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The Szeged, vertex PI, first and second Zagreb indices of corona product of graphs

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Abstract. The corona product GoH of two graphs G and H is defined as the graph obtained by taking one copy of G and $|V(G)|$ copies of H and joining the i-th vertex of G to every vertex in the $i-$th copy of H. In this paper, the Szeged, vertex PI and the first and second Zagreb indices of corona product of graphs are computed.

1. Introduction

Let G be a connected graph with vertex and edge sets $V(G)$ and $E(G)$, respectively. The distance between the vertices u and v of G is denoted by $d_G(u, v)$ and it is defined as the number of edges in a shortest path connecting the vertices u and v. A topological index is a numerical quantity related to a graph which is invariant under graph automorphisms. One of the most famous topological indices is the Wiener index introduced by Harold Wiener [25] as an aid to determining the boiling point of paraffin. Since then, the index has been shown to correlate with a host of other properties of molecules (viewed as graphs). For more information about the Wiener index in chemistry and mathematics see $[4-6, 8-11]$. The Wiener index of G is the sum of distances between all unordered pairs of vertices of G, $W(G) = \sum_{\{u,v\}\subseteq V(G)} d_G(u, v)$. The Szeged index $Sz(G)$ is another topological index was introduced by Ivan Gutman [9]. It is defined as $Sz(G) = \sum_{e=uv\in E(G)} n_u(e|G)n_v(e|G)$, where $n_u(e|G)$ is the number of vertices of G lying closer to u than v and $n_u(e|G)$ is defined analogously, see [1, 2, 18, 20] for mathematical properties and chemical meaning of this topological index. It is a well-known fact that for an acyclic graph T, $Sz(T) = W(T)$. The vertex PI index is a recently introduced topological index defined as, $PI_v(G) = \sum_{e=uv\in E(G)}[n_u(e|G) + n_v(e|G)]$, [1, 17]. Notice that for computing Szeged and vertex PI indices, vertices equidistant from u and v are not taken into account. In general, if G is a bipartite graph then $PI_v(G) = |V(G)||E(G)|$. This shows that the vertex PI index is the same for bipartite graphs with n vertices and q edges. On the other hand, the vertex PI index of bipartite graphs has the maximum value between graphs with exactly n vertices and q edges. Finally, the first and second Zagreb indices are defined as $M_1(G) = \sum_{u\in V(G)} \deg_G^2 u$ and $M_2(G) = \sum_{e=uv\in E(G)} \deg_G u \ \deg_G v$, respectively, where $deg_G u$ is the degree of vertex u in G. The interested readers for more information on Zagreb indices can be referred to [12, 13, 16].

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Graph operations play an important role in the study of graph decompositions into isomorphic subgraphs. Let G and H be two simple graphs. If $|V(G)| = n$ and $|E(G)| = q$, we say that G is an $(n, q)-$graph. We also say that G is of order n. The corona product GoH of two graphs G and H is an important graph operation defined as the graph obtained by taking one copy of G and $|V(G)|$ copies of H and joining the $i-$th vertex of G to every vertex in $i-$th copy of H. If G is an $(n, q)-$graph and H is an $(n, q)-$graph then $|V(GoH)| = n + nn'$ and $|E(GoH)| = q + nq' + nn'$. The $i-$th copy of H is denoted by H_i, $1 \le i \le n$ as shown in Fig. 1. It is clear from the definition that corona product of two graphs is not commutative. Obviously, GoH is connected if and only if G is connected. Also if H contains at least one edge then GoH is not bipartite graph.

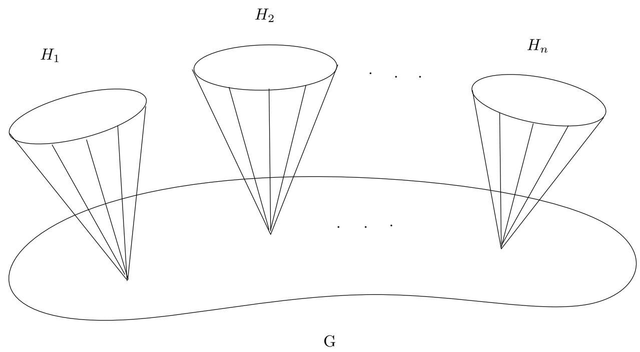

Figure 1: The corona product of two graphs

In this paper we study some topological indices of a graph under corona product. We encourage the reader to consult [3, 15] for our notation and [7, 14, 19 − 24] for more information on graph operations under some topological indices.

2. Main Results

In this section some topological indices of corona product of two graphs are computed. We start by computing the Szeged index of corona product. In what follows, the number of triangles containing an edge $e = uv$ is denoted by t_{uv}.

Theorem 2.1. *Let G be a connected graph of order n. For every (m, q)-graph H, the Szeged index of GoH is given by*

$$Sz(GoH) = nM_2(H) + n \sum_{e=uv\in E(H)} t_{uv}(t_{uv} - \deg_H u - \deg_H v) + (m + 1)^2 Sz(G) + mn(mn + n - 1) - 2nq.$$

Proof. By definition of Szeged index,

$$Sz(GoH) = \sum_{e=uv\in E(GoH)} n_u(e|GoH)n_v(e|GoH).$$

We partition the edges of GoH in to three subset E_1, E_2 and E_3, as follows:

$$E_1 = \{e \in E(GoH) \,|\, e \in E(H_i)\,, 1 \le i \le n\},$$
$$E_2 = \{e \in E(GoH) \,|\, e \in E(G)\},$$
$$E_3 = \{e \in E(GoH) \,|\, e = uv\,, u \in V(H_i)\,, 1 \le i \le n\,, v \in V(G)\}.$$

Therefore,

$$Sz(GoH) = \sum_{e\in E_1} n_u(e\,|GoH)n_v(e\,|GoH) + \sum_{e\in E_2} n_u(e\,|GoH)n_v(e\,|GoH) + \sum_{e\in E_3} n_u(e\,|GoH)n_v(e\,|GoH).$$

For every $e = uv \in E(H)$ if there exists $w \in V(H)$ such that $uw \notin E(H)$ and $vw \notin E(H)$ then $d_{GoH}(u,w) = d_{GoH}(v,w) = 2$. Also if there exists $w \in V(H)$ such that $uw \in E(H)$ and $vw \in E(H)$ then $d_{GoH}(u,w) = d_{GoH}(v,w) = 1$. Hence $n_u(e|GoH) = \deg_H u - t_{uv}$ and so

$$\sum_{e\in E_1} n_u(e|GoH)n_v(e|GoH) = n \sum_{e=uv\in E(H)} (\deg_H u - t_{uv})(\deg_H v - t_{uv}). \tag{1}$$

We now assume that $e = uv \in E_2$. Then for each vertex w closer to u than v, the vertices of the copy of H attached to w are also closer to u than v. Since each copy of H has exactly m vertices, $n_u(e|GoH) = (m+1)n_u(e|G)$. Similarly, $n_v(e|GoH) = (m+1)n_v(e|G)$. Therefore,

$$\sum_{e\in E_2} n_u(e|GoH)n_v(e|GoH) = \sum_{e\in E(G)} (m+1)^2 n_u(e|G)n_v(e|G). \tag{2}$$

Finally, we assume that $e = uv \in E_3$, $deg_H u = k$ and $\{u_1, u_2, ..., u_k\}$ are adjacent vertices of u in H_i. By definition of corona product of graphs, v is adjacent to vertices $u_1, ..., u_k$. Thus for each j, $1 \le j \le k$, u_j is equidistant from u and v. On the other hand, every vertex of GoH other than $u, u_1, .., u_k$ are closer to v than u. This implies that $n_v(e|GoH) = |V(GoH)| - (deg_H u + 1)$ and $n_u(e|GoH) = 1$. Therefore,

$$\sum_{e\in E_3} n_u(e|GoH)n_v(e|GoH) = \sum_{e\in E_3} [|V(GoH)| - (deg_H u + 1)]. \tag{3}$$

We now apply Equations 1-3, we have:

$$\begin{aligned} Sz(GoH) &= n \sum_{e=uv\in E(H)} (\deg_H u - t_{uv})(\deg_H v - t_{uv}) + \sum_{e=uv\in E_2} (1+m)^2 n_u(e|G)n_v(e|G) \\ &\quad + \sum_{e=uv\in E_3} [|V(GoH)| - (\deg_H u + 1)] \\ &= nM_2(H) + n \sum_{e=uv\in E(H)} t_{uv}(t_{uv} - \deg_H u - \deg_H v) + (m+1)^2 Sz(G) + mn(mn+n-1) - 2nq. \end{aligned}$$

By above calculations, one can see that,

$$Sz(GoH) = nM_2(H) + n \sum_{\substack{e=uv \\ e\in E(H)}} t_{uv}(t_{uv} - \deg_H u - \deg_H v) + (m+1)^2 Sz(G) + mn(mn+n-1) - 2nq.$$

□

Corollary 2.2. *Let G be a connected graph of order n and H be a triangle-free (m, q)-graph. Then,*

$$Sz(GoH) = nM_2(H) + (m+1)^2 Sz(G) + mn(mn+n-1) - 2nq.$$

Proof. Substitute $t_{uv} = 0$, for every edge $e = uv \in E(H)$, in the statement of Theorem 1. □

Let P_n, $n \ge 2$, C_n and S_n denote the path, the cycle and the star on n vertices, respectively.

Corollary 2.3. *The following equalities are hold:*

a. $Sz(P_n o P_m) = \begin{cases} \frac{1}{6}n(n^2-1)(m+1)^2 + mn(mn+n+1) - 6n & m \ne 2 \\ \frac{3}{2}n(n^2+4n-3) & m = 2 \end{cases}$,

b. $Sz(S_n o P_m) = \begin{cases} (n-1)^2(m+1)^2 + mn(mn+n+1) - 6n & m \ne 2 \\ 3(5n^2-7n+3) & m = 2 \end{cases}$,

c. $Sz(P_noC_m) = \begin{cases} \frac{1}{6}n(n^2-1)(m+1)^2 + mn(mn+n+1) & m \neq 3 \\ \frac{8}{3}n(n^2-1) + 6n(2n-1) & m = 3 \end{cases}$,

d. $Sz(C_noC_m) = \begin{cases} \frac{1}{4}n^3(m+1)^2 + mn(mn+n+1) & 2|n,\ m \neq 3 \\ \frac{1}{4}n(n-1)^2(m+1)^2 + mn(mn+n+1) & 2 \not| n,\ m \neq 3 \end{cases}$,

e. $Sz(P_noS_m) = \frac{1}{6}n(n^2-1)(m+1)^2 + n(m-1)(m-3) + mn(mn+n-1)$.

Corollary 2.4. *Let $G = P_n$ and $H = K_m^c$ be an empty graph of order m. Then GoH is a Caterpillar tree and $Sz(P_noH) = \frac{1}{6}n(n^2-1)(m+1)^2 + mn(mn+n-1)$.*

In the following theorem, we apply a similar reasoning as in the proof of Theorem 1 to calculate the vertex PI index of corona product of graphs.

Theorem 2.5. *Let G be a connected graph of order n and H be (m,q)-graph, then the vertex PI index of GoH is given by*

$$PI_v(GoH) = (m+1)PI_v(G) + nM_1(H) + n^2m(m+1) - 2n(q+3t),$$

where t is the number triangles of H.

Proof. By definition

$$PI_v(GoH) = \sum_{e\in E_1}[n_u(e|GoH) + n_v(e|GoH)] + \sum_{e\in E_2}[n_u(e|GoH) + n_v(e|GoH)] + \sum_{e\in E_3}[n_u(e|GoH) + n_v(e|GoH)].$$

We compute each summation as follows:

$$\begin{aligned} PI_v(GoH) &= n\sum_{e=uv\in E(H)}[(\deg_H u - t_{uv}) + (\deg_H v - t_{uv})] + \sum_{e=uv\in E(G)}[n_u(e|G) + n_v(e|G)](m+1) \\ &\quad + \sum_{e=uv\in E_3}[|V(GoH)| - \deg_H u]. \\ &= n\sum_{e=uv\in E(H)}[(\deg_H u + \deg_H v) - 2n\sum_{e=uv\in E(H)} t_{uv} + (m+1)PI_v(G) + mn\,|V(GoH)| - \sum_{e\in E_3}\deg_H u. \end{aligned}$$

By above calculations, $PI_v(GoH) = (m+1)PI_v(G) + nM_1(H) + n^2m(m+1) - 2(nq+3nt)$. □

Corollary 2.6. *Suppose H is triangle-free (m,q)-graph and G is a connected graph of order n. Then*

$$PI_v(GoH) = (m+1)PI_v(G) + nM_1(H) + n^2m(m+1) - 2nq.$$

Corollary 2.7. *The following equalities are hold:*

a. $PI_v(P_noP_m) = mn(mn+2n+1) + n(n-5)$,

b. $PI_v(P_noS_m) = n^2(m+1)^2 + n(m-2)^2 - 3n$,

c. $PI_v(P_noC_m) = \begin{cases} n^2(m+1)^2 + n(m-1) & m \neq 3 \\ 4n(4n-1) & m = 3 \end{cases}$,

d. $PI_v(C_noP_m) = \begin{cases} mn(mn+2n+2) + n(n-4) & 2|n \\ mn(mn+2n+1) + n(n-5) & 2 \not| n \end{cases}$.

We end this section by computing the Zagreb indices of corona products.

Theorem 2.8. *Let G be (n,q')-graph and H be (m,q)-graph then*

$$M_1(GoH) = M_1(G) + nM_1(H) + 4(mq' + nq) + mn(m+1),$$

$$M_2(GoH) = n[M_1(H) + M_2(H) + q] + (2q+m)(2q' + mn) + mM_1(G) + M_2(G) + m^2q'.$$

Proof. By definition,

$$\begin{aligned}
M_1(GoH) &= \sum_{u\in V(GoH)} \deg^2_{GoH} u \\
&= n\sum_{u\in V(H)} \deg^2_{GoH} u + \sum_{u\in V(G)} \deg^2_{GoH} u \\
&= n\sum_{u\in V(H)} (\deg_H u + 1)^2 + \sum_{u\in V(G)} (\deg_G u + m)^2 \\
&= n\sum_{u\in V(H)} (\deg^2_H u + 2\deg_H u + 1) + \sum_{u\in V(G)} (\deg^2_G u + 2m\deg_G u + m^2) \\
&= nM_1(H) + 4nq + mn + M_1(G) + 4mq' + m^2 n \\
&= M_1(G) + nM_1(H) + 4(mq' + nq) + mn(m+1).
\end{aligned}$$

In order to compute the second Zagreb index, suppose that $V(G) = \{v_1, v_2, ..., v_n\}$ and $V(H) = \{u_1, u_2, ..., u_m\}$. We partition of the set $E(GoH)$ into three parts and evaluate the resulting sums:

$$\begin{aligned}
M_2(GoH) &= \sum_{e=uv} \deg_{GoH} u \deg_{GoH} v = n\sum_{\substack{e=uv\\ e\in E(H)}} \deg_{GoH} u \deg_{GoH} v \\
&+ \sum_{\substack{e=uv\\ u\in V(H)\\ v\in V(G)}} \deg_{GoH} u \deg_{GoH} v + \sum_{\substack{e=uv\\ e\in E(G)}} \deg_{GoH} u \deg_{GoH} v \\
&= n\sum_{\substack{e=uv\\ e\in E(H)}} (\deg_H u + 1)(\deg_H v + 1) \\
&+ \sum_{i=1}^{n}\sum_{j=1}^{m} (\deg_H u_j + 1)(\deg_G v_i + m) \\
&+ \sum_{\substack{e=uv\\ e\in E(G)}} (\deg_G u + m)(\deg_G v + m) \\
&= n[\sum_{\substack{e=uv\\ e\in E(H)}} \deg_H u \deg_H v + \sum_{\substack{e=uv\\ e\in E(H)}} (\deg_H u + \deg_H v) + \sum_{\substack{e=uv\\ e\in E(H)}} 1] \\
&+ \sum_{i=1}^{n} (\deg_G v_i + m)\sum_{j=1}^{m} (\deg_H u_j + 1)
\end{aligned}$$

$$\begin{aligned}
&+ \sum_{\substack{e=uv\\ e\in E(G)}} \deg_G u \deg_G v + m\sum_{\substack{e=uv\\ e\in E(G)}} (\deg_G u + \deg_G v) + \sum_{\substack{e=uv\\ e\in E(G)}} m^2 \\
&= n[M_1(H) + M_2(H) + q] + (2q + m)\sum_{i=1}^{n} (\deg_G v_i + m) \\
&+ M_2(G) + mM_1(G) + m^2 q'.
\end{aligned}$$

From these equations,

$$M_2(GoH) = n[M_1(H) + M_2(H) + q] + (2q + m)(2q' + mn) + mM_1(G) + M_2(G) + m^2 q'$$

which completes the proof. □

Corollary 2.9. *The following equalities are hold:*

a. $M_1(P_noP_m) = nm^2 + (13n-4)m - 6n - 6$,

b. $M_2(P_noP_m) = (4n-1)m^2 + (17n-12)m - 15n - 4 \qquad m, n \neq 2$,

c. $M_1(C_noC_m) = n(m^2 + 13m + 4)$,

d. $M_2(C_noC_m) = mn(4m+19) + 4n$,

e. $M_1(P_noC_m) = mn(m+13) + 2(2n - 2m - 3)$,

f. $M_2(P_noC_m) = \begin{cases} mn(4m+19) - m(m+12) + 4(n-2) & n \neq 2 \\ 7m^2 + 26m + 1 & n = 2 \end{cases}$.

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References

[1] A. R. Ashrfi, M. Gorbani, M. Jalali, The vertex PI and Szeged indices of an infinite family of fullerenes, Journal of Theoretical and Computational Chemistry 7(2) (2008) 221–231.
[2] A. R. Ashrfi, B. Manoochehrian, H. Yousefi-Azari, On Szeged polynomial of a graph, The Bulletin of the Iranian Mathematical Society 33 (2007) 37–46.
[3] F. Buckley, F. Harary, Distance in Graphs, Addison-Wesely, Redwood, Calif, USA, 1990.
[4] M.V. Diudea, I. Gutman, L. Jantschi, Molecular Topology, Huntington, NY, 2001.
[5] A. A. Dobrynin, R. Entringer, I. Gutman, Wiener index of trees: theory and applications, Acta Applicandae Mathematicae 66 (2001) 211–249.
[6] A.A. Dobrynin, I. Gutman, S. Klavzar, P. Zigert, Wiener index of hexagonal systems, Acta Applicandae Mathematicae 72 (2002) 247–294.
[7] G. H. Fath-Tabar, A. Hamzeh, S. Hossein-Zadeh, GA_2 index of some graph operations, Filomat 24(1) (2010) 21–28.
[8] A. Graovac, T. Pisanski, On the Wiener index of a graph, Journal of Mathematical Chemistry 8 (1991) 53–62.
[9] I. Gutman, A formula for the Wiener number of trees and its extension to graphs containing cycles, Graph Theory Notes of New York 27 (1994) 9–15.
[10] I. Gutman, S. Klavzar, B. Mohar (eds.), Fifty Years of the Wiener Index, MATCH Communications in Mathematical and in Computer Chemistry 35 (1997) 1–259.
[11] I. Gutman, S. Klavzar, B. Mohar (eds.), Fiftieth Anniversary of the Wiener Index, Discrete Applied Mathematics 80(1) (1997) 1–113.
[12] I. Gutman, N. Trinajstic, Graph theory and molecular orbitals, Total π electron energy of alternant hydrocarbons, Chemical Physics Letters 17 (1972) 535–538.
[13] I. Gutman, K.C. Das, The first Zagreb index 30 years after, MATCH Communications in Mathematical and in Computer Chemistry 50 (2004) 83–92.
[14] S. Hossein-Zadeh, A. Hamzeh, A. R. Ashrafi, Wiener-type invariants of some graph operations, Filomat 23 (2009) 103–113.
[15] W. Imrich, S. Klavzar, Product Graphs, Structure and Recognition, John Wiley & Sons, New York, USA, 2000.
[16] M. Khalifeh, H. Yousefi-Azari, A. R. Ashrafi, The first and second Zagreb indices of some graph operations, Discrete Applied Mathematics 157 (2009) 804–811.
[17] M. H. Khalifeh, H. Yousefi-Azari, A. R. Ashrafi, Vertex and edge PI indices of Cartesian product graphs, Discrete Applied Mathematics 156 (2008) 1780–1789.
[18] S. Klavzar, A. Rajapakse, I. Gutman, The Szeged and the Wiener index of graphs, Applied Mathematics Letter 9 (1996) 45–49.
[19] S. Klavzar, On the PI Index: PI-Partitions and Cartesian Product Graphs, MATCH Communications in Mathematical and in Computer Chemistry 57(3) (2007) 573–586.
[20] T. Mansour, M. Schork, The vertex PI and Szeged index of bridge and chain graphs, Discrete Applied Mathematics 157 (2009) 1600–1606.
[21] D. Stevanović, Hosoya polynomial of composite graphs, Discrete Applied Mathematics 235 (2001) 237–244.
[22] Z. Yarahmadi, T. Došlić, A. R. Ashrafi, The bipartite edge frustration of composite graphs, Discrete Applied Mathematics 158 (2010) 1551–1558.
[23] Y. N. Yeh, I. Gutman, On the sum of all distances in composite graphs, Discrete Mathematics 135 (1994) 359–365.
[24] H. Yousefi-Azari, B. Manoochehrian, A. R.Ashrafi, PI index of product graphs, Applied Mathematics Letter 21 (2008) 624–627.
[25] H. Wiener, Structural determination of the paraffin boiling points, Journal of the American Chemical Society 69 (1947) 17–20.