 0$, one obtains $(N-1)\underset{1}{\psi}_{(jm)} = 2L^\alpha_{\underset{\vee}{m\alpha}}\psi_j + 2L^\alpha_{\underset{\vee}{j\alpha}}\psi_m$. On the other hand

$$\begin{aligned}\underset{1}{\psi}_{(jm)} &= \psi_{j,m} - L^p_{jm}\psi_p + \psi_{m,j} - L^p_{mj}\psi_p - 2\psi_m\psi_j\\ &= \psi_{j,m} - L^p_{\underline{jm}}\psi_p + \psi_{m,j} - L^p_{\underline{mj}}\psi_p - 2\psi_m\psi_j\\ &= \psi_{jm} + \psi_{mj} = \psi_{(jm)}.\end{aligned}$$

Obviously, ψ_{ij} is anti-symmetric if and only if the condition (4.4) is satisfied. On the same way, if we start from $\underset{\theta}{R}_{jm}$, $\theta \in \{2,3,4\}$, we get the same condition. □

The Corollaries 4.1 and 4.2 don't hold for the other Ricci tensors in general case, but they do in the case of special equitorsion geodesic mappings. Namely,

Theorem 4.3. *Let $f : \mathbb{GA}_N \to \mathbb{G}\overline{\mathbb{A}}_N$ be equitorsion geodesic mapping of two general affine connection spaces which satisfies*

$$L^\alpha_{\underset{\vee}{m\alpha}}\psi_j + L^\alpha_{\underset{\vee}{j\alpha}}\psi_m = 0. \tag{4.6}$$

Then the next conditions are equivalent:

$$\begin{aligned}&a)\quad \overline{R}_{(jm)} = R_{(jm)}, \quad b)\quad \underset{1}{\overline{R}}_{(jm)} = \underset{1}{R}_{(jm)}, \quad c)\quad \underset{2}{\overline{R}}_{(jm)} = \underset{2}{R}_{(jm)},\\ &d)\quad \underset{3}{\overline{R}}_{(jm)} = \underset{3}{R}_{(jm)}, \quad e)\quad \underset{4}{\overline{R}}_{(jm)} = \underset{4}{R}_{(jm)}, \quad f)\quad \underset{5}{\overline{R}}_{(jm)} = \underset{5}{R}_{(jm)}.\end{aligned}$$

Proof. Let us prove the equivalence $\overline{R}_{(jm)} = R_{(jm)} \Leftrightarrow \underset{1}{\overline{R}}_{(jm)} = \underset{1}{R}_{(jm)}$. According to (2.3), (2.6) and (4.1) we have

$$\underset{1}{\overline{R}}_{(jm)} = \overline{R}_{(jm)} - \overline{L}^p_{\underset{\vee}{jp}\bar{;}m} - \overline{L}^p_{\underset{\vee}{mp}\bar{;}j} - 2\overline{L}^p_{\underset{\vee}{jq}}\overline{L}^q_{\underset{\vee}{pm}} = \overline{R}_{(jm)} - L^p_{\underset{\vee}{jp};m} - L^p_{\underset{\vee}{mp};j} - 2L^p_{\underset{\vee}{jq}}L^q_{\underset{\vee}{pm}} + 2P^q_{jm}L^p_{\underset{\vee}{qp}},$$

where $2P^q_{jm}L^p_{\underset{\vee}{qp}} = 2(\delta^q_j\psi_m + \delta^q_m\psi_j)L^p_{\underset{\vee}{qp}} = 2(\psi_m L^p_{\underset{\vee}{jp}} + \psi_j L^p_{\underset{\vee}{mp}}) = 0$, i.e. $\underset{1}{\overline{R}}_{(jm)} = \overline{R}_{(jm)} - L^p_{\underset{\vee}{jp};m} - L^p_{\underset{\vee}{mp};j} - 2L^p_{\underset{\vee}{jq}}L^q_{\underset{\vee}{pm}}$. It is easy to see that $\overline{R}_{(jm)} = R_{(jm)} \Leftrightarrow \underset{1}{\overline{R}}_{(jm)} = \underset{1}{R}_{(jm)}$ holds. On the same way we get the other equivalences. □

Immediately follows

Corollary 4.3. *Let $f : \mathbb{GA}_N \to \mathbb{G}\overline{\mathbb{A}}_N$ be equitorsion geodesic mapping of two anti-equiaffine spaces of θ-kind, $\theta \in \{0,\ldots,5\}$, which satisfies the condition (4.6), then $R_{(jm)}, \underset{1}{R}_{(jm)}, \underset{2}{R}_{(jm)}, \underset{3}{R}_{(jm)}, \underset{4}{R}_{(jm)}, \underset{5}{R}_{(jm)}$ are invariants of this mapping.*

Specially, in the generalized Riemannian space $\mathbb{GR}_N(\mathbb{G}\overline{\mathbb{R}}_N)$ the equations (4.1) reduce to

$$
\begin{aligned}
&\underset{1}{R}_{(jm)} = 2R_{jm} - 2\Gamma^p_{\underset{\vee}{jq}}\Gamma^q_{\underset{\vee}{pm}}, \qquad &&\underset{1}{\overline{R}}_{(jm)} = 2\overline{R}_{jm} - 2\overline{\Gamma}^p_{\underset{\vee}{jq}}\overline{\Gamma}^q_{\underset{\vee}{pm}},\\
&\underset{2}{R}_{(jm)} = 2R_{jm} - 2\Gamma^p_{\underset{\vee}{jq}}\Gamma^q_{\underset{\vee}{pm}}, \qquad &&\underset{2}{\overline{R}}_{(jm)} = 2\overline{R}_{jm} - 2\overline{\Gamma}^p_{\underset{\vee}{jq}}\overline{\Gamma}^q_{\underset{\vee}{pm}},\\
&\underset{3}{R}_{(jm)} = 2R_{jm} - 2\Gamma^p_{\underset{\vee}{jq}}\Gamma^q_{\underset{\vee}{pm}}, \qquad &&\underset{3}{\overline{R}}_{(jm)} = 2\overline{R}_{jm} - 2\overline{\Gamma}^p_{\underset{\vee}{jq}}\overline{\Gamma}^q_{\underset{\vee}{pm}},\\
&\underset{4}{R}_{(jm)} = 2R_{jm} + 6\Gamma^p_{\underset{\vee}{jq}}\Gamma^q_{\underset{\vee}{pm}}, \qquad &&\underset{4}{\overline{R}}_{(jm)} = 2\overline{R}_{jm} + 6\overline{\Gamma}^p_{\underset{\vee}{jq}}\overline{\Gamma}^q_{\underset{\vee}{pm}},\\
&\underset{5}{R}_{jm} = R_{jm} + \Gamma^p_{\underset{\vee}{jq}}\Gamma^q_{\underset{\vee}{pm}}, \qquad &&\underset{5}{\overline{R}}_{jm} = \overline{R}_{jm} + \overline{\Gamma}^p_{\underset{\vee}{jq}}\overline{\Gamma}^q_{\underset{\vee}{pm}}.
\end{aligned}
$$

Corollary 4.4. *In the space $\mathbb{GR}_N$ is valid $\underset{1}{R}_{(jm)} = \underset{2}{R}_{(jm)} = \underset{3}{R}_{(jm)}$.*

Corollary 4.5. *Under equitorsion geodesic mapping of two generalized Riemannian spaces $\mathbb{GR}_N$ and $\mathbb{G}\overline{\mathbb{R}}_N$ the next tensors are invariants of this mapping: $\underset{1}{R}_{(jm)} - 2R_{jm}$, $\underset{2}{R}_{(jm)} - 2R_{jm}$, $\underset{3}{R}_{(jm)} - 2R_{jm}$, $\underset{4}{R}_{(jm)} - 2R_{jm}$, $\underset{5}{R}_{jm} - R_{jm}$.*

Let $f : \mathbb{GR}_N \to \mathbb{G}\overline{\mathbb{R}}_N$ be equitorsion geodesic mapping of two generalized Riemannian spaces $\mathbb{GR}_N$ and $\mathbb{G}\overline{\mathbb{R}}_N$. In this case vector ψ_j is always gradient and R_{jm}, $\overline{R}_{jm}$, $\underset{5}{R}_{jm}$, $\underset{5}{\overline{R}}_{jm}$ are symmetric. The equation (4.4) always holds.

Suppose that some other pair of Ricci tensors is anti-symmetric, for example $\underset{1}{\overline{R}}_{jm}$ and $\underset{1}{R}_{jm}$. Then we have $\psi_{(jm)} = 0$, and according to Corollary 4.3, R_{jm} and $\underset{5}{R}_{jm}$ are invariants of that mapping. As ψ_j is gradient, we get $0 = \psi_{jm} = \psi_{j,m} - L^p_{jm}\psi_p - \psi_j\psi_m + L^p_{\underset{\vee}{jm}}\psi_p = \underset{1}{\psi}_{jm} + L^p_{\underset{\vee}{jm}}\psi_p$, i.e. $\underset{1}{\psi}_{jm} = \underset{4}{\psi}_{jm} = -L^p_{\underset{\vee}{jm}}\psi_p$, and, on the same way $\underset{2}{\psi}_{jm} = \underset{3}{\psi}_{jm} = -L^p_{\underset{\vee}{mj}}\psi_p$. Using these facts and (2.6), (2.7), (2.10), (2.4) and (2.5), we get the connections between the curvature tensors

$$
\begin{aligned}
&\underset{1}{\overline{R}}{}^i_{jmn} = \underset{1}{R}{}^i_{jmn} - \delta^i_m\Gamma^p_{\underset{\vee}{jn}}\psi_p + \delta^i_n\Gamma^p_{\underset{\vee}{jm}}\psi_p + 2\Gamma^i_{\underset{\vee}{mn}}\psi_j,\\
&\underset{2}{\overline{R}}{}^i_{jmn} = \underset{2}{R}{}^i_{jmn} - \delta^i_m\Gamma^p_{\underset{\vee}{nj}}\psi_p + \delta^i_n\Gamma^p_{\underset{\vee}{mj}}\psi_p + 2\Gamma^i_{\underset{\vee}{nm}}\psi_j,\\
&\underset{3}{\overline{R}}{}^i_{jmn} = \underset{3}{R}{}^i_{jmn} - 2\delta^i_j\Gamma^p_{\underset{\vee}{nm}}\psi_p - \delta^i_m\Gamma^p_{\underset{\vee}{nj}}\psi_p - \delta^i_n\Gamma^p_{\underset{\vee}{mj}}\psi_p + 2\Gamma^i_{\underset{\vee}{mj}}\psi_n + 2\Gamma^i_{\underset{\vee}{nj}}\psi_m,\\
&\underset{4}{\overline{R}}{}^i_{jmn} = \underset{4}{R}{}^i_{jmn} - 2\delta^i_j\Gamma^p_{\underset{\vee}{nm}}\psi_p - \delta^i_m\Gamma^p_{\underset{\vee}{nj}}\psi_p - \delta^i_n\Gamma^p_{\underset{\vee}{mj}}\psi_p + 2\Gamma^i_{\underset{\vee}{mj}}\psi_n + 2\Gamma^i_{\underset{\vee}{nj}}\psi_m,\\
&\underset{5}{\overline{R}}{}^i_{jmn} = \underset{5}{R}{}^i_{jmn}, \qquad \overline{R}^i_{jmn} = R^i_{jmn}.
\end{aligned}
\tag{4.7}
$$

If we start from the tensors $\underset{\theta}{\overline{R}}_{jm}$ and $\underset{\theta}{R}_{jm}$, $\theta \in \{2, 3, 4\}$, and suppose their anti-symmetry, we get the same equations (4.7). Obviously,

Theorem 4.4. *Let $f : \mathbb{GR}_N \to \mathbb{G}\overline{\mathbb{R}}_N$ be equitorsion geodesic mapping of two generalized Riemannian spaces $\mathbb{GR}_N$ and $\mathbb{G}\overline{\mathbb{R}}_N$ which are anti-equiaffine of θ-kind, $\theta \in \{1, ..., 4\}$, then the tensors $\underset{1}{R}{}^i_{jmn} + \underset{2}{R}{}^i_{jmn}$, $\underset{3}{R}{}^i_{jmn} - \underset{4}{R}{}^i_{jmn}$, $\underset{5}{R}{}^i_{jmn}$, R^i_{jmn} are the invariants of that mapping.*

In [17] several Ricci type identities are obtained by using non-symmetric affine connection. In these identities 12 curvature tensors $\underset{1}{R}, \underset{2}{R}, \underset{3}{R}, \underset{4}{R}, \underset{1}{\widetilde{R}}, \underset{2}{\widetilde{R}}, ..., \underset{8}{\widetilde{R}}$ appear. In (2.4) five independent tensors are given, and the rest can be expressed as their linear combinations and of the tensor R given at (2.5). Following the notation in [17], we get

Corollary 4.6. *Under equitorsion geodesic mapping of two anti-equiaffine generalized Riemannian spaces of θ-kind, $\theta \in \{1, ..., 4\}$, the next curvature tensors are invariant:*

$$R^i_{jmn}, \quad \underset{1}{\widetilde{R}}{}^i_{jmn} = 2R^i_{jmn} - \frac{1}{2}(\underset{1}{R}{}^i_{jmn} + \underset{2}{R}{}^i_{jmn}), \quad \underset{2}{\widetilde{R}}{}^i_{jmn} = \underset{5}{R}{}^i_{jmn}, \quad \underset{3}{\widetilde{R}}{}^i_{jmn} = 2R^i_{jmn} - \underset{2}{\widetilde{R}}{}^i_{jmn}.$$

5. Conclusion

It is clearly, that a curve γ is a geodesic of connection L^i_{jk} if and only if it is a geodesic of the symmetrized connection $L^i_{\underline{jk}} = \frac{1}{2}(L^i_{jk} + L^i_{kj})$. Any invariant object of the projective class of the connection $L^i_{\underline{jk}}$, is also invariant object of the projective class of the connection L^i_{jk}, but it is not valid inversely.

Since the tensors $\underset{\theta}{R}{}^i_{jmn}$, $\theta \in \{1, \ldots, 5\}$ are generalizations of Riemannian curvature tensor, then the magnitudes $\underset{\theta}{\mathcal{E}}$, $\theta \in \{1, \ldots, 5\}$ are generalizations of the Weyl projective curvature tensor [29]. That means than in geodesic mapping for finding some others invariants the antisymmetric part of connection would be included (2.7). This fact means that geodesic mapping of two non-symmetric affine connection spaces has a sense.

In the case of symmetric affine connection spaces, Ricci tensors of the θ-kind, $\theta \in \{1, ..., 5\}$, reduce to the Ricci tensor obtained from the Riemannian curvature tensor (2.5). Geodesic mapping of two equiaffine spaces of the θ-kind reduces to the geodesic mapping of two equiaffine spaces. Under such mapping, vector ψ_i is always gradient.

Anti-equiaffine spaces of the θ-kind, $\theta \in \{1, ..., 5\}$ reduce to the anti-equiaffine spaces of symmetric connection, when Ricci tensor is anti-symmetric.

Specially, generalized Riemannian spaces are interesting, in which Ricci tensor R_{jm} and Ricci tensor of the fifth kind $\underset{5}{R}{}_{jm}$ are always symmetric. Under equitorsion geodesic mapping of two anti-equiaffine generalized Riemannian spaces of the θ-kind, $\theta \in \{1, ..., 4\}$, the curvature tensors $\underset{5}{R}{}^i_{jmn}$ and R^i_{jmn} are invariants.

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