Filomat 26:3 (2012), 479–510
DOI 10.2298/FIL1203479M

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Quasiconformality of harmonic mappings between Jordan domains

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Abstract. Suppose that h is a harmonic mapping of the unit disc onto a $C^{1,\alpha}$ domain D. Then h is q.c. if and only if it is bi-Lipschitz. In particular, we consider sufficient and necessary conditions in terms of boundary function that h is q.c. We give a review of recent related results including the case when domain is the upper half plane. We also consider harmonic mapping with respect to ρ metric on codomain.

1. Introduction

This is mainly a review paper which is an extension of [52]. We mainly consider some results, related to harmonic quasiconformal mappings, obtained by the participants of Belgrade Analysis Seminar. Here and in [11, 52] we outline a proof of Theorem 1.2, which currently seems to be one of the main new result in this area. We give proof of Theorem 4.15 stated in [30] and we also announce and outline proofs of some new results: see for instance Theorems 5.10, 5.11, 5.13, 5.14 and 5.15. In particular, in subsection 3.3 we refine and generalize some results from [26], related to the geometry of $C^{1,\mu}$ curves.

The first characterization of harmonic quasiconformal mappings with respect to the Euclidean metric for the unit disc was given by O. Martio [47]. Thereafter this area has mainly been studied by the participants of Belgrade Seminar for Analysis; for a partial review and further results see for example [37, 50], references cited in this article and [37, 50].

Throughout this paper, $\mathbb{D}$ will denote the unit disc $\{z : |z| < 1\}$, $\mathbb{T}$ the unit circle, $\{z : |z| = 1\}$ and we will use notation $z = re^{i\theta}$.

By $\partial_\theta h$ and $\partial_r h$ (or sometimes by h'_r and h'_θ), h'_x and h'_y we denote partial derivatives with respect to θ and r, x and y respectively.

Every harmonic function h in $\mathbb{D}$ can be written in the form $h = f + \bar{g}$, where f and g are holomorphic functions in $\mathbb{D}$. Then an easy calculation shows $\partial_\theta h(z) = i(zf'(z) - \overline{zg'(z)})$, $h'_r = e^{i\theta} f' + \overline{e^{i\theta} g'}$, $h'_\theta + irh'_r = 2izf'$ and therefore rh'_r is the harmonic conjugate of h'_θ. We also use notation $p = f'$, $q = g'$, $\Lambda_h = |f'| + |g'|$, $\lambda_h = |f'| - |g'|$ and $\mu_h = q/p$.

We need some facts related to Hardy spaces (for more details see for example [14, 15, 19, 42, 59]).

For $f : \mathbb{U} \to \mathbb{C}$, define

$$f_*(\theta) = f^*(e^{i\theta}) = \lim_{r\to 1} f(re^{i\theta})$$

when this limit exists. For $f : \mathbb{T} \to \mathbb{C}$, define $\underline{f}(\theta) = f(e^{i\theta})$. Sometimes we use the notation f_* instead of $\underline{f}$.

2010 *Mathematics Subject Classification*. Primary 30C55; Secondary 31C62
Keywords. Complex functions, planar convex harmonic mappings, universal Teichmüller space
Received: 01 June 2010; Accepted: 16 June 2010
Communicated by Dragan S. Djordjević
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The class of all holomorphic (respectively harmonic, complex valued continuous) functions in a plane domain D will be denoted by $H(D)$ (respectively $h(D)$,$C(D)$).

For $f \in C(\mathbb{D})$, define

$$\begin{aligned} M_p(f;r) &= \left(\frac{1}{2\pi}\int_{-\pi}^{\pi}|f(re^{it}|dt\right)^{1/p}, \quad 0<p<\infty \\ M_\infty(f;r) &= \sup_t |f(re^{it}| \end{aligned}$$

and put $\|f\|_p = \lim_{r\to1} M_p(f;r)$. The class H^p (respectively h^p) consists of all $f \in H(\mathbb{D})$ (respectively $f \in h(\mathbb{D})$) for which $\|f\|_p < \infty$.

Let $\mathfrak{M}$ be a σ-algebra in a set X. For a complex Borel measure μ we define a set function $|\mu|$ by

$$|\mu|(E) = \sup\sum_1^\infty |\mu(E_i)|,$$

where the supremum is taken over all partitions E_i of E. It turns out that $|\mu|$ actually is a measure. The set function $|\mu|$ is called the *total variation* of μ, or sometimes, to avoid misunderstanding, the *total variation measure*. The term "total variation of μ" is also frequently used to denote the number $|\mu|(X)$.

A complex Borel measure has it range in the complex plane, but our usage of the term "positive measure" includes ∞ as an admissible value. Thus the positive measures do not form a subclass of the complex ones. If μ is a complex Borel measure on X, then $|\mu|(X) < \infty$.

If μ is a real measure we define $\mu^+ = \frac{1}{2}(|\mu| + \mu)$ and $\mu^- = \frac{1}{2}(|\mu| - \mu)$. Then both μ^+ and μ^- are positive measures and they are bounded. Also, $\mu = \mu^+ - \mu^-$ and $|\mu| = \mu^+ + \mu^-$. This representation of μ is known as the Jordan decomposition of μ and has a certain minimum property which is a corollary of the Hahn decomposition theorem: if $\mu = \lambda_1 - \lambda_2$, where λ_1 and λ_2 are positive measures, then $\lambda_1 \geq \mu^+$ and $\lambda_2 \geq \mu^-$.

The measures μ^+ and μ^- are called the positive and negative variations of μ, respectively.

1. Suppose that μ is a real Borel measure on $\mathbb{T}$ and $F = P[d\mu]$. Then

   $$\lim_{r\to1} F(re^{it}) = (D\mu)(t)$$

   exists and is finite for almost all t (with respect to Lebesgue measure). Note that $(D\mu)(t)$ is here defined as the symmetric derivative of μ at t.
   It turns out that there is one-to-one correspondence between the set of all positive finite Borel measure on $\mathbb{T}$ and the set of all positive harmonic functions in $\mathbb{D}$.
   It is clear that the Poisson integral $F = P[d\mu]$ of every positive finite Borel measure on the unit circle $\mathbb{T}$ is a positive harmonic function in the open disk $\mathbb{D}$.
   If $F = P[d\mu]$, where μ is any complex Borel measure on $\mathbb{T}$, Fubini's theorem shows that

   $$\int_{-\pi}^{\pi} |F(re^{it}|dt \leq |\mu|(\mathbb{T})$$

   where $|\mu|$ is the total variation of μ on $\mathbb{T}$.
   Now if F is a positive harmonic functions in the open disk $\mathbb{D}$, then $|F| = F$, so the first integral is $2\pi F(0)$, for every $r \in [0, 1)$. Thus positive harmonic functions satisfy the necessary condition which we just found, and we are led to the following stronger theorem.
2. The mapping $\mu \to P[d\mu]$ is a linear one-to-one correspondence between the space of all complex Borel measure on $\mathbb{T}$ and the space h^1.
   There is one-to-one correspondence between the set of all positive finite Borel measure on $\mathbb{T}$ and the set of all positive harmonic functions in $\mathbb{D}$.
3. If $f \in H^1$, then $f^*(e^{i\theta}) = \lim_{r\to1} f(re^{i\theta})$ exists at almost all point of $\mathbb{T}$, and

   $$\lim_{r\to1}\int_{-\pi}^{\pi} |f^*(e^{it}) - f(re^{it})|\,dt = 0\,.$$

4. If $f \in H^1$, then f is the Poisson integral and the Cauchy integral (see formulas (1.2) and (1.1) below) of f_*.
5. If $f \in H^p$, $p > 0$, then $f^*(e^{i\theta}) = \lim_{r\to 1} f(re^{i\theta})$ exists at almost all point of $\mathbb{T}$, and
$$\lim_{r\to 1} \int_{-\pi}^{\pi} |f^*(re^{it})|^p\, dt = \int_{-\pi}^{\pi} |f^*(e^{it})|^p\, dt,$$
$$\lim_{r\to 1} \int_{-\pi}^{\pi} |f^*(e^{it}) - f(re^{it})|^p\, dt = 0\,.$$
6. Every real harmonic function u in the unit disk $\mathbb{D}$ is the real part of one $f \in H(\mathbb{D})$ such that $f(0) = u(0)$. If $f + u + iv$, the last requirement can also be stated in the form $v(0) = 0$. The function v is called the harmonic conjugate of u, or the conjugate function of u. $M_p(u; r)$ is monotonically increasing function of r in $[0, 1)$.
To each $p \in (1, \infty)$ there corresponds a constant A_p such that $\|u\|_p \le A_p \|v\|_p$ for every real harmonic function u if v is the conjugate function of u.
7. Let f be holomorphic on $\mathbb{D}$. Then $f' \in H^1$ iff f has continuous extension to $\overline{\mathbb{D}}$ and $\underline{f}$ is absolutely continuous.
If $f' \in H^1$, then $\underline{f}' = ie^{it} f'(e^{it})$ a.e. Here $f'(e^{it})$ denotes the non tangential limit when z tends to e^{it}.
8. Consider now domain D bounded by a rectifiable Jordan curve. Let ω be a conformal mapping of $\mathbb{D}$ onto D. Then
   (a) ω has continuous extension to $\overline{\mathbb{D}}$ and $\underline{\omega}$ is absolutely continuous on $[0, 2\pi]$,
   (b) $\omega' \in H^1$,
   (c) $\underline{\omega}'(t) = ie^{it}\omega'(e^{it})$ a.e.
   (d) length $s(t_1, t_2)$ of arc $w = \omega(e^{it})$, $t_1 \le t \le t_2$, is $s(t_1, t_2) = \int_{t_1}^{t_2} |\omega'(e^{it})|\, dt$,
   (e) ω is conformal a.e. on $\mathbb{T}$.

   Hint: Let A be the set of points $t \in [0, 2\pi)$ for which there exists $\underline{\omega}'(t) \ne 0$, let $t_0 \in A$ and $z_0 = e^{it_0}$. The function
$$u(z, z_0) = arg \frac{\omega(z) - \omega(z_0)}{z - z_0}$$
   is continuous in $\overline{\mathbb{D}} \setminus \{z_0\}$, bounded on $\mathbb{D}$, u^* is continuous and $u = P[u^*]$.
9. Let D be a domain bounded by a smooth Jordan curve C and ω conformal mapping of $\mathbb{D}$ onto D and ϕ the inverse mapping.
   (i) Then there is a branch $arg\omega'$ of $Arg\omega'$ on $\mathbb{D}$. The functions $arg\omega'$ and $arg\phi'$ have continuous extension to $\overline{\mathbb{D}}$ and $\overline{D}$ respectively, and
   (ii) $arg\,\omega'(z) = arg\,\mathrm{T}(w) - arg z - \pi/2$, $z \in \mathbb{T}$, where T is the tangent of the curve C, parameterized by $t \mapsto \omega(e^{it})$, at $w = \omega(e^{it})$; more precisely there is a branch $argT$ of multi-valued function $ArgT$ on $C \setminus \{\omega(1)\}$ such that (ii) holds for $z \in \mathbb{T} \setminus \{1\}$.

   Hint: (i) For a fixed τ the function
$$u(z, \tau) = arg \frac{\omega(ze^{i\tau}) - \omega(z)}{ze^{i\tau} - z}$$
   is harmonic in $z \in \overline{\mathbb{D}}$ and there is K such that $|u(z, \tau)| < K$ for $z \in \overline{\mathbb{D}}$ and $\tau \in (0, \pi)$.
   (ii) Use $\mathrm{T}(\omega(e^{it})) = \underline{\omega}'(t) = ie^{it}\omega'(e^{it})$.
10. Let γ be a Jordan curve. By the Riemann mapping theorem there exists a Riemann conformal mapping of the unit disk onto a Jordan domain $\Omega = \mathrm{int}\ \gamma$. By Caratheodory's theorem it has a continuous extension to the boundary. Moreover if $\gamma \in C^{n,\alpha}$, $n \in \mathbb{N}$, $0 \le \alpha < 1$, then the Riemann conformal mapping has $C^{n,\alpha}$ extension to the boundary (this result is known as Kellogg's theorem), see [61].

Let D be a plane domain and F a family of functions $f : D \to \mathbb{C}$. If F is uniformly bounded on each compact subset of the region D we say that F is uniformly bounded inside D.

Let $\mathcal{F} \subset \mathcal{H}(D)$. The following conditions are equivalent

1. For every sequence $f_n \in \mathcal{F}$, there is subsequence f_{n_k} which converges uniformly inside D to a holomorphic function.
2. $\mathcal{F}$ is uniformly bounded inside D.

Now we list some properties of harmonic functions. For the proof of the next statements see for example [19].

1. If a sequence of harmonic function u_n, $n = 1, 2, \dots$ in D, converges uniformly inside D to a function u, then u is harmonic in D.
2. If u_n, $n = 1, 2, \dots,$ is a sequence of harmonic function in D, which is uniformly bounded inside D, then there is subsequence u_{n_k} which converges uniformly inside D to a harmonic function.
3. Harnack theorem: If u_n, $n = 1, 2, \dots,$ is a sequence of harmonic function in D, which is not decreasing ($u_n(z) \le u_{n+1}(z)$, $z \in D$), then it converges uniformly inside D to a harmonic function or $+\infty$.

Let

$$P_r(t) = \frac{1 - r^2}{1 - 2r\cos(t) + r^2}$$

denote the Poisson kernel.

If $\psi \in L^1[0, 2\pi]$ and

$$h(z) = \frac{1}{2\pi} \int_0^{2\pi} P_r(\theta - t)\, \psi(t)\, dt, \tag{1.1}$$

then the function $h = P[\psi]$ so defined is called Poisson integral of ψ.

If ψ is of bounded variation, define $T_\psi(x)$ as variation of ψ on $[0, x]$, and let $V(\psi)$ denote variation of ψ on $[0, 2\pi]$ (see, for example, [59] p.171).

Define

$$h_*(\theta) = h^*(e^{i\theta}) = \lim_{r \to 1} h(re^{i\theta})$$

when this limit exists.

If $\psi \in L^1[0, 2\pi]$ (or $L^1[\mathbb{T}]$), then the Cauchy transform $C(\psi)$ is defined as

$$C(\psi)(z) = \frac{1}{2\pi} \int_0^{2\pi} \frac{\psi(t)e^{it}}{e^{it} - z}\, dt \tag{1.2}$$

with its kernel

$$K(z, t) = \frac{e^{it}}{e^{it} - z}\,.$$

While the Hilbert transform $H(\psi)$ is defined as

$$H(\psi)(\varphi) = -\frac{1}{2\pi} \int_{0_+}^{\pi} \frac{\psi(\varphi + t) - \psi(\varphi - t)}{\tan t/2}\, dt,$$

where we abuse notation by extending ψ to be 2π periodic, or consider it to be a function from $L^1(\mathbb{T})$. The following property of the Hilbert transform is also sometimes taken as the definition:
If $u = P[\psi]$ and v is the harmonic conjugate of u, then $v_* = H(\psi)$ a.e.

Note that, if ψ is 2π-periodic, absolutely continuous on $[0, 2\pi]$ (and therefore $\psi' \in L^1[0, 2\pi]$), then

$$h'_\theta = P[\psi']. \tag{1.3}$$

Hence, since rh'_r is the harmonic conjugate of h'_θ, we find

$$\begin{aligned} &rh'_r = P[H(\psi')], \\ &(h'_r)^*(e^{i\theta}) = H(\psi')(\theta) \ \text{ a.e.} \end{aligned} \tag{1.4}$$

It is clear that $K(z,t) + \overline{K(z,t)} - 1 = P_r(\theta - t)$.

Recall, for $f : \mathbb{U} \to \mathbb{C}$, define

$$f_*(\theta) = f^*(e^{i\theta}) = \lim_{r\to 1} f(re^{i\theta})$$

when this limit exists. For $f : \mathbb{T} \to \mathbb{C}$, define $\underline{f}(\theta) = f(e^{i\theta})$.

If f is a bounded harmonic map defined on the unit disc $\mathbb{U}$, then f^* exists a.e., f^* is a bounded integrable function defined on the unit circle $\mathbb{T}$, and f has the following representation

$$f(z) = P[f^*](z) = \frac{1}{2\pi}\int_0^{2\pi} P(r, t-\varphi) f^*(e^{it})dt,$$

where $z = re^{i\varphi}$.

A homeomorphism $f\colon D \to G$, where D and G are subdomains of the complex plane $\mathbb{C}$, is said to be K-quasiconformal (K-q.c or k-q.c), $K \geq 1$, if f is absolutely continuous on a.e. horizontal and a.e. vertical line in D and there is $k \in [0,1)$ such that

$$|f_{\bar z}| \leq k|f_z| \quad \text{a.e. on } D, \tag{1.5}$$

where $K = (1+k)/(1-k)$, i.e. $k = (K-1)/(K+1)$.

Note that the condition (1.5) can be written as

$$D_f := \frac{\Lambda}{\lambda} = \frac{|f_z| + |f_{\bar z}|}{|f_z| - |f_{\bar z}|} \leq K,$$

where $K = \frac{1+k}{1-k}$ i.e. $k = \frac{K-1}{K+1}$.

Note that if γ is 2π-periodic absolutely continuous on $[0, 2\pi]$ (and therefore $\gamma' \in L^1[0, 2\pi]$) and $h = P[\gamma]$, then

$$(h'_r)^*(e^{i\theta}) = H(\gamma')(\theta) \,\text{a.e.},$$

where H denotes the Hilbert transform.

Let Γ be a curve of $C^{1,\mu}$ class and $\gamma : \mathbb{R} \to \Gamma^*$ be arbitrary topological (homeomorphic) parametrization of Γ and $s(\varphi) = \int_0^\varphi |\gamma'(t)|dt$. It is convenient to abuse notation and to denote by $\Gamma(s)$ natural parametrization.

For $\Gamma(s) = \gamma(\varphi)$, we define $n_\gamma(\varphi) = i\Gamma'\big(s(\varphi)\big)$ and $R_\gamma(\varphi, t) = (\gamma(t) - \gamma(\varphi), n_\gamma(\varphi))$.

For $\theta \in \mathbb{R}$ and $h = P[\gamma]$, define

$$\begin{aligned} &E_\gamma(\theta) = \big((h'_r)^*(e^{i\theta}), n_\gamma(\theta)\big) = \big(H(\gamma')(\theta), n_\gamma(\theta)\big), \ a.e. \\ &v(z,\theta) = v_\gamma(z,\theta) = \big(rh'_r(z), n_\gamma(\theta)\big), \quad z \in \mathbb{D}. \end{aligned}$$

Note that $v_*(t,\theta) = \big(H(\gamma'_*)(t), n_\gamma(\theta)\big)$ a.e.

1.1. Background

To each mapping (in particular closed curve Γ) given by $\gamma : \mathbb{T} \to \mathbb{C}$, we associate a function $\gamma_* : \mathbb{R} \to \mathbb{C}$ defined by $\gamma_*(t) = \gamma(e^{it})$; we also call $\gamma_* : [0, 2\pi] \to \Gamma^*$ a parametrization of Γ. Harmonic quasiconformal (abbreviated by HQC) mappings are now very active area of investigation (see for example [29, 35, 36, 55] and the references cited there).

Let $\mathcal{D}_1$ (respectively $\mathcal{D}_2$) be the family of all Jordan domains in the plane which are of class $C^{1,\mu}$ (res $C^{2,\mu}$) for some $0 < \mu < 1$.

In [26] the following result is proved:

Theorem A. *Let Ω and Ω_1 be Jordan domains, let $\mu \in (0, 1]$, and let $f : \Omega \to \Omega_1$ be a harmonic homeomorphism. Then:*

(*a*) *If f is q.c. and $\partial\Omega, \partial\Omega_1 \in \mathcal{D}_1$, then f is Lipschitz;*

(*b*) *if f is q.c., $\partial\Omega, \partial\Omega_1 \in \mathcal{D}_1$ and Ω_1 is convex, then f is bi-Lipschitz; and*

(*c*) *if Ω is the unit disk, Ω_1 is convex, and $\partial\Omega_1 \in C^{1,\mu}$, then f is quasiconformal if and only if its boundary function is bi-Lipschitz and the Hilbert transform of its derivative is in L^∞.*

In [27] it is proved that the convexity hypothesis can be dropped if codomain is in $\mathcal{D}_2$:

(b1) if f is q.c., $\partial\Omega \in \mathcal{D}_1$ and $\partial\Omega_1 \in \mathcal{D}_2$, then f is bi-Lipschitz.

Similar results were announced in [51]. These extend the results obtained in [24, 38, 47, 56].

The proof of the part (*a*) of Theorem A in [26] is based on an application of Mori's theorem on quasiconformal mappings, which has also been used in [56] in the case $\Omega_1 = \Omega = \mathbb{D}$, and the following lemma.

Lemma 1.1. *Let Γ be a curve of class $C^{1,\mu}$ and $\gamma : \mathbb{T} \to \Gamma^*$ be arbitrary topological (homeomorphic) parametrization of Γ. Then*

$$|R_\gamma(\varphi, t)| \le A\, |\gamma((e^{i\varphi}) - \gamma(e^{it})|^{1+\mu},$$

where $A = A(\Gamma)$.

In [30], we prove a version of "inner estimate" for quasi-conformal diffeomorphisms, which satisfies a certain estimate concerning their Laplacian. As an application of this estimate, we show that quasiconformal harmonic mappings between smooth domains (with respect to the approximately analytic metric), have bounded partial derivatives; in particular, these mappings are Lipschitz. Our discussion in [30] includes harmonic mappings with respect to (a) spherical and euclidean metrics (which are approximately analytic) as well as (b) the metric induced by the holomorphic quadratic differential.

1.2. HQC are bi-Lipschitz

We now announce some results obtained in [11] and outline their proofs. The results make use of the Gehring-Osgood inequality [16], as we are going to explain; see Section 5 for more details.

Let Ω be Jordan domain in $\mathcal{D}_1$, γ curve defined by $\partial\Omega$ and h K-qch from $\mathbb{D}$ onto Ω and $h(0) = a_0$. Then h is L-Lipschitz, where L depends only on $K, dist(a_0, \partial\Omega)$ and $\mathcal{D}_1$ constant C_γ. In [11] we give an explicit bound for the Lipschitz constant.

Let h be a harmonic quasiconformal map from the unit disk onto D in class $\mathcal{D}_1$. Examples show that a q.c. harmonic function does not have necessarily a C^1 extension to the boundary as in conformal case. In [11] it is proved that the corresponding functions E_{h_*} are continuous on the boundary and for fixed θ_0, $v_{h_*}(z, \theta_0)$ is continuous in z at $e^{i\theta_0}$ on $\mathbb{D}$.

The main result in [11] is

Theorem 1.2. *Let Ω and Ω_1 be Jordan domains in $\mathcal{D}_1$, and let $h : \Omega \to \Omega_1$ be a harmonic q.c. homeomorphism. Then h is bi-Lipschitz.*

It seems that we use a new idea. We reduce proof to the case when $\Omega = H$. Suppose that $h(0) = 0 \in \Omega_1$. We show that there is a convex domain $D \subset \Omega_1$ in $\mathcal{D}_1$ such that $\gamma_0 = \partial D$ touch the boundary of Ω_1 at 0. Since there is qc extension h_1 of h to $\mathbb{C}$, we can apply the Gehring-Osgood theorem to $h_1 : \mathbb{C}^* \to \mathbb{C}^*$. This gives estimate for $arg\gamma_1(z)$ for z near 0, where $\gamma_1 = h^{-1}(\gamma_0)$, and we show that there is a domain $D_0 \subset \mathbb{H}$ in $\mathcal{D}_1$ such that $h(D_0) \subset D$. Finally, we combine the convexity type argument and noted continuity of functions E and v to finish the proof.

2. Preliminary results

We first give an extension of Proposition 1 [49]:

Proposition 2.1. *Suppose that $\psi : [0, 2\pi] \to \mathbb{C}$ is of bounded variation and $h = P[\psi]$. Then*

(1) *$l(r) = \int_0^{2\pi} |h'_\theta(z)|\, d\theta \leq V(\psi)$ and $l(r)$ is increasing,*

(2) *$f', g' \in H^p$ for every $0 < p < 1$,*

(3) *h_r^* exists a.e.*

$$\partial_r h(e^{it}) = \lim_{r\to 1_{-0}} \frac{h^*(e^{it}) - h(re^{it})}{1-r}$$

exists a.e. and $(\partial_r h)^*(e^{it}) = \partial_r h(e^{it})$ a.e. $(\partial_\theta h)^* = \psi'$ a.e.

(4) *If ψ is absolutely continuous, then $C[\psi'](z) = izf'(z)$ and $izf'(z)$ is the analytic part of $\partial_\theta h$. Also, $C[\overline{\psi'}](z) = izg'(z)$,*

(5) *If $h = P[\psi]$ is K-q.c., then h_* is absolutely continuous and $h'_*(t) \neq 0$ a.e.*

Proof. (1) If ψ is of bounded variation, then

$$\partial_\theta h(z) = \frac{1}{2\pi} \int_0^{2\pi} P_r(\theta - t)\, d\psi(t)$$

and hence

$$l(r) = l(r, h) = \int_0^{2\pi} |h'_\theta(z)|\, d\theta \leq \frac{1}{2\pi} \int_0^{2\pi} \int_0^{2\pi} P_r(\theta - t)\, dT_\psi(t) d\theta\,,$$

where $T_\psi(x)$ is total variation of ψ on $[0, x]$. Thus $l(r) \leq \int_0^{2\pi} dT_\psi(t) = T_\psi(2\pi) = V(\psi)$.

Since h'_θ is harmonic, $|h'_\theta|$ is subharmonic and therefore $l(r)$ is increasing.

(2) Since $h'_\theta \in h^1$, then the Cauchy transform $C[h_\theta] \in H^p$ for every $0 < p < 1$.

(3) We leave the proof of (3) to the interested reader.

(4) Note that $d\psi(t) = \psi'(t)dt + d\sigma(t)$, where σ is a singular measure with respect to Lebesgue measure, i.e., one supported on a set of Lebesgue measure zero. If ψ is absolutely continuous, then $d\psi(t) = \psi'(t)dt$. Hence, we find $h'_\theta = P[\psi']$ and therefore (4). Note that here absolute continuity of ψ is essential.

If $M \subset \mathbb{T}$ is Lebesgue measurable, we denote by $|M|$ its Lebesgue measure.

(5) Since $|f'| + |g'| \leq K |h'_\theta(z)|$, then $f', g' \in H^1$. Hence, we conclude that h_* is absolutely continuous.

Let $A_0 = \{e^{it} : h'_*(t) = 0\}$, $dh = pdz + qd\overline{z}$ and $E_0 \subset \mathbb{T}$ the set on which p^*, q^* exist and $|p^*| \leq K|q^*|$. If $z_0 \in A_0 \cap E_0$, then $p = q = 0$ at z_0.

Since p and $\overline{q}$ are analytic on $\mathbb{D}$ and belong to H^1, we conclude that $|A_0 \cap E_0| = 0$. Since $\mathbb{T} \setminus E_0$ has measure 0, we conclude that $|A_0| = 0$. □

3. Characterizations of HQC

3.1. *The half plane*

By $\mathbb{H}$ we denote the upper-half plane and $\Pi^+ = \{z : \mathrm{Re} z > 0\}$.

The first characterizations of the HQC conditions have been obtained by Kalaj in his thesis research.

In the case of the upper half plane, the following known fact plays an important role, cf for example [41]:

Lemma B. *Let f be an euclidean harmonic $1-1$ mapping of the upper half-plane $\mathbb{H}$ onto itself, continuous on $\overline{\mathbb{H}}$, normalized by $f(\infty) = \infty$ and $v = \mathrm{Im} f$. Then $v(z) = c\,\mathrm{Im} z$, where c is a positive constant. In particular, v has bounded partial derivatives on $\mathbb{H}$.*

The lemma is a corollary of the Herglotz representation of the positive harmonic function v (see for example [7]).

Theorem 3.1. ([52]) *Let $h : \mathbb{H} \to \mathbb{H}$ be harmonic function. Then h is orientation preserving harmonic diffeomorphism of $\mathbb{H}$ onto itself, continuous on $\mathbb{H} \cup \mathbb{R}$ such that $h(\infty) = \infty$ if and only if there are analytic function $\phi : \mathbb{H} \to \Pi^+$ and constants $c > 0$ and $c_1 \in \mathbb{R}$ such that $\lim_{z\to\infty} \Phi_1(z) = \infty$, where*

$$\Phi(z) = \int_i^z \phi(\zeta) d\zeta, \ \Phi_1 = \mathrm{Re}\,\Phi, \ \textit{and}$$

$$h(z) = h^{\phi}(z) = \Phi_1(z) + icy + c_1, \ z \in \mathbb{H}. \tag{*1}$$

Let χ denote restriction of h on $\mathbb{R}$. In this setting, $h(z) = h^{\phi}(z) = P[\chi] + icy$, $z \in \mathbb{H}$, where $P = P_{\mathbb{H}}$ denotes the Poisson kernel for the upper half-plane $\mathbb{H}$.

A version of this result is proved in [24].

Let $h = u + iv$. By Lemma B, $u = \mathrm{Re}\,\Phi$ and $v = cy$, where $c > 0$ and Φ is analytic function in $\mathbb{H}$. Since $\Phi'_y = i\Phi'$ and $h(z) = h^{\phi}(z) = (\Phi(z) + \overline{\Phi(z)})/2 + icy + c_1$, we find

$$h'_y(z) = (i\Phi'(z) + \overline{i\Phi'(z)})/2 + ic = (i\phi - i\overline{\phi})/2 + ic = -\mathrm{Im}\,\phi(z) + ic.$$

Hence

($X1$) $h'_x(z) = \mathrm{Re}\,\phi(z)$ and $h'_y(z) = -\mathrm{Im}\,\phi(z) + ic$. Since $h_z = (h'_x - ih'_y)/2 = \phi/2 + c/2$ is analytic, $-h'_y$ is harmonic conjugate of h'_x and therefore

($X2$) $h'_y = H(h'_*) = \mathrm{Im}\,\phi(z) - ic$,

where h_* denotes the restriction of h on $\mathbb{R}$.

By $HQC_0(\mathbb{H})$ (respectively $HQC_0^k(\mathbb{H})$) we denote the set of all qc (respectively k-qc) harmonic mappings h of $\mathbb{H}$ onto itself for which $h(\infty) = \infty$.

Recall by χ we denote restriction of h on $\mathbb{R}$. If $h \in HQC_0(\mathbb{H})$ it is well-known that $\chi : \mathbb{R} \to \mathbb{R}$ is a homeomorphism and $\mathrm{Re}\, h = P[\chi]$. Now we give characterizations of $h \in HQC_0(\mathbb{H})$ in terms of its boundary value χ.

Theorem 3.2. *The following condition are equivalent*

($A1$) $h \in HQC_0(\mathbb{H})$,

($A2$) *there is analytic function $\phi : \mathbb{H} \to \Pi^+$ such that $\phi(\mathbb{H})$ is relatively compact subset of Π^+ and $h = h^{\phi}$.*

Proof. Suppose ($A1$). We can suppose that h is K-qc and $c = 1$ in the representation (∗1). Since $v(z) = \mathrm{Im}\, h(z) = y$, we have $\lambda_h \geq 1/K$. Let $z_0 \in \mathbb{H}$ and define the curve $L = \{z : \Phi_1(z) = \Phi_1(z_0)\}$ and denote by l_0 the unit tangent vector to the curve L at z_0. Since $|dh_{z_0}(l_0)| \leq 1$, we have $\Lambda_h \leq K$ on $\mathbb{H}$. Hence absolute values of partial derivatives of h are bounded from above and below by two positive constants. Thus, by ($X1$) and ($X2$), ϕ is bounded on $\mathbb{H}$.

In particular, $(A1)$ implies that h is bi-Lipschitz. Hence there two positive constants s_1 and s_2 such that $s_1 \le \chi'(x) \le s_2$, a.e.

Since $\chi'(x) = \operatorname{Re}\phi^*(x)$ a.e. on $\mathbb{R}$ and ϕ is bounded on $\mathbb{H}$, we find $s_1 \le \operatorname{Re}\phi(z) \le s_2$, $z \in \mathbb{H}$ and $(A2)$ follows.

We leave to the reader to prove that $(A2)$ implies (A1) and using equation (3.1) below to prove $(A1)$ implies $(A2)$. □

It is clear that the conditions $(A1)$ and $(A2)$ are equivalent to

$(A3)$ there is an analytic function $\phi \in H^\infty(\mathbb{H})$ and there exist two positive constants s_1 and s_2 such that $s_1 \le \operatorname{Re}\phi(z) \le s_2$, $z \in \mathbb{H}$.

Since $\chi'(x) = \operatorname{Re}\phi^*(x)$ a.e. on $\mathbb{R}$ and $H\chi' = \operatorname{Im}\phi^*(x) - ic$ a.e. on $\mathbb{R}$, we get characterization in terms of the Hilbert transform:

$(A4)$ χ is absolutely continuous, and there exist two positive constants s_1 and s_2 such that $s_1 \le \chi'(x) \le s_2$, a.e. and $H\chi'$ is bounded.

A similar characterization holds for smooth domains and in particular in the case of the unit disk; see Theorems 3.14 and 3.5 below.

From the proof of Theorem 3.3 below, cf [41], it follows that if we set $c = 1$ in the representation $(*1)$, then $h = h^\phi \in HQC_0^k(\mathbb{H})$ if and only $\phi(\mathbb{H})$ is in a disk $B_k = B(a_k; R_k)$, where $a_k = \frac{1}{2}(K + 1/K) = \frac{1+k^2}{1-k^2}$ and $R_k = \frac{1}{2}(K - 1/K) = \frac{2k}{1-k^2}$.

First, we need to introduce some notation: For $a \in \mathbb{C}$ and $r > 0$ we define $B(a;r) = \{z : |z - a| < r\}$. In particular, we write $\mathbb{D}_r$ instead of $B(0;r)$.

Theorem 3.3. ([41]), the half plane Euclidean-hqc version *Let f be a K-qc Euclidean harmonic diffeomorphism from $\mathbb{H}$ onto itself. Then f is a $(1/K, K)$ quasi-isometry with respect to Poincaré distance.*

For higher dimension version of this result see [4, 49, 53].

Proof. We first show that, by pre composition with a linear fractional transformation, we can reduce the proof to the case $f(\infty) = \infty$. If $f(\infty) \ne \infty$, there is the real number a such that $f(a) = \infty$. On the other hand, there is a conformal automorphism A of $\mathbb{H}$ such that $A(\infty) = a$. Since A is an isometry of $\mathbb{H}$ onto itself and $f \circ A$ is a K-qc Euclidean harmonic diffeomorphism from $\mathbb{H}$ onto itself, the proof is reduced to the case $f(\infty) = \infty$.

It is known that f has a continuous extension to $\overline{\mathbb{H}}$ (see for example [43]). Hence, by Lemma B, $f = u + ic \operatorname{Im} z$, where c is a positive constant. Using the linear mapping B, defined by $B(w) = w/c$, and a similar consideration as the above, we can reduce the proof to the case $c = 1$. Therefore we can write f in the form $f = u + i\operatorname{Im} z = \frac{1}{2}(F(z) + z + \overline{F(z) - z})$, where F is a holomorphic function in $\mathbb{H}$. Hence,

$$\mu_f(z) = \frac{F'(z) - 1}{F'(z) + 1} \quad \text{and} \quad F'(z) = \frac{1 + \mu_f(z)}{1 - \mu_f(z)}, \quad z \in \mathbb{H}. \tag{3.1}$$

Define $k = \frac{K-1}{K+1}$ and $w = S\zeta = \frac{1+\zeta}{1-\zeta}$. Then, $S(U_k) = B_k = B(a_k; R_k)$, where $a_k = \frac{1}{2}(K + 1/K) = \frac{1+k^2}{1-k^2}$ and $R_k = \frac{1}{2}(K - 1/K) = \frac{2k}{1-k^2}$.

Since f is k-qc, we see that $\mu_f(z) \in U_k$ and therefore $F'(z) \in B_k$ for $z \in \mathbb{H}$. This yields, first,

$$K + 1 \ge |F'(z) + 1| \ge 1 + 1/K, \quad K - 1 \ge |F'(z) - 1| \ge 1 - 1/K,$$

and then, $1 \le \Lambda_f(z) = \frac{1}{2}(|F'(z) + 1| + |F'(z) - 1|) \le K$.

So we have $\lambda_f(z) \ge \Lambda_f(z)/K \ge 1/K$. Thus, we find

$$1/K \le \lambda_f(z) \le \Lambda_f(z) \le K. \tag{3.2}$$

Let λ denote the hyperbolic density on $\mathbb{H}$. Since $\lambda(f(z)) = \lambda(z)$, $z \in \mathbb{H}$, using (3.2) and the corresponding versions of 3A and 3B for $\mathbb{H}$, cf [41], we obtain

$$\frac{1-k}{1+k}\, d_h(z_1, z_2) \le d_h(f(z_1), f(z_2)) \le \frac{1+k}{1-k}\, d_h(z_1, z_2).$$

It also follows from (3.2) that

$$\frac{1}{K}\,|z_2 - z_1| \le |f(z_2) - f(z_1)| \le K\,|z_2 - z_1|, \quad z_1, z_2 \in \mathbb{H}.$$

We leave to the reader to prove this inequality as an exercise.

This estimate is sharp (see also [38] for an estimate with some constant $c(K)$). $\square$

Using a similar approach as in the proof of Theorem 3.3 one can show.

Proposition 3.4. *Let $f : \mathbb{R} \xrightarrow{onto} \mathbb{R}$ be an increasing homeomorphism and $F(z) = f(x) + iy$. Then the following condition are equivalent (a) f is bi-Lipschitz, (b) F is bi-Lipschitz and (c) F is quasiconformal. In this setting, F is quasi-isometry with respect to Euclidean and Poincaré distance.*

For further results of this type see [33].

3.2. *The unit disc*

Suppose that h is an orientation preserving diffeomorphism of $\mathbb{H}$ onto itself, continuous on $\mathbb{H} \cup \mathbb{R}$ such that $h(\infty) = \infty$ and χ the restriction of h to $\mathbb{R}$. Recall $h \in HQC_0(\mathbb{H})$ iff there is analytic function $\phi : \mathbb{H} \to \Pi^+$ such that $\phi(\mathbb{H})$ is relatively compact subset of Π^+ and $\chi'(x) = \operatorname{Re}\phi^*(x)$ a.e.
We give similar characterizations in the case of the unit disk and for smooth domains (see below).

Theorem 3.5. *Let ψ be a continuous increasing function on $\mathbb{R}$ such that $\psi(t + 2\pi) - \psi(t) = 2\pi$, $\gamma(t) = e^{i\psi(t)}$ and $h = P[\gamma]$. Then h is q.c. if and only if*

1. $\operatorname{ess\,inf}\psi' > 0$
2. *there is analytic function $\phi : U \to \Pi^+$ such that $\phi(U)$ is relatively compact subset of Π^+ and $\psi'(x) = \operatorname{Re}\phi^*(e^{ix})$ a.e.*

In the setting of this theorem we write $h = h^\phi$. The reader can use the above characterization and functions of the form $\phi(z) = 2 + M(z)$, where M is an inner function, to produce examples of HQC mappings $h = h^\phi$ of the unit disk onto itself so the partial derivatives of h have no continuous extension to certain points on the unit circle. In particular we can take $M(z) = \exp\frac{z+1}{z-1}$, cf [10].

In the next subsection we extend the above theorem to smooth domains. Note that the proof that a HQC mapping between the unit disk and $\mathcal{D}_1$ domain is bi-Lipschitz is more delicate than in the case of the upper half plane. Instead of Lemma B we use Theorem A and Theorem 1.2.

3.3. *The geometry of $C^{1,\mu}$ curves*

In this section we refine and generalize some results from [26].

If X is a topological space, a curve in X is a continuous mapping γ of a compact interval $[\alpha, \beta] \subset \mathbb{R}$ (here $\alpha < \beta$) into X. We call $[\alpha, \beta]$ the parameter interval of γ and denote the range of γ by $\operatorname{tr}(\gamma)$. Thus γ is a mapping, and $\operatorname{tr}(\gamma)$ is the set of all points $\gamma(t)$, for $\alpha \le t \le \beta$. Suppose that γ is a rectifiable curve. For $t \in [\alpha, \beta]$, denote by $s = s(t) = s_\gamma(t)$ the length of the curve which is the restriction of γ on $[\alpha, t]$. Then $l = s(\beta)$ is the length of γ and there exists a function g defined on $[0, l]$ such that $\gamma(x) = g(s(x))$, for all $x \in [\alpha, \beta]$. We call g an arc-length parametrization of γ and $s = s_\gamma$ a natural (or arc-length) parameter function (associated to γ). If $f : [\alpha_1, \beta_1] \to \operatorname{tr}(\gamma)$ is another parametrization of γ in a similar way we define the function s_f. We call $s = s_f$ a arc-length parameter function (associated to f).

Suppose now that γ is a rectifiable, oriented, differentiable curve given by its arc-length parametrization $g(s)$, $0 \le s \le l$, where l is the length of γ. If γ is differentiable, then $|g'(s)| = 1$ and $s = \int_0^s |g'(t)|dt$, for all $s \in [0, l]$.

If γ is a twice-differentiable curve, then the curvature of γ at a point $p = g(s)$ is given by $\kappa_\gamma(p) = |g''(s)|$. Since $|g'(s)| = 1$, then $< g''(s), g'(s) >= 0$ and consequently $g'(s) = \alpha\, i\, g''(s)$, $\alpha \in \mathbb{R}$, and $|\alpha| = \kappa_\gamma^{-1}(p)$.

If g'' exists and it is bounded on $[0, l]$, we say that γ has a bounded curvature and define

$$\kappa_0 = \kappa_0(\gamma) = |\kappa_\gamma|_\infty = \sup\left\{|\kappa_\gamma(g(s))| : s \in [0, l]\right\} < \infty\,,$$

and $C_\gamma^* = \kappa_0/2$.

Let $K(s,t) = \mathrm{Re}\,[\overline{(g(t) - g(s))} \cdot ig'(s)]$. We say that γ is $C^{1,\mu}$, $0 < \mu \le 1$, at $w_0 = g(s_0)$ if $g \in C^1$ (more generally if there exists g' on $[0, l]$) and

$$C_\gamma(w_0) = \sup_{t\in[0,l]} \frac{|g'(t) - g'(s_0)|}{|t - s_0|^\mu} < \infty.$$

It is convenient to define $C_\gamma(\mu; w_0) = \frac{1}{1+\mu} C_\gamma(w_0)$ and $C_\gamma(\mu) = \sup\{C_\gamma(\mu; w) : w \in \mathrm{tr}(\gamma)\}$.

We say that $\gamma \in C^1$ if $g \in C^1$ and that $\gamma \in C^{1,\mu}$, $0 < \mu \le 1$, if $g \in C^1$ and $C_\gamma(\mu) < \infty$, i.e.

$$\sup_{t,s} \frac{|g'(t) - g'(s)|}{|t - s|^\mu} < \infty.$$

Note that the following conditions are then equivalent:

$a1)$ γ is $C^{1,1}$,

$a2)$ γ' is Lipschitz,

$a3)$ γ' is absolutely continuous and $\gamma'' \in L^\infty[0, l]$.

If γ is C^2, then

$a4)$ γ has a bounded curvature,

Condition $a4)$ implies $a1)$: that is if γ has a bounded curvature, then γ is $C^{1,1}$.

For $f : \mathbb{T} \to \mathbb{C}$, we define $\underline{f}$ on $[0, 2\pi]$ by $\underline{f}(t) = f(e^{it})$; also, we write simply $\underline{f}(t)$ instead of $f(e^{it})$ and $f'(x)$ instead of $\underline{f}'(x)$ if the meaning of it is clear from the context. The function $\underline{f}$ has periodic extension on $\mathbb{R}$ defined by $\underline{f}(t) = \underline{f}(t + 2\,\pi)$, $t \in \mathbb{R}$.

Suppose now that $f : \mathbb{R} \to \gamma$ is an arbitrary 2π periodic C^1 (more generally absolutely continuous) function such that $f|_{[0,2\pi)} : [0, 2\pi) \to \mathrm{tr}(\gamma)$ is an orientation preserving bijective function. Then the function $s = s_f$ (the arc-length parameter function) given by $s(\varphi) = \int_0^\varphi |f'(t)|dt$ is an increasing continuous function from $[0, 2\pi]$ onto $[0, l]$; and if g is arc-length parametrization then $f(\varphi) = g(s(\varphi))$, and therefore if f is C^1, we find $|f'(\varphi)| = |g'(s(\varphi))| \cdot |s'(\varphi)| = s'(\varphi)$.

Define $d_\Gamma(f(e^{i\varphi}), f(e^{ix})) := \min\{|s(\varphi) - s(x)|, (l - |s(\varphi) - s(x)|)\}$. If γ is C^1 and f a homeomorphism, it is easily verified that this expression is the distance (shorter) between $f(e^{i\varphi})$ and $f(e^{ix})$ along γ.

More generally, if we suppose only that the curve is rectifiable we can define the distance along it. Let C be a rectifiable Jordan closed curve and z_1, z_2 finite points of C. They divide C into two arc, and we consider one with smaller euclidean length and denote its length with $d_C(z_1, z_2)$. The curve C is said to satisfy the arc-chord condition if the ratio of this length to the distance $|z_1 - z_2|$ is bounded by a fixed number $b_C = b_C^{arc}$ for all finite $z_1, z_2 \in C$.

The curve C is said to satisfy the arc-chord condition at a fixed point $z_1 \in C$ if the ratio of the length $d_C(z_1, z)$ to the distance $|z_1 - z|$ is bounded by a fixed number $b_C(z_1) = b_C^{arc}(z_1)$ for all finite $z \in C$. We will prove that a C^1 curve satisfies the arc-chord condition.

To get some idea about C^{1,μ_1} curve we give a basic example:

Example 3.6. For $c > 0$, $0 < \mu < 1$, and $x_0 > 0$ the curve

$$y = c|x|^{1+\mu}, \quad |x| < x_0 \tag{3.3}$$

is $C^{1,\mu}$ at origin but it is not C^{1,μ_1} for $\mu_1 > \mu$. It is convenient to write this equation using polar coordinates $z = re^{i\varphi}$: $r\sin\varphi = cr^{1+\mu}(\cos\varphi)^{1+\mu}$ and we find $\sin\varphi = cr^{\mu}(\cos\varphi)^{1+\mu}$, $0 \le r < r_0$, where r_0 is a positive number. Since $\sin\varphi = \varphi + o(\varphi)$ and $\cos\varphi = 1 = o(1)$, when $\varphi \to 0$, the curve $\gamma(c,\mu)$ defined by joining curves $\varphi = cr^{1+\mu}$ and $\pi - \varphi = cr^{1+\mu}$, $0 \le r < r_0$, which share the origin, has similar properties near the origin as the curve defined by (3.3). The reader can check that $\gamma(c,\mu)$ is $C^{1,\mu}$ at origin but it is not C^{1,μ_1} for $\mu_1 > \mu$. Note that if a curve satisfies $\varphi \le cr^{1+\mu}$, then it is below the curve $\gamma(c,\mu)$.

Let $K(s,t) = \operatorname{Re}[\overline{(g(t) - g(s))} \cdot ig'(s)]$.
Along with the function K we will also consider the function K_f defined by

$$K_f(\varphi, x) = \operatorname{Re}[\overline{(f(x) - f(\varphi))} \cdot if'(\varphi)] = (f(x) - f(\varphi), if'(\varphi)).$$

If $f : U \to \mathbb{C}$ and there exist $\omega = f^*(e^{i\varphi})$ and $f'(\varphi)$, we define

$$K_f(\varphi, z) = \operatorname{Re}[\overline{(f(z) - f(\varphi))} \cdot if'(\varphi)] = (f(z) - f(\varphi), if'(\varphi)).$$

Suppose in addition that f has continuous extension to $\mathbb{D} \cup l$, where $l = l_\epsilon = \{e^{it} : -\epsilon \le t - t_0 \le \epsilon\}$, $t_0 \in \mathbb{R}$ and $\epsilon > 0$, and let f_1 be a curve defined by $f \circ l$ on $[-\epsilon, \epsilon]$. If f_1 is rectifiable and C^1 at $\omega = e^{i\varphi}$, we define $n = n_f(\omega) = ig'(s)$, where $s = s_{f_1}(\varphi)$. We also define $T = T_f(\omega) = f'(\varphi)/|f'(\varphi)|$, if $f'(\varphi) \ne 0$; note $T_f(\omega) = g'(s)$ and

$$R_f(\varphi, x) = \operatorname{Re}[\overline{(f(x) - f(\varphi))} \cdot n] = (f(x) - f(\varphi), n_f(\varphi)).$$

Note that $K_f(\varphi, x) = |f'(\varphi)| R_f(\varphi, x)$ and there is an obvious geometric interpretation of some notion defined the above: for $\omega = e^{i\varphi}$ if $f'(\varphi)$ exists it is tangent vector of γ at $f(\omega)$, $T_f(\omega) = g'(s)$ is the unit tangent vector and $n_f(\varphi) = ig'(s)$ is the unit normal vector.

Let γ be a curve of $C^{1,\mu}$ class and $f : \mathbb{R} \to \operatorname{tr}(\gamma)$ be arbitrary topological (homeomorphic) parametrization of γ and denote by $\Gamma(s)$ natural parametrization.

For $\Gamma(s) = f(\varphi)$, we define $n_f(\varphi) = i\Gamma'(s)$ and $R_f(\varphi, t) = (f(t) - f(\varphi), n_f(\varphi))$.

Let γ be a closed Jordan rectifiable curve and $f : \mathbb{R} \to \operatorname{tr}(\gamma)$ be arbitrary topological (homeomorphic) parametrization of γ and denote by $\Gamma(s)$ its natural arc-length parametrization (in order to emphasize it).

The following two lemmas are basically proved in [26].

Lemma 3.7. *Let $f : \mathbb{T} \to \operatorname{tr}(\gamma)$ be arbitrary topological (homeomorphic) parametrization of γ and γ be a curve of class $C^{1,\mu}$, $0 < \mu \le 1$, at $w = f((e^{i\varphi})$. Then*

$$\begin{aligned} K_f(\varphi, t) &\le A|\underline{f}'(\varphi)||(f(e^{i\varphi}) - f(e^{it})|^{1+\mu} \\ |R_\gamma(\varphi, t)| &\le A\,|f((e^{i\varphi}) - f(e^{it})|^{1+\mu}, \end{aligned}$$

where $A = A(\gamma, \mu, \varphi) = C_\gamma(\varphi) b_\gamma(\varphi)^{1+\mu}$.

Proof. By notation $\underline{f}(\varphi) = g(s(\varphi))$ and $\underline{f}(x) = g(s(x))$, we have $R_f(\varphi, t) = (\underline{f}(x) - \underline{f}(\varphi), n_\gamma(\varphi)) = K(s(\varphi), s(x))$. Then, by Lemma 3.8, $R_f(\varphi, t) \le c|s(x) - s(\varphi)|^{1+\mu} \le c_1|\underline{f}(x) - \underline{f}(\varphi)|^{1+\mu}$, where $c = C_\gamma(w)$ and $c_1 = A$. $\square$

Lemma 3.8. *Let γ be a Jordan closed rectifiable curve, $\Gamma : [0, l] \to \operatorname{tr}(\gamma)$ be its natural parametrization and let $f : [0, 2\pi] \to \operatorname{tr}(\gamma)$ be arbitrary topological parametrization of* $\operatorname{tr}(\gamma)$. *Suppose that γ is a $C^{1,\mu}$ at $w_0 = \Gamma(s_0)$, where $s_0 = s_f(\varphi_0)$. Then*

$$|K(s_0, t)| \le C_\gamma(w_0) \min\{|s_0 - t|^{1+\mu}, (l - |s_0 - t|)^{1+\mu}\} \tag{3.4}$$

for all t and

$$|R_f(\varphi_0, x)| \le C_\gamma(w_0)\min\{|s(\varphi_0) - s(x)|^{1+\mu}, (l - |s(\varphi_0) - s(x)|)^{1+\mu}\}, \tag{3.5}$$

for all x, where recall

$$C_\gamma(w_0) = \frac{1}{1+\mu}\sup_{0\le t\ne s_0\le l}\frac{|\Gamma'(t) - \Gamma'(s_0)|}{|t-s_0|^\mu}.$$

Proof. Let γ be a Jordan closed rectifiable curve, $\Gamma : [0, l] \to \mathrm{tr}(\gamma)$ be a natural parametrization and let γ be a curve of $C^{1,\mu}$, $0 < \mu \le 1$, class at $w_0 = \Gamma(s_0)$, where $s_0 = s(\varphi_0)$. Since $R(s) = R_\Gamma(s, s_0) = (\Gamma(s) - \Gamma(s_0), n)$ and $(\Gamma'(s_0), n) = 0$, we find $R'(s) = (\Gamma'(s), n) = (\Gamma'(s) - \Gamma'(s_0), n)$ and $|R'(t)| \le c|t - s_0|^\mu$, where $c = (1 + \mu)C_\gamma(s_0)$. Hence

$$|R(s)| \le \int_{s_0}^{s}|R'(t)|dt \le c\int_{s_0}^{s}|t-s_0|^\mu dt = c\int_0^{s-s_0}|t|^\mu dt = C_\gamma(w_0)|s-s_0|^{1+\mu}.$$

□

Recall $C_\gamma = C_\gamma^\mu = \sup\{C_\gamma(\mu; w) : w \in \mathrm{tr}(\gamma)\}$.

Moreover if γ has a bounded curvature then the relations (3.4) and (3.5) are true for

$$C_\gamma = \sup\left\{|\kappa_\gamma(g(s))|/2 : s \in [0, l]\right\}$$

and $\mu = 1$. In this case

$$\lim_{t\to s}\frac{K(s,t)}{(s-t)^2} = \frac{|\kappa_\gamma(g(s))|}{2} \text{ and } \lim_{x\to\varphi}\frac{K_f(\varphi, x)}{(s(x) - s(\varphi))^2} = \frac{|\kappa_\gamma(g(s))|}{2}s'(\varphi),$$

and the constant C_γ is the best possible, cf [26].

Lemma 3.9. *Let γ be a closed curve of class C^1 and $f : \mathbb{T} \to \mathrm{tr}(\gamma)$ be arbitrary topological (homeomorphic) parametrization of γ. Then*

$$|f(e^{i\varphi}) - f(e^{ix}))| \le d_\gamma(f(e^{i\varphi}), f(e^{ix})) \le b_\gamma|(f(e^{i\varphi}) - f(e^{ix})|. \tag{3.6}$$

Proof. Suppose that $\Gamma = (\Gamma_1, \Gamma_2)$, an arbitrary point $s_0 \in [0, l)$ and s_1, s_2 approaches s_0. Since

$$A(s_1, s_2) := \frac{|\Gamma(s_2) - \Gamma(s_1)|}{|s_2 - s_1|} = |\Gamma_1'(s_3) + i\Gamma_2'(s_4)|,$$

we find that $A(s_1, s_2)$ tends to 1 when s_1, s_2 approaches s_0. Using this, one can show that that the function B defined by

$$B(s_1, s_2) := \frac{d_\gamma(\Gamma(s_1), \Gamma(s_2))}{|\Gamma(s_2) - \Gamma(s_1)|}$$

is continuous on $[0, l] \times [0, l]$. Hence we get (3.6), where b_γ denote the maximum of B on $[0, l] \times [0, l]$. □

Recall if b_γ is finite, we say that γ satisfies chord-arc condition with constant $b_\gamma = b_\gamma^a rc$ (or it is of length-bounded turning with constant b_γ). For fixed s_1, we define $b_\gamma(s_1) = \max\{B(s_1, s) : s \ne s_1\}$. Note $b_\gamma = \max\{B(s_1, s_2) : s_2 \ne s_1\}$. See also bounded turning [43] p. 102-105. We have proved if γ is a C^1 Jordan curve, then γ satisfies chord-arc condition. We leave to the interested reader to prove:

Lemma 3.10. *Suppose that γ is a rectifiable Jordan closed curve and let $\Gamma : [0, l] \to \gamma$ be a natural parametrization of γ. If the curve γ is C^1 at $w_0 = \Gamma(s_0)$, then γ satisfies the arc-chord condition at $\Gamma(s_0)$. If the curve γ is $C^{1,\mu}$ at $w_0 = \Gamma(s_0)$, then*

$$|K(s_0, t)| \leq C_\gamma(w_0) \min\{|s_0 - t|^{1+\mu}, (l - |s_0 - t|)^{1+\mu}\}$$

for all $t \in [0, l]$.

Suppose that γ is a closed rectifiable Jordan curve, Γ its arc-length parametrization and D domain bounded by γ. If $|R(s)| = |R_\Gamma(s, s_0)| \leq C(s_0)|s - s_0|^{1+\mu}$ (more generally $|R(s)| = |R_\Gamma(s, s_0)| \leq C(s_0)|\Gamma(s) - \Gamma(s_0)|^{1+\mu}$), we say that D is close to convex of order μ at $\Gamma(s_0)$. If there is a uniform constant independent of s_0 we say that D is close to convex of order μ.

The following two basic theorems are important for our research.

Theorem 3.11. (Kellogg. See for example [19]) *If a domain $D = Int(\Gamma)$ is $C^{1,\alpha}$ and ω is a conformal mapping of $\mathbb{D}$ onto D, then ω' and $\ln \omega'$ are in Lip_α. In particular, $|\omega'|$ is bounded from above and below on $\mathbb{U}$.*

Theorem 3.12. (Kellogg and Warschawski. See [58, Theorem 3.6]]) *If a domain $D = Int(\Gamma)$ is $C^{2,\alpha}$ and ω is a conformal mapping of $\mathbb{D}$ onto D, then $|\omega''|$ has a continuous extension to the boundary. In particular it is bounded from above on $\mathbb{D}$.*

3.4. HQC and convex smooth codomains

We need the following result related to convex codomains.

Theorem 3.13. ([49]) *Suppose that h is a Euclidean harmonic mapping from $\mathbb{D}$ onto a bounded convex domain $D = h(\mathbb{D})$, which contains the disc $B(h(0); R_0)$. Then*

(1) *$d(h(z), \partial D) \geq (1 - |z|)R_0/2$, $z \in \mathbb{D}$.*

(2) *Suppose that $\omega = h^*(e^{i\theta})$ and $h^*_r = h'_r(e^{i\theta})$ exist at a point $e^{i\theta} \in \mathbb{T}$, and there exists the unit inner normal $n = n_\omega$ at $\omega = h^*(e^{i\theta})$ with respect to ∂D. Then $E = (h^*_r, n_{h_*}) \geq c_0$, where $c_0 = \frac{R_0}{2}$.*

(3) *In addition to the hypothesis stated in the item (2), suppose that h'_* exists at the point $e^{i\theta}$. Then $|J_h| = |(h^*_r, N)| = \big|(h^*_r, n)\big||N| \geq c_0|N|$, where $N = i h'_*$ and the Jacobian is computed at the point $e^{i\theta}$ with respect to the polar coordinates.*

If in addition D is of $C^{1,\mu}$ class and h qc, using the result that the function E is continuous [11], we find

(4) $|E| \geq c_0$.

Outline of proof of (1). To every $a \in \partial D$ we associate a nonnegative harmonic function $u = u_a$. Since D is convex, for $a \in \partial D$, there is a supporting line Λ_a defined by $(w - a, n_a) = 0$, where $n = n_a$ is a unimodular complex number such that $(w - a, n_a) \geq 0$ for every $w \in \overline{D}$. Define $u(z) = (h(z) - a, n_a)$ and $d_a = d(h(0), \Lambda_a)$. Then $u(0) = (h(0) - a, n_a) = d(h(0), \Lambda_a)$ and therefore, by the mean value theorem,

$$\frac{1}{2\pi} \int_0^{2\pi} u(e^{it}) dt = u(0) = d_a = d(h(0), \Lambda_a).$$

Since $u = u_a$ is a nonnegative harmonic function, for $z = re^{i\varphi} \in \mathbb{D}$, we obtain

$$u(z) \geq \frac{1 - r}{1 + r} \frac{1}{2\pi} \int_0^{2\pi} u(e^{it}) dt.$$

Hence $u(re^{i\varphi}) \geq d_a(1 - r)/2$, and therefore $|h(z) - a| \geq d_a(1 - r)/2 \geq (1 - r)R_0/2$. Thus $|h(z) - a| \geq (1 - r)R_0/2$ for every $a \in \partial D$ and therefore we obtain (1): $d(h(z), \partial D) \geq (1 - r)R_0/2$. ◻

Theorem 3.14. *Suppose that $C^{1,\alpha}$ domain D is convex and denote by γ positively oriented boundary of D. Let $h_0 : \mathbb{T} \to \gamma$ be an orientation preserving homeomorphism and $h = P[h_0]$. The following conditions are then equivalent*

a) *h is K-qc mapping.*

b) *h is bi-Lipschitz in the Euclidean metric.*

c) *the boundary function h_* is bi-Lipschitz in the Euclidean metric and the Cauchy transform $C[h_*']$ of its derivative is in L^∞.*

d) *the boundary function h_* is absolutely continuous,* $\operatorname{ess\,inf}|h_*'| > 0$ *and the Cauchy transform $C[h_*']$ of its derivative is in L^∞.*

e) *the boundary function h_* is bi-Lipschitz in the Euclidean metric and the Hilbert transform $H[h_*']$ of its derivative is in L^∞.*

f) *the boundary function h_* is absolutely continuous,* $\operatorname{ess\,sup}|h_*'| < +\infty$, $\operatorname{ess\,inf}|h_*'| > 0$ *and the Hilbert transform $H[h_*']$ of its derivative is in L^∞.*

We refer to Theorem 3.14 as the main characterization. Note that here, by our notation, $\underline{h_0} = h_*$ and $h_0 = h^*$.

Proof. By the fundamental theorem of Rado, Kneser and Choquet, h is an orientation preserving harmonic mapping of the unit disc onto D.

If D is $C^{1,\alpha}$, it has been shown in [11] that *a*) implies *b*) even without hypothesis that D is a convex domain. Note that an arbitrary bi-Lipschitz mapping is quasiconformal. Hence the conditions *a*) and *b*) are equivalent.

The Hilbert transform of a derivative of HQC boundary function will be in L^∞, and hence *a*) implies *e*).

Recall, we use notation $p = f'$, $q = g'$, $\Lambda_h = |f'| + |g'|$, $\lambda_h = |f'| - |g'|$.

If h_* is absolutely continuous, since $h_\theta'(z) = i(zf'(z) - \overline{zg'(z)})$, we find $C[h_*'](z) = izf'(z)$. It follows that *a*) implies *c*) and *d*).

Since bi-Lipschitz condition implies absolute continuity, *c*) implies *d*) and *e*).

Let us show *d*) implies *a*). Hypothesis $C[h_*'] \in L^\infty$ implies that $f' \in L^\infty$ and therefore since h is orientation preserving and $|f'| \geq |g'|$, we find $g' \in L^\infty$. This shows that Λ_h is bounded from above.

We will show that $|p^*|$ is bounded from above, $\lambda_h^* = |p^*|(1 - |\mu^*|)$ is bounded from below, and therefore that $(1 - |\mu^*|)$ is bounded from below.

Let $N = i\,h_*'$ and $N = n|N|$. Since D is a convex domain $|f'|$ and (h_r^*, n) are bounded from below with positive constant (for an outline of proof see [48, 49]).

Condition $C[h_*'] \in L^\infty$ implies that $f' \in H^\infty$. Hence, since $|f'|$ is bounded from below with positive constant, it follows that Λ_h is bounded from above and below with two positive constants.

By assumption *d*), $|h_*'|$ is bounded essentially from below. Since, $J_h = \Lambda_h \lambda_h$ and by Theorem 3.13

$$J_h^* = (h_r^*, N) = (h_r^*, n)|N| \geq c_0|N|,$$

where $n = n_{h_*}$ and $N = n|N|$ and $N = i\,h_*'$, we conclude that λ_h^* is bounded from above and below with two positive constants. It follows from $\lambda_h^* = |p^*|(1 - |\mu^*|)$, that $(1 - |\mu^*|)$ is bounded from below with positive constant c_1 and therefore $k_1 = (1 - c_1) \geq |\mu^*|$. By maximum principle, $\|\mu\|_\infty \leq k_1$.

Note that hypothesis *d*) implies that $|h_*'|$ is bounded from above and therefore the boundary function h_* is bi-Lipschitz. Thus, we have that *a*) and *b*) follow from *d*).

Let us prove that *f*) implies *d*). This will finish the proof, since *e*) implies *f*) and we have already established that *d*) implies *a*). Since the boundary function h_* is absolutely continuous, recall that, by (1.3), we have

$$\partial_\theta h(z) = P[h_*'](z) = i(zf'(z) - \overline{zg'(z)}),$$

and, by (1.4), that its harmonic conjugate is $zf'(z) + \overline{zg'(z)} = rh_r'(z) = P[H(h_*')]$.

Thus if h_* is Lipschitz and $H(h'_*)$ is bounded, then $\partial_\theta h$ and $irh'_r(z)$ are bounded on $\mathbb{D}$ so by adding these two together we conclude that $h'_\theta + irh'_r = 2izf' = 2C[h'_*]$ is bounded and therefore the Cauchy transform $C[h'_*]$ is bounded, and d) follows.

Note that we have here $|f'|$ is bounded and therefore all partial derivatives of h are bounded, and $H(h'_*) = zp^* + \overline{zq^*}$ a.e. on $\mathbb{T}$, where $p = f'$ and $q = g'$. □

A version of the part of the main characterization (that (a) is equivalent to (f)) has been stated in [26].

Theorem 3.15. ([26]) *Let $f : \mathbb{T} \to \gamma$ be an orientation preserving homeomorphism of the unit circle onto the Jordan convex curve $\gamma = \partial\Omega \in C^{1,\mu}$. Then $h = P[f]$ is a quasiconformal mapping if and only if*

$$0 < \operatorname{ess\,inf} |f'(\varphi)|, \tag{3.7}$$

$$\operatorname{ess\,sup} |f'(\varphi)| < \infty \tag{3.8}$$

and

$$\operatorname*{ess\,sup}_{\varphi} |H(f')(\varphi)| < \infty, \tag{3.9}$$

where

$$H(f')(\varphi) = -\frac{1}{2\pi}\int_{0_+}^{\pi} \frac{f'(\varphi+t) - f'(\varphi-t)}{\tan t/2} dt,$$

denotes the Hilbert transformations of f'.

Note that the condition f) (in Theorem 3.14) contains the hypothesis that f is absolutely continuous which does not appear in the above formulation. It seems, however, that this hypothesis is essential for validity of the proof given in [26]. Indeed, it is easy to find an example of a function f satisfying conditions (3.7), (3.8) and (3.9), such that the corresponding harmonic map $h = P[f]$ is not q.c., cf [10].

Our characterization works only for convex domains. If all conditions are kept, but convexity is dropped, then there can be found examples of maps which are not HQC, cf [10]. We can get HQC characterization of general harmonic maps to $C^{1,\alpha}$ domains, if we set apart the condition which depends on the convexity of the domain as one of the requirements.

Theorem 3.16. *Suppose that D is $C^{1,\alpha}$ domain. Let h be a harmonic orientation preserving map of the unit disc onto D and homeomorphism of $\overline{\mathbb{D}}$ onto $\overline{D}$. The following conditions are equivalent*

*a*1) *h is K-qc mapping*

*a*2) *the boundary function h_* is absolutely continuous,* $\operatorname{ess\,sup}|h'_*| < +\infty$, $Hh'_* \in L^\infty$ *and* $s_0 = \operatorname{ess\,inf}|(Hh'_*, ih'_*)| > 0$.

We only outline the proof of the this theorem.

Proof. Put $\mu = \mu_h$. Clearly $a2$) implies $\operatorname{ess\,inf}|h'_*| > 0$. We leave to the reader to check that $2zp^* = H(h'_*) - ih'_*$, $2zq^* = H(h'_*) + ih'_*$, $J_h^* = (h_r^*, ih'_*) = (H(h'_*), ih'_*) \geq 0$ a.e. on $\mathbb{T}$ and $J_h > 0$ on $\mathbb{D}$. Hence $|\mu| < 1$ and $\Lambda_h^* \lambda_h^* = J_h^* \geq s_0 > 0$. Similarly like in the proof of the main characterization theorem $a2$) implies $|\mu^*|_\infty = k < 1$ and so we have $a1$). The converse is straightforward. □

Our last example in this subsection shows that in the mentioned characterization of quasiconformality one cannot drop the convexity hypothesis.

Example 3.17. ([10], three-cornered hat domain) Let $h(z) = z + \overline{z}^2/2$, $\gamma(t) = h(e^{it})$, $z_k = e^{\pi/3 + 2k\pi/3}$ and $A = \{z_0, z_1, z_2\}$. Suppose also that Γ is a smooth Jordan closed curve, $G = Int(\Gamma)$ such that $G \subset \mathbb{D}$, $A \subset Ext(\Gamma)$ and Γ has a joint arc I_0 with $\mathbb{T}$. Let ϕ be conformal mapping of $\mathbb{D}$ onto G, $z = \phi(\zeta)$, and $\check{h} = h \circ \phi$. Then

$d\check{h} = Pd\zeta + Qd\overline{\zeta}$ and $H(\check{h}'_*) = \zeta P^* + \overline{\zeta Q^*}$ a.e. on $\mathbb{T}$, where $P = \phi'$ and $Q = \phi\phi'$. Since $h_*(t) = h(e^{it}) = e^{it} - e^{-2it}/2$, we find $h'_*(t) = ie^{-2it}(e^{3it} - 1)$ and $h'_*(t) = 0$ if and only if $t = t_k = 2k\pi/3$.

It is easily to check that $\check{h}$ satisfies the all the hypothesis in the characterization except for the convexity condition, but $\check{h}$ is not a qc on $\mathbb{D}$ because $J_{\check{h}}$ is zero on I_0. We can chose Γ to be C^∞ curve, then $\check{h}_*$ is C^∞; in particular $H(\check{h}'_*)$ is in L^∞.

The domain $h(\mathbb{D})$ is known as a three-cornered hat domain.

3.5. Application to the Universal Teichmüller space

For $\zeta = \xi + i\eta$ we use notation $|d\zeta|^2 = d\xi\, d\eta$. Here we apply our characterization to the problem of minimizing functional

$$\mathbb{K}(f) = \int_U \frac{|f_z(\zeta)|^2 + |f_{\overline{z}}(\zeta)|^2}{|f_z(\zeta)|^2 - |f_{\overline{z}}(\zeta)|^2} |d\zeta|^2$$

over all quasiconformal maps $f : \mathbb{D} \to \mathbb{D}$ with the same boundary condition, i.e. belonging to the same class in the Universal Teichmüller space. Existence of minimizers of functional $\mathbb{K}$ in the Teichmüller spaces have been of considerable recent interest. For instance, in [46] it has been proved that minimizers do not exist in the case of punctured disc.

From results in [5], it follows that the minimizer will exist in the Universal Teichmüller class if and only if the inverse map on the boundary induces harmonic quasiconformal map, i.e. if $P[f^{-1}]$ is quasiconformal. Applying our results, we get the following characterization, cf also [5]:

Theorem 3.18. *Let $f : \mathbb{T} \to \mathbb{T}$ be a homeomorphism of $\mathbb{T}$, that satisfies the M-conditon i.e. that has quasiconformal extension to $\mathbb{D}$. Then in the Universal Teichmüller class of f there is minimiser of the functional $\mathbb{K}$ if and only if*

- *b1) f is bi-Lipschitz and $H[(f^{-1})'] \in L^\infty(\mathbb{T})$, or*
- *b2) f is bi-Lipschitz and $C[(f^{-1})'] \in L^\infty(\mathbb{T})$.*

Also, we can get the result about minimisers of $\mathbb{K}$ functional of maps from the convex $C^{1,\alpha}$ domains to the unit disc.

Theorem 3.19. *Let D be a convex $C^{1,\alpha}$ domain and $f : \partial D \to \mathbb{T}$ an orientation preserving homeomorphism that has quasiconformal extension to D. Then functional $\mathbb{K}$ is minimised in the class of all qc maps with the same boundary condition if and only if*

- *b1) f is bi-Lipschitz and $H[(f^{-1})'] \in L^\infty(\mathbb{T})$, or*
- *b2) f is bi-Lipschitz and $C[(f^{-1})'] \in L^\infty(\mathbb{T})$.*

Note that with the second type of condition we can have more general codomains, applying our theory.

4. Inner estimate and quasiconformal harmonic maps between smooth domains

4.1. Basic facts and notations

Let $\mathbb{U}$ and $\mathbb{H}$ denote, the unit disc and the upper half plane, respectively. By Ω, Ω' and D we denote simply connected domains. Suppose that γ is a rectifiable curve in the complex plane. Denote by l the length of γ and let $\Gamma : [0, l] \to \gamma$ be the natural parametrization of γ, i.e. the parametrization satisfying the condition $|\dot{\Gamma}(s)| = 1$ for all $s \in [0, l]$.

We will say that γ is of class $C^{n,\mu}$, for $n \in \mathbb{N}$, $0 < \mu \leq 1$, if Γ is of class C^n and

$$\sup_{t,s} \frac{|\Gamma^{(n)}(t) - \Gamma^{(n)}(s)|}{|t - s|^\mu} < \infty.$$

We will call Jordan domains in $\mathbb{C}$ bounded by $C^{n,\mu}$ Jordan curves, $C^{n,\mu}$ domains or smooth ones.

A Riemannian metric given by the fundamental form $ds^2 = \rho(dx^2+dy^2)$ or $ds = \sqrt{\rho}|dz|$, $\rho > 0$, is conformal with the Euclidean metric.

If we write the metric in the form $ds = \sqrt{\rho}|dz|$, we call shortly $\sqrt{\rho}$ a metric density (scale) and ρ a metric.

Let $\rho(w)|dw|^2$ be an arbitrary conformal C^1-metric defined on D. If $f : \Omega \to D$ is a C^2 mapping between the Jordan domains Ω and D, the energy integral of f is defined by the formula:

$$E[f,\rho] = \int_\Omega \rho \circ f(|f_z|^2 + |f_{\bar z}|^2)dx\,dy.$$

The stationary points of the energy integral satisfy the Euler Lagrange equation

$$f_{z\bar z} + (\log\rho)_w \circ f f_z f_{\bar z} = 0, \tag{4.1}$$

and a C^2 solution of this equation is called a *harmonic mapping* (more precisely a *ρ-harmonic mapping*).

It is known that f is a harmonic mapping if and only if the mapping $\Psi = \Psi_f = \rho \circ f f_z \overline{f_{\bar z}}$ is analytic; and we say that Ψ is the *Hopf differential* of f and we write Ψ =Hopf(f).

If φ is a holomorphic mapping and different from 0 on D and $\rho = |\varphi|$ on D, we call ρ a $\varphi-$ metric. We will call the corresponding harmonic mapping a *$\varphi-$harmonic*.

Notice that for $\rho = 1$, a ρ-harmonic mapping is an Euclidean harmonic function.

Since we consider diffeomorphisms or C^2-mappings, we can use the following definition of quasiconformal mappings. Let $0 \le k < 1$ and $K = (1+k)/(1-k)$. An orientation preserving diffeomorphism $f : \Omega \to D$ between two domains $\Omega, D \subset \mathbb{C}$ is called a *K -quasiconformal mapping* or shortly a q.c. mapping if it satisfies the condition:

$$|f_{\bar z}(z)| \le k|f_z(z)| \text{ for each } z \in \Omega.$$

Occasionally, in this setting, it is also convenient to say that f is a *k -quasiconformal mapping*.

In this paper we will mainly consider harmonic quasiconformal mappings between smooth domains. Let Ω, Ω' be two domains in plane. By $QC(\Omega,\Omega')$ we denote the family of all qc mappings f of Ω onto Ω'; if $\Omega' = \Omega$ we write $QC(\Omega)$ here and use this convention in similar situation. By $QC_K(\Omega,\Omega')$ we denote the family of all K-qc mappings f of Ω onto Ω' and by $H_\rho QC(\Omega,\Omega')$ the family of all mappings in $QC(\Omega,\Omega')$ which are ρ-harmonic on Ω; if $\rho = 1$ we can drop ρ.

4.2. Results

The following proposition has an important role in the proofs concerning results obtained in [24], [38] and [41].

Proposition 4.1. *Let f be an Euclidean harmonic $1-1$ mapping of the upper half-plane $\mathbb{H}$ onto itself, continuous on $\overline{\mathbb{H}}$, normalized by $f(\infty) = \infty$ and $v = \mathrm{Im} f$. Then $v(z) = cy$, where c is a positive constant. In particular, v has bounded partial derivatives on $\mathbb{H}$.*

Suppose that f is a harmonic Euclidean mapping of the unit disc onto a smooth domain D and ψ is a conformal mapping of D onto $\mathbb{H}$, then the composition $\psi \circ f$ is very rarely Euclidean harmonic, so we can not apply Proposition 4.1. However, the composition satisfies a simple equation (see (4.8), Section 4) and it is harmonic with respect to the other metric density ρ defined on $\mathbb{H}$ by $\rho(\zeta)|d\zeta| = |dw|$, where $\zeta = \psi(w)$. Having this in mind, our idea is to apply the following result instead of Proposition 4.1 in more complicated cases:

Proposition 4.2. (Inner estimate. Heinz-Bernstein, see [20]). *Let $s : \overline{\mathbb{D}} \to \mathbb{R}$ be a continuous function from the closed unit disc $\overline{\mathbb{D}}$ into the real line satisfying the conditions:*

1. *s is C^2 on $\mathbb{D}$,*

2. $\dot{s}(\theta) = s(e^{i\theta})$ *is* C^2 *and*
3. $|\Delta s| \le c_0 |\nabla s|^2$ *on* $\mathbb{D}$ *for some constant* c_0.

Then the function $|\nabla s| = |\mathrm{grad}\, s|$ *is bounded on* $\mathbb{D}$.

We refer to this result as the *inner estimate*. Applying this estimate and Kellogg-Warschawski results we prove the main result of the paper [30].

Theorem 4.3. ([30]) *Let f be a quasiconformal C^2 diffeomorphism from the $C^{1,\alpha}$ Jordan domain Ω onto the $C^{2,\alpha}$ Jordan domain D. If there exists a constant M such that*

$$|\Delta f| \le M |f_z \cdot f_{\bar{z}}|, \quad z \in \Omega, \tag{4.2}$$

then f has bounded partial derivatives. In particular, it is a Lipschitz mapping.

If ϕ and η are conformal, $\hat{f} = \phi \circ f \circ \eta$, we obtain the equality

$$\frac{\hat{f}_{z\bar{z}}}{\hat{f}_z \cdot \hat{f}_{\bar{z}}} = \left(\frac{\phi''}{\phi'^2} + \frac{1}{\phi'} \frac{\partial \bar{\partial} f}{\partial f \cdot \bar{\partial} f} \right). \tag{4.3}$$

Note that equation (4.3) (see below) shows how to transform condition (4.2) if we consider compositions of the mapping f by conformal mappings. In particular, Theorem 4.3 holds if h is quasiconformal ρ-harmonic and the metric ρ is *approximately analytic*, i.e. $|\bar{\partial}\rho| \le M|\rho|$ on Ω, (see Theorems 3.1-3.2, 3.4, 4.4 below).
Notice that

(*a*) Theorem 3.1 can be considered as a special case of Theorem 3.4 and 4.4 and
(*b*) Euclidean and spherical metrics are approximately analytic, so our results can be considered as extensions of the corresponding ones proved in [47], [56], [38] and [24].

The paper [30] is organized as follows. The main result is proved in Section 2 and its applications are given in Section 3. In Section 4, we show that the composition of a conformal mapping ψ and a $\varphi-$ harmonic mapping satisfies a certain equation (see Theorem 4.12); and in particular, if ψ is a natural parameter, we obtain a representation of $\varphi-$harmonic mappings by means of Euclidean harmonics.

In space, instead of Proposition 4.2, we can use the following result of Gilbarg-Hörmander [18] (see also [17]):

Proposition 4.4. *Let Ω be of class $C^{1,\alpha}$. If $f \in C^{1,\alpha}(\overline{\Omega})$, then $P[f] \in C^{1,\alpha}(\overline{\Omega})$.*

4.3. Applications

Let D be a domain in $\mathbb{C}$ with a Riemannian metric given by the fundamental form $ds^2 = \rho |dz|^2$ in D (we say shortly ρ is a conformal metric in D). The Gaussian curvature on the domain is given by

$$K_D = -\frac{1}{2} \frac{\triangle \log \rho}{\rho}.$$

If, in particular, the domain D is simply connected in $\mathbb{C}$ and the Gaussian curvature $K_D = 0$ on D, then $\Delta \log \rho = 0$ and therefore $\rho = |e^{\omega}|$, where ω is a holomorphic function on D.

Thus the metric $\rho = |\varphi|$ is induced by a non-vanishing holomorphic function $\varphi(z) = e^{\omega(z)}$ defined on the domain D; in this setting we call ρ a $\varphi-$ metric.

The corresponding harmonic mapping we will call $\varphi-$*harmonic*. Roughly speaking, $\varphi-$*harmonic* maps arise if the curvature of the target is 0.

Since $\rho^2 = \varphi\overline{\varphi}$, a short computation yields $2\rho\rho_w = \varphi'\overline{\varphi}$ and therefore $2(\log \rho)_w = (\log \varphi)'$, i.e.

$$2(\log |\varphi|)_w = (\log \varphi)'. \tag{4.4}$$

Hence, by (4.1) we obtain: if f is $\varphi-$ harmonic, then

$$f_{z\bar{z}} + \frac{\varphi'}{2\varphi} \circ f f_z f_{\bar{z}} = 0. \tag{4.5}$$

As a direct application of Theorem 4.3 (the main result), using the equality (4.5), we obtain the following theorem:

Theorem 4.5. *Let f be a $\varphi-$harmonic mapping of the unit disc U onto a $C^{2,\alpha}$ Jordan domain D. If $M = \|(\log \varphi)'\|_\infty < \infty$ and f is quasiconformal, then f has bounded partial derivatives and in particular, it is a Lipschitz mapping.*

Proof. It is enough to notice that the hypothesis $M = \|(\log \varphi)'\|_\infty < \infty$ and equality (4.5) imply that the crucial hypothesis (4.2) of the main theorem is satisfied. □

Recall $\mathbb{T} = \partial\mathbb{D}$.

Theorem 4.6. (Local version) *Let f be a C^2 $\varphi-$harmonic mapping of the unit disc $\mathbb{D}$ onto the $C^{2,\alpha}$ Jordan domain D having continuous extension $\tilde{f}$ to the boundary such that $\tilde{f}(\mathbb{T}) = \partial D$. If f is quasiconformal in some neighborhood of a point $z_0 \in \mathbb{T}$ and $(\ln \varphi)'$ is bounded in some neighborhood of $w_0 = f(z_0)$, then f has bounded partial derivatives and in particular, it is a Lipschitz mapping in a neighborhood of the point z_0.*

Proof. Let $r > 0$ be such that f is q.c. in $U_0 = D(z_0, r) \cap \mathbb{U}$. Then $\gamma_0 = f(\mathbb{T} \cap D(z_0, r))$ is a $C^{2,\alpha}$ Jordan arc in ∂D containing w_0. Now following the proof of Theorem 4.3, we obtain that the function f has bounded partial derivatives near the arc $\gamma = f(\mathbb{T} \cap \bar{D}(z_0, r/2))$ and therefore in some neighborhood of the point z_0. □

Definition 4.7. A function χ which is of class C^1 and satisfies the inequality $|\bar{\partial}\chi| \le M|\chi|$ in a domain D is said to be *approximately analytic* in D with the constant M.

If a φ-metric satisfies the hypothesis $M = \|(\log \varphi)'\|_\infty < \infty$ on a domain D, then, by equation (4.4), it is approximately analytic with the constant $M/2$.

Hence, the following theorem, concerning an approximately analytic metric, is a generalization of Theorem 4.5.

Theorem 4.8. *Let f be a C^2 $\rho-$harmonic mapping of the unit disc $\mathbb{D}$ onto the $C^{2,\alpha}$ Jordan domain D. If the metric ρ is approximately analytic in D and f is quasiconformal, then f has bounded partial derivatives; and, in particular, it is a Lipschitz mapping.*

The proof of the Theorem 4.8 follows directly from Theorem 4.3(the main result), using the fact that the equation $|\bar{\partial}\chi| = |\partial\chi|$ holds for all real functions χ. The following theorem can be proved in the same way as Theorem 4.6.

Theorem 4.9. (Local version) *Let f be a C^2 $\rho-$harmonic mapping of the unit disc $\mathbb{D}$ onto the $C^{2,\alpha}$ Jordan domain D having a continuous extension $\tilde{f}$ to the boundary such that $\tilde{f}(\partial\mathbb{D}) = \partial D$. If f is quasiconformal in some neighborhood of a point $z_0 \in \mathbb{T} = \partial\mathbb{D}$, and the metric ρ is approximately analytic in some neighborhood of $w_0 = f(z_0)$, then f has bounded partial derivatives, and in particular it is a Lipschitz mapping in a neighborhood of the point z_0.*

The harmonic and q.c. mappings between Riemann surfaces

Similarly as in the case of domains of the complex plane we define a quasiconformal mapping and a harmonic mapping $f : R \to S$ between the Riemann surfaces R and S with the metrics ϱ and ρ, respectively.

If f is a harmonic mapping then $\varphi dz^2 = \rho \circ f f_z \overline{f}_{\bar{z}} dz^2$ is a holomorphic quadratic differential on R, and we say that φ is the *Hopf differential* of f and we write $\varphi =$Hopf(f).

Lemma 4.10. *Let (S_1, ρ_1) and (S_2, ρ_2) and (R, ρ) be three Riemann surfaces. Let g be an isometric transformation of the surface S_1 onto the surface S_2:*

$$\rho_1(\omega)|d\omega|^2 = \rho_2(w)|dw|^2,\ w = g(\omega).$$

Then $f : R \to S_1$ is ρ_1- harmonic if and only if $g \circ f : R \to S_2$ is ρ_2- harmonic. In particular, if g is an isometric self-mapping of S_1, then f is ρ_1- harmonic if and only if $g \circ f$ is ρ_1- harmonic.

Proof. If f is a harmonic map then $\varphi dz^2 = \rho \circ f\, p\, \overline{q}\, dz^2$ is a holomorphic quadratic differential in R, i.e., the mapping $\rho \circ f\, p\, \overline{q}$ is analytic near to the parameter $z = z(\zeta)$, $\zeta \in R$. Let $\omega = f(z)$, $F = g \circ f$, $P = (g \circ f)_z$ and $Q = (g \circ f)_{\bar{z}}$. Then $P = g'(\omega) \cdot p$ and $Q = g'(\omega) \cdot q$. Since $\rho_1(\omega) = \rho_2(w)|g'(\omega)|^2$, we obtain

$$\rho_2 \circ F P\bar{Q} = \rho_2 \circ g \circ f \cdot |g'(\omega)|^2 p\bar{q} = \rho_1 \circ f p\bar{q}.$$

Hence φ_1 =Hopf$(g \circ f)$ is a holomorphic differential, i.e., $g \circ f$ is harmonic with respect to the metric ρ_2. □

Instead of an arbitrary Riemann surface we consider here only the Riemann sphere. Note that most of the arguments work for an arbitrary compact Riemann surface.

We call the metric ρ defined on $S^2 = \overline{\mathbb{C}}$ by

$$\rho|dz|^2 = \frac{4|dz|^2}{(1 + |z|^2)^2}$$

the spherical metric. The corresponding distance function is

$$d_S(z, w) = \frac{2|z - w|}{\sqrt{(1 + |z|^2)(1 + |w|^2)}},\ d_S(z, \infty) = \frac{2}{\sqrt{(1 + |z|^2)}}.$$

We can verify that the orientation preserving isometries of the Riemann sphere S^2 with respect to the spherical metric are described by Möbius transformations of the form

$$g(z) = \frac{az + b}{\bar{a} - \bar{b}z},\ a, b \in \mathbb{C},\ |a|^2 + |b|^2 \neq 0. \tag{4.6}$$

The Euler-Lagrange equation for spherical harmonic mappings is

$$f_{z\bar{z}} + \frac{2\bar{f}}{1 + |f|^2} f_z \cdot f_{\bar{z}} = 0.$$

It is easy to verify that the spherical metric density is approximately analytic in $\mathbb{C}$ with constant 1; more precisely one can verify

$$\frac{\rho_{\bar{z}}}{\rho} = -\frac{2z}{1 + |z|^2}.$$

If f is a diffeomorphism of the Riemann sphere (or of a compact Riemann surface M) onto itself, then f is quasi-isometry with respect to the corresponding metric and consequently, it is quasiconformal.

A natural question is what we can say for harmonic q.c. diffeomorphisms defined in some sub-domain of the Riemann sphere.

Using Theorem 4.8, Lemma 4.10 and the isometries defined by (4.6) we can prove:

Proposition 4.11. *Let the domains $\Omega, D \subset \overline{\mathbb{C}}$ have $C^{1,\alpha}$ and $C^{2,\alpha}$ Jordan boundary on $S^2 = \overline{\mathbb{C}}$, respectively. Then any q.c. spherical harmonic diffeomorphism of Ω onto D is Lipschitz with respect to the spherical metric.*

4.4. Representation of φ−harmonic mappings

If f is φ−harmonic, and ϕ is a so called natural parameter defined by φ then the mapping $F = \phi \circ f$ is an euclidean harmonic. Application of Theorem K (see the introduction) to $F = \phi \circ f$ leads to Theorem 4.15(the main result of this section), which shows that Theorem 4.5 holds for more general domains than the unit disk, too.

Recall that if f is φ−*harmonic*, then it satisfies the equation (4.5). If $\varphi(w_0) \neq 0$, then there is a neighborhood V of w_0 such that there is a branch $\sqrt{\varphi}$ in V such that

$$\phi = \int \sqrt{\varphi(z)}\, dz$$

is conformal on V.

In this setting, we refer to $\phi = \int \sqrt{\varphi(z)}\, dz$ as a natural parameter on V.

Theorem 4.12. *If f is φ−harmonic and ψ is conformal on the co-domain of f, then the mapping $F = \psi \circ f$ satisfies the following equation:*

$$F_{z\bar{z}} = \left[\frac{\psi''(w)}{\psi'(w)^2} - \frac{\varphi'(w)}{2\psi'(w) \cdot \varphi(w)} \right] \cdot F_z \cdot F_{\bar{z}}\,, \tag{4.7}$$

where $w = f(z)$.

Proof. Since ϕ is analytic we have that $F_z = \psi'(w) \cdot f_z$ and that $F_{\bar{z}} = \psi'(w) \cdot f_{\bar{z}}$. Hence $F_{z\bar{z}} = \psi''(w) f_z f_{\bar{z}} + \psi'(w) f_{z\bar{z}}$. On the other hand, f is φ− harmonic and therefore:

$$f_{z\bar{z}} = -\frac{1}{2}\frac{\varphi'}{\varphi} \circ f \cdot f_z\, f_{\bar{z}}.$$

Combining those facts, we obtain (4.7). □

Notice that if $\varphi = 1$, then the φ-metric is reduced to the Euclidean metric; so if f is a Euclidean harmonic mapping, then

$$F_{z\bar{z}} = \frac{\psi''}{\psi'^2} F_z \cdot F_{\bar{z}}\,. \tag{4.8}$$

Let φ be an analytic function on a domain D. If $z_0 \in D$, $B = B(z_0; R) \subset D$ and $\varphi \neq 0$ on B, there exists a branch $\sqrt[0]{\varphi}$ of $\sqrt{\varphi(z)}$ on B. Define

$$\phi(w) = \int_0^w \sqrt[0]{\varphi(z)}\, dz,$$

for $w \in B$.

If D is simply connected $\varphi \neq 0$ on D, ϕ has analytic continuation on D. Recall, in this setting, we refer to $\phi = \int \sqrt{\varphi(z)}\, dz$ as a natural parameter on D.

Corollary 4.13. *Let φ be an analytic function such that there exists a branch of $\int \sqrt{\varphi(z)}\,\mathrm{dz}$ in some domain D. If $f : \Omega \to D$ is φ−harmonic and*

$$\phi = \int \sqrt{\varphi(z)}\, dz,$$

then the mapping $F = \phi \circ f$ is harmonic with respect to the Euclidean metric.

Proof. We easily obtain

$$\frac{\phi''(w)}{\phi'(w)^2} - \frac{\varphi'(w)}{2\phi'(w)\cdot\varphi(w)} = 0.$$

It follows from (4.7) that $F_{z\bar{z}} \equiv 0$. Hence F is harmonic. □

Using (4.8) we obtain:

Corollary 4.14. *Let h be a Euclidean harmonic mapping, let ψ be conformal on the co-domain of h; and let $\varphi = ((\psi^{-1})')^2$. Then the mapping $\hat{h} = \psi \circ h$ is φ- harmonic.*

Now we prove that Theorem 4.5 holds for more general domains.

Theorem 4.15. *Suppose that*

1. *D is a Jordan plane domain and φ be an analytic non-vanishing function on D.*

2. *f is a $\varphi-$harmonic quasiconformal mapping of the $C^{1,\alpha}$ domain Ω onto the $C^{1,\alpha}$ Jordan domain D.*

3. *$M = \|(\log\varphi)'\|_\infty < \infty$*

Then

(A) *ϕ has continuous non-vanishing extension to $\overline{D}$, is locally univalent on $\overline{D}$.*

(B) *f has bounded partial derivatives and in particular, it is a Lipschitz mapping.*

Note that the hypothesis 3. implies

4. φ has continuous non-vanishing extension to $\overline{D}$.

If we substitute the hypothesis 3. with 4. the conclusion of theorem holds.

Proof. In particular, if φ is an analytic non-vanishing function on $\overline{D}$ then the hypothesis 3. holds. Since D is simply connected and φ an analytic non-vanishing function on D, there is a branch $\log\varphi$ of multiply valued function $Ln\varphi$ and a natural parameter ϕ defined by φ on D. Recall, if f is $\varphi-$ harmonic, and ϕ is a natural parameter defined by φ then the mapping $F = \phi \circ f$ is Euclidean harmonic.

The hypothesis 3. implies that $\log\varphi$ is Lipschitz on D and in particular that $\log\varphi$ and φ have continuous non-vanishing extension to $\overline{D}$ for which we use the same notation. Hence there are $a, b > 0$ such that $a < |\varphi| < b$ on $\overline{D}$, and ϕ' has continuous non-vanishing extension to $\overline{D}$.

Proof of (*A*). Contrary, suppose that ϕ is not locally univalent at a point $z_0 \in \partial D$. Then there are points $a_n, b_n \in \overline{D}$, $a_n \neq b_n$ that converge to z_0 such that $\phi(a_n) = \phi(b_n)$ and there are points $z_n, w_n \in [a_n, b_n]$ such that $\mathrm{Re}\, \mathrm{z}'_\mathrm{n} = 0$ and $\mathrm{Im}\, \mathrm{w}'_\mathrm{n} = 0$, where $z'_n = ie^{i\alpha_n}\phi'(z_n)$ and $w'_n = ie^{i\alpha_n}\phi'(w_n)$. Hence $z'_n = i\,\mathrm{Im}\,(\mathrm{z}'_\mathrm{n} - \mathrm{w}'_\mathrm{n})$ and $|z'_n| = |\phi'(z_n)|$ converges to $|\phi'(z_0)| \neq 0$; since $z'_n - w'_n$ converges to 0, we get a contradiction. Thus we showed that ϕ is locally univalent on $\overline{D}$.

Proof of (*B*). Since $\phi' = \sqrt{\varphi}$ and $a < |\varphi| < b$, we have on $\sqrt{a} < |\phi'| < \sqrt{b}$ on $\overline{D}$, and therefore ϕ is locally by-lipschitz on $\overline{D}$.

Applying a local version of Theorem A (a) (see the introduction) to a locally $C^{1,\alpha}$ co-domain $D' = \phi(D)$ and a Euclidean harmonic mapping $F = \phi \circ f$ (note that ϕ is not 1-1 in general), we can prove (B) and the proof of theorem is complete. □

Assume that $\varphi(z) \neq 0$ and that the natural parameter

$$\phi(z) = \int \sqrt{\varphi(z)}\, dz$$

is well defined on a domain D; and let ϕ map D onto the convex domain $D' = \phi(D)$. We now show that ϕ transforms the φ-metric to the Euclidean metric (see Proposition 4.16 below).

By the definition of φ- metric, we have that:

$$d(z, w) = \inf_{z,w\in\gamma\subset D} \int_\gamma \sqrt{|\varphi(\zeta)|}|d\zeta|.$$

Since $\sqrt{|\varphi(\zeta)|}|d\zeta| = |d(\phi(\zeta))|$, setting $A = \phi(z)$, $B = \phi(w)$ and $\xi = \phi(\zeta)$, by the chain rule we obtain that

$$d(z, w) = \inf_{A,B\in\gamma'\subset D'} \int_{\gamma'} |d\xi|,$$

where $D' = \phi(D)$.

Now it is clear that the segment $[A, B]$ that belongs to D' (because D' is convex), is the curve that minimizes the previous functional. Hence $d(z, w) = |A - B| = |\phi(z) - \phi(w)|$. Thus we have proved the following proposition:

Proposition 4.16. *If $D' = \phi(D)$ is convex, then ϕ transforms the φ-metric to the Euclidean metric; i.e. the distance function defined by φ- metric is given by the formula $d(z, w) = \left|\phi(z) - \phi(w)\right|$.*

Example 4.17. Let $\varphi_0(w) = (w - c_0)^{-2}$ and let us consider the harmonic maps between two domains Ω and D with respect to the following metric density on D:

$$\rho_0(w) = |\varphi_0(w)| = |w - c_0|^{-2}, \ w \in D, \tag{4.9}$$

where $c_0 \notin \overline{D}$ is a given point. If $D' = \log(D - \{c_0\})$ is a convex domain, then the metric defined by the metric density (4.9) is

$$d_0(z, w) = \left|\log \frac{z - c_0}{w - c_0}\right|.$$

It is easy to verify that the conformal mappings A:

$$A(z) = c_0 + re^{i\alpha(\varepsilon-1)/2}(z - c_0)^\varepsilon, \ r \in \mathbb{R}, \ \varepsilon = \pm1, \tag{4.10}$$

describe the orientation preserving isometries of the domain $D_\alpha = \mathbb{C} \setminus \{c_0 + te^{i\alpha}, t \in \mathbb{R}^+\}$, with respect to the metric d_0 given by (4.9).

Let f be φ_0−harmonic between Ω and D, where $D \subset D_\alpha$ for some α. The natural parameter is $\phi_0(w) = \pm\log(w - c_0)$. Hence, as an application of Corollary 4.13, we obtain that $F(z) = \log(f(z) - c_0)$ is a harmonic function defined on the simply connected domain Ω. Hence we have

$$f(z) - c_0 = e^{g_0(z)+\overline{h_0(z)}} = g_1(z) \cdot \overline{h_1(z)} = \Big(\sqrt{c_0} - \frac{1}{\sqrt{c_0}}g(z)\Big) \cdot \Big(-\overline{\sqrt{c_0}} + \frac{1}{\overline{\sqrt{c_0}}}h(z)\Big),$$

which yields the representation:

$$f(z) = g(z) + \overline{h(z)} - c_0{}^{-1}g(z) \cdot \overline{h(z)}, \tag{4.11}$$

where g and h are analytic mappings, which map Ω into $\mathbb{C} \setminus \{c_0\}$.

It is easy to see that the family of mappings defined by (4.11) is closed under transformations given by (4.10) (see Lemma 4.10).

The above example provides the motivation for the following result.

Theorem 4.18. *Let g and h be analytic functions and let $f = g + \overline{h} - c_0{}^{-1} g\overline{h}$, $c_0 \neq 0$, be a diffeomorphism of the $C^{1,\beta}$ domain Ω onto the $C^{1,\alpha}$ Jordan domain D such that $c_0 \in \overline{\mathbb{C}} \setminus \overline{D}$. If f is a q.c. mapping, then it has bounded partial derivatives and the analytic functions g' and h' are bounded.*

Proof. The case $c_0 = \infty$ is proved by Theorem 4.15 and therefore we can assume that $c_0 \neq \infty$. Put

$$g_1 = \sqrt{c_0} - \frac{1}{\sqrt{c_0}} g \quad \text{and} \quad h_1 = -\overline{\sqrt{c_0}} + \frac{1}{\overline{\sqrt{c_0}}} h.$$

Then $f - c_0 = g_1 \cdot \overline{h}_1$. Since $f(z) \neq c_0$ it follows that $h_1(z) \neq 0$ and $g_1(z) \neq 0$. Therefore we can take the mapping $F = \log(f - c_0)$ which can be written as $F = \log g_1 + \overline{\log h_1}$ on Ω. Hence F is a harmonic mapping of Ω onto $C^{1,\alpha}$ domain $D' = \log(D - c_0)$. We obtain from Theorem 4.15 that there exists a constant M such that

$$\left|\frac{h_1'}{h_1}\right|^2 + \left|\frac{g_1'}{g_1}\right|^2 < M. \tag{4.12}$$

Thus $(\log h_1)'$ is bounded on Ω and consequently $\log h_1$ has a continuous extension to the boundary of Ω. Thus h_1 has a continuous and non-vanishing extension to $\overline{\Omega}$. The same holds for G.

Now, by 4.12, we obtain that h_1' and g_1' are bounded mappings. Thus h' and g' are bounded. □

Example 4.19. A harmonic mapping u with respect to the hyperbolic metric on the unit disk satisfies the following equation

$$u_{z\bar{z}} + \frac{2\bar{u}}{1 - |u|^2} u_z \cdot u_{\bar{z}} = 0.$$

As far as we know this equation cannot be solved using known methods of PDE; however, we can produce some examples; more precisely, we characterize real hyperbolic harmonic mappings.

Let

$$\varphi_1(w) = \frac{4}{(1 - w^2)^2}.$$

Using a natural parameter, i.e. a branch of $\phi_1(z) = \log \frac{z+1}{z-1} = 2\,arc \tanh z$, one can verify that f is φ_1−harmonic if and only if $f = \tanh g$, where g is Euclidean harmonic. Since the metric $\rho = |\varphi(w)|$ coincides with the Poincaré metric

$$\lambda = \frac{4}{(1 - |w|^2)^2}$$

for real w we obtain that *f is real λ- harmonic (hyperbolic harmonic) if and only if $f = 2\tanh g$, where g is real Euclidean harmonic.* Since the mappings

$$w = e^{i\varphi} \frac{z - a}{1 - \bar{a}z}, \quad a \in \mathbb{D},$$

are the isometries of the Poincaré disc, because of Lemma 4.10, we obtain the following claim: If h is real harmonic defined on some domain Ω, then the function

$$w = e^{i\varphi} \frac{\tanh(h(z)) - a}{1 - \bar{a}\tanh(h(z))} \quad (|a| < 1) \tag{4.13}$$

is harmonic with respect to the hyperbolic metric . Note that the mappings given by (4.13) have the rank 1 and they map Ω into circular arcs orthogonal on the unit circle $\mathbb{T}$.

Moreover, if a circle S orthogonal on the unit circle is given and $\Lambda = S \cap \mathbb{T}$, we can use (4.13) to describe all λ- harmonic mappings between Ω and Λ.

5. HQC and the Gehring-Osgood inequality

For a domain $G \subset \mathbb{R}^n$ let $\rho : G \to (0, \infty)$ be a function. We say that ρ is a weight function or a metric density if for every locally rectifiable curve γ in G, the integral

$$l_\rho(\gamma) = \int_\gamma \rho(x) ds$$

exists.

In this case we call $l_\rho(\gamma)$ the ρ-length of γ. A metric density defines a metric $d_\rho : G \times G \to (0, \infty)$ as follows. For $a, b \in G$, let

$$d_\rho(a, b) = \inf_\gamma l_\rho(\gamma)$$

where the infimum is taken over all locally rectifiable curves in G joining a and b. It is an easy exercise to check that d_ρ satisfies the axioms of a metric. For instance, the hyperbolic (or Poincaré) metric of $\mathbb{D}$ is defined in terms of the density $\rho(x) = c/(1 - |x|^2)$ where $c > 0$ is a constant.

The quasi-hyperbolic metric $k = k_G$ of G is a particular case of the metric d_ρ when $\rho(x) = \frac{1}{d(x, \partial G)}$.

Suppose that $G \subset \mathbb{R}^n$, $f : G \to \mathbb{R}^n$ is K-qr and $G' = f(G)$. Let $\partial G'$ be a continuum containing at least two distinct points. By the Gehring-Osgood inequality [16], there exists a constant $c > 0$ depending only on n and K such that

$$k_{G'}(fy, fx) \le c \max\{k_G(y, x)^\alpha, k_G(y, x)\}, \alpha = K^{1/(1-n)}, \quad x, y \in G\,.$$

Example 5.1. Suppose that $B(x_0, r) \subset G$.

a) If $|x_1 - x_0| \le r/2$, then $k_G(x_0, x_1) \le 2|x_0 - x_1|/r$.

b) If $x_1 \in B(x_0, r)$, then $k_G(x_0, x_1) \ge |x_0 - x_1|/2r$.

Hint a) for $x \in [x_0, x_1]$, we find $d(x) = d(x, \partial G) \ge r/2$ and therefore $k_G(x_0, x_1) \le \int_0^{|x_0 - x_1|} |dx|/d(x) \le 2|x_0 - x_1|/r$. b) If $x_1 \in B(x_0, r)$, then for $x \in B(x_0, r)$, $d(x) \le 2r$ and therefore $1/d(x) \ge 1/2r$.

The proof of the Gehring-Osgood inequality is based on:

Lemma 5.2. ([16]) *There is a constant a which depends only on n with the following property. If f is qc of D onto D', then*

$$\frac{|f(x_2) - f(x_1)|}{d(f(x_1), \partial D')} \le a(\frac{|x_2 - x_1|}{d(x_1, \partial D)})^\alpha, \quad \alpha = K^{1/(1-n)}$$

for all $x_1, x_2 \in D$ with $|x_2 - x_1| \le a^{-1/\alpha} d(x_1, \partial D)$.

Theorem 5.3. (The Gehring-Osgood inequality) *Suppose that $D \subset \mathbb{R}^n$, $f : D \to \mathbb{R}^n$ is K-qc and $D' = f(D)$. There exists a constant $c_2 = 4(2a)^{1/\alpha} > 0$ depending only on n and K such that*

$$k_{D'}(f(x_1), f(x_2)) \le c_2 \max\{k_D(x_1, x_2)^\alpha, k_D(x_1, x_2)\}, \alpha = K^{1/(1-n)}, \quad x_1, x_2 \in D\,.$$

Proof. We consider two cases.

Case (A). Suppose that

$$|x_2 - x_1| \le (2a)^{-1/\alpha} d(x_1, \partial D) < 1\,. \tag{5.1}$$

By lemma,

$$\frac{|f(x_2) - f(x_1)|}{d(f(x_1), \partial D')} \le a \left(\frac{|x_2 - x_1|}{d(x_1, \partial D)} \right)^\alpha \le \frac{1}{2}\,.$$

Put $y_1 = f(x_1), y_2 = f(x_2)$, $d = d(x_1, \partial D)$ and $d' = d(y_1) = d(f(x_1), \partial D')$. Then, by Example 5.1, $k_{D'}(y_1, y_2) \leq 2|y_2 - y_1|/d'$ and $k_D(x_1, x_2) \geq |x_2 - x_1|/2d$.

In particular, $k_{D'}(y_1, y_2) \leq 1$.

Case (B). Suppose that (5.1) is not true. Then join x_1 and x_2 with geodesic line γ and choose $z_1 = x_1, \cdots, z_{m+1} = x_2$ such that

$$\frac{|z_j - z_{j+1}|}{d(z_j, \partial D)} = (2a)^{-1/\alpha}, \quad \frac{|z_m - z_{m+1}|}{d(z_m, \partial D)} \leq (2a)^{-1/\alpha}$$

for $j = 1, \cdots, m-1$. Then $k_{D'}(y_1, y_2) \leq m$ and $k_D(x_1, x_2) \geq \frac{m-1}{2}(2a)^{-1/\alpha}$. □

Let f be a map from a metric space M to another metric space N. We say that f is a pseudo -isometry if there exist two positive constants a and b such that for all $x, y \in M$

$$a^{-1} d_M(x, y) - b \leq d_N(f(x), f(y)) \leq a d_M(x, y).$$

For for studying pseudo -isometries it is convenient to have the following lemma, cf [53].

Lemma 5.4. *Let G and G' be two domains in $\mathbb{R}^n$. Let σ and ρ be two metric densities on G and G', respectively, which define the corresponding metrics $ds = \sigma(z)|dz|$ and $ds = \rho(w)|dw|$; and $f : G \to G'$, C^1-mapping. If $\rho(f(z))\,|f'(z)| \leq c\sigma(z), z \in G$, then $\rho(f(z_2), f(z_1)) \leq c\,\sigma(z_2, z_1)$, $z_1, z_2 \in G$.*

The proof of this result is straightforward and it is left to the reader as an exercise.

Using the Gehring-Osgood inequality, the following results are proved, cf [53]:

Theorem 5.5. *Suppose that Ω is a proper subset of $\mathbb{R}^n$, $h : \Omega \to \mathbb{R}^n$ is a harmonic K-qc mapping. Then h is pseudo-isometry with respect to quasi-hyperbolic metrics on Ω and $\Omega' = h\Omega$.*

In particular:

Corollary 5.6. *Under the above condition, $f : (\Omega, k_\Omega) \to (\Omega', k_{\Omega'})$ is Lipschitz.*

Theorem 5.7. *Suppose that Ω is a proper subset of $\mathbb{R}^n$, $f : \Omega \to \mathbb{R}^n$ is K-qr and $\Omega' = f(\Omega)$. Let $\partial\Omega'$ be a continuum containing at least two distinct point.*
If f is a vector harmonic map, then f is Lipschitz with respect to quasi-hyperbolic metrics on Ω and Ω'.

5.1. HQC are bi-Lipschitz

We can compute the quasihyperbolic metric k on $\mathbb{C}^*$ by using the covering $\exp : \mathbb{C} \to \mathbb{C}^*$, where exp is exponential function. Let $z_1, z_2 \in \mathbb{C}^*$, $z_1 = r_1 e^{it_1}, z_2 = r_2 e^{it_2}$ and $\theta = \theta(z_1, z_2) \in [0, \pi]$ the measure of convex angle between z_1, z_2. We will prove

$$k(z_1, z_2) = \sqrt{|\ln \frac{r_2}{r_1}|^2 + \theta^2}.$$

This well-known formula is due to Martin and Osgood, see [[60],(3.12)].

Let $l = l(z_1)$ be line defined by 0 and z_1. Then z_2 belongs to one half-plane, say M, on which $l = l(z_1)$ divides $\mathbb{C}$.

Locally denote by ln a branch of Log on M. Note that ln maps M conformally onto horizontal strip of with π. Since $w = \ln z$, we find the quasi-hyperbolic metric

$$|dw| = \frac{|dz|}{|z|}.$$

Note that $\rho(z) = \frac{1}{|z|}$ is the quasi-hyperbolic density for $z \in \mathbb{C}^*$ and therefore $k(z_1, z_2) = |w_1 - w_2| = |\ln z_1 - \ln z_2|$. Let $z_1, z_2 \in \mathbb{C}^*$, $w_1 = \ln z_1 = \ln r_1 + it_1$. Then $z_1 = r_1 e^{it_1}$; there is $t_2 \in [t_1, t_1 + \pi)$ or $t_2 \in [t_1 - \pi, t_1)$ and $w_2 = \ln z_2 = \ln r_2 + it_2$. Hence

$$k(z_1, z_2) = \sqrt{|\ln \frac{r_2}{r_1}|^2 + (t_2 - t_1)^2}\,,$$

and therefore as a corollary of the Gehring-Osgood inequality, we have

Proposition 5.8. *Let f be a K-qc mapping of the plane such that $f(0) = 0$, $f(\infty) = \infty$ and $\alpha = K^{-1}$. If $z_1, z_2 \in \mathbb{C}^*$, $|z_1| = |z_2|$ and $\theta \in [0, \pi]$* (respectively $\theta^* \in [0, \pi]$) is the measure of convex angle between z_1, z_2 (respectively $f(z_1), f(z_2)$), then $\theta^* \leq c \max\{\theta^\alpha, \theta\}$, where $c = c(K)$. In particular, if $\theta \leq 1$, then $\theta^* \leq c\theta^\alpha$.

We announce some results obtained in [11]. The results make use of Proposition 5.8, which is a corollary of the Gehring-Osgood inequality [16], as we are going to explain.

Let Ω be Jordan domain in $\mathcal{D}_1$, γ curve defined by $\partial\Omega$ and h K-hqc from $\mathbb{D}$ onto Ω and $h(0) = a_0$. Then h is L-Lipschitz, where L depends only on $K, dist(a_0, \partial\Omega)$ and $\mathcal{D}_1$ constant C_γ. In [11] we give an explicit bound for the Lipschitz constant.

Let h be a harmonic quasiconformal map from the unit disk onto D in class $\mathcal{D}_1$. Examples show that a q.c. harmonic function does not have necessarily a C^1 extension to the boundary as in conformal case. In [11] it is proved that the corresponding function E_{h_*} is continuous on the boundary and for fixed θ_0, $v_{h_*}(z, \theta_0)$ is continuous in z at $e^{i\theta_0}$ on $\mathbb{D}$.

The main result in [11] is (stated in the introduction as Theorem 1.2):

Theorem 5.9. *Let Ω and Ω_1 be Jordan domains in $\mathcal{D}_1$, and let $h : \Omega \to \Omega_1$ be a harmonic q.c. homeomorphism. Then h is bi-Lipschitz.*

It seems that we use a new idea here. Let Ω_1 be $C^{1,\mu}$ curve. We reduce proof to the case when $\Omega = H$. Suppose that $h(0) = 0 \in \Omega_1$. We show that there is a convex domain $D \subset \Omega_1$ in $\mathcal{D}_1$ such that $\gamma_0 = \partial D$ touch the boundary of Ω_1 at 0 and that the part of γ_0 near 0 is a curve $\gamma(c) = \gamma(c, \mu)$. Since there is qc extension h_1 of h to $\mathbb{C}$, we can apply Proposition 5.8 to $h_1 : \mathbb{C}^* \to \mathbb{C}^*$. This gives estimate for $arg\gamma_1(z)$ for z near 0, where $\gamma_1 = h^{-1}(\gamma(c))$, and we show that there exist constants $c_1 > 0$ and μ_1 such that the graph of the curve $h^{-1}(\gamma(c))$ is below of the graph of the curve $\gamma(c_1) = \gamma(c_1, \mu_1)$. Therefore there is a domain $D_0 \subset \mathbb{H}$ in $\mathcal{D}_1$ such that $h(D_0) \subset D$. Finally, we combine the convexity type argument and noted continuity of functions E and v to finish the proof.

5.2. ρ-HQC are bi-Lipschitz

We announce the following results:

Theorem 5.10. *Let f be a C^2 K quasiconformal mapping of the unit disk onto a $C^{2,\alpha}$ Jordan domain Ω. Let ρ be a C^1 metric on $\overline{\Omega}$ and suppose that f is ρ-harmonic, that is $w_{z\bar{z}} + 2(\log \rho)_w w_z w_{\bar{z}} = 0$. Then $J_f \neq 0$ and f is Euclidean bi-Lipschitz.*

Suppose that $ds = \rho|dz|$ is metric on domain $G \subset \mathbb{C}$, $\rho \in C^1(G)$, the Gaussian curvature $K_\rho < 0$, $S = (G, \rho|dz|)$, and $h : \overline{\mathbb{D}} \to G$ is continuous. Define $\Gamma(t) = h(e^{it})$.

We also announce a generalization of G. Alessandrini, V. Nesi [3] theorem:

Theorem 5.11. *Suppose the above notation and*

1. *h is univalent on $\mathbb{T}$ and $D = Int(\Gamma)$*
2. *$h \in C^1(\overline{\mathbb{D}})$*
3. *h is ρ-harmonic on $\mathbb{D}$ and $\overline{D} \subset G$*
4. *$|p| > |q|$ on $\mathbb{T}$, where $p = f_z$ and $q = f_{\bar{z}}$.*
   Then
5. *h is univalent on $\mathbb{D}$ and $h(\mathbb{D}) = D$.*

5.3. Univalent harmonic qc mapping of strip

Define $\mathbb{K}_2 = \{w : |\mathrm{Im} w| < 1\}$ (in some preprints we also use the notation $\mathbb{S}_0$ for $\mathbb{K}_2$). If h is univalent harmonic of $\mathbb{K}_2$ with $h(\infty) = \infty$, we say that $h \in H_0(\mathbb{K}_2)$ and if it is in addition qc that $h \in HQC_0(K_2)$. We can use the shearing method to study univalent harmonic of $\mathbb{S}_0 = \mathbb{K}_2$ and show that:

Proposition 5.12. *Suppose that*

a) h is a homeomorphism of $\overline{\mathbb{K}_2}$, and

b) $h \in H_0(\mathbb{K}_2)$.

Then $H(z) = f(z) - g(z) = z$ and $h = z + 2\mathrm{Re}\, g$, where $\mathrm{Re}\, g' > -1/2$ on $\mathbb{K}_2$, and $\lim_{z\to\infty}(x + 2\mathrm{Re}\, g(z)) = \infty$.

Whether the conclusion of this proposition holds if the hypothesis a) is replaced by

c) h is continuous on $\overline{\mathbb{K}_2}$.

If h is defined by $h(z) = \arg(iz+1)+iy$, where arg is branch of argument on the right half-plane determined by values in $(-\pi/2, \pi/2)$, then it maps $\mathbb{K}_2$ onto the rectangle $(-\pi/2, \pi/2) \times (-1, 1)$.

If $h \in HQC_0(\mathbb{K}_2)$, then $v = \mathrm{Im}\, h(z) = \pm y$ and $h(z) = \mathrm{Re} F \pm iy$, where F is analytic on $\mathbb{K}_2$. If we suppose that $\mathrm{Im}\, h(z) = y$, then $h(z) = \left(F + z + \overline{(F - z)}\right)/2$, where F is analytic on $\mathbb{K}_2$.

Note that h is orientation preserving if and only if $|F' + 1| > |F' - 1|$, that is $\mathrm{Re} F' > 0$ on $\mathbb{K}_0$.

If $h \in HQC_0(\mathbb{K}_2)$ is a K-qc, since $|v'_y| = 1$, we have $\Lambda_h \le K$ and $\lambda_h \ge 1/K$; hence we find

Theorem 5.13. *If $h \in HQC_0(S_0)$, then*

$$\frac{1}{K}|z_1 - z_2| \le |h(z_1) - h(z_2)| \le K|z_1 - z_2|.$$

for $z_1, z_2 \in \mathbb{K}_2$.

5.4. The upper half space $\mathbb{H}^n$.

Here we follow Subsection 2.7 [53]. Let $\mathbb{H}^n$ denote the half-space in $\mathbb{R}^n$. If D is a domain in $\mathbb{R}^n$, by $HQC(D)$ we denote the set of Euclidean harmonic quasiconformal mappings of D onto itself.

In particular if $x \in \mathbb{R}^3$, we use notation $x = (x_1, x_2, x_3)$ and we denote by $\partial_{x_k} f = f'_{x_k}$ the partial derivative of f with respect to x_k .

A fundamental solution in space $\mathbb{R}^3$ of the Laplace equation is $\frac{1}{|x|}$. Let $U_0 = \frac{1}{|x+e_3|}$, where $e_3 = (0, 0, 1)$. Define $h(x) = (x_1 + \varepsilon_1 U_0, x_2 + \varepsilon_2 U_0, x_3)$. It is easy to verify that $h \in HQC(\mathbb{H}^3)$ for small values of ε_1 and ε_2.

Using the Herglotz representation of a nonnegative harmonic function u (see Theorem 7.24 and Corollary 6.36 [7]), one can get:

Lemma A. If u is a nonnegative harmonic function on a half space $\mathbb{H}^n$, continuous up to the boundary with $u = 0$ on $\mathbb{H}^n$, then u is (affine) linear.

In [49], the first author has outlined a proof of the following result:

Theorem A. If h is a quasiconformal harmonic mapping of the upper half space $\mathbb{H}^n$ onto itself and $h(\infty) = \infty$, then h is a quasi-isometry with respect to both the Euclidean and the Poincaré distance.

Note that the outline of proof in [49] can be justified by Lemma A.

In [53], we show that the analog statement of this result holds for p-harmonic vector functions (solutions of p-Laplacian equations) using the mentioned result obtained in the paper [40], stated here as:

Theorem B. If u is a nonnegative p-harmonic function on a half space $\mathbb{H}^n$, continuous up to the boundary with $u = 0$ on $\mathbb{H}^n$, then u is (affine) linear.

We plan further to investigate methods developed in this article (specifically see the proofs of Theorems 3.1, 3.2, 3.3 and Theorem A) and the sketch of proofs of Theorem 5.13 given in this subsection. Here are a few comments and questions.

1. We can derive an analogy of Theorem 3.1 in the case of $\mathbb{K}_2$.
2. Describe an analogy of Theorem 3.1 in the case of $\mathbb{H}^3 = \{(x, y, z) : x, y \in \mathbb{R}, z > 0\}$.
3. We can derive an analogy of Theorem 5.13 in the case $\mathbb{K}_3 = \{(x, y, z) : x, y \in \mathbb{R}, 0 < z < 1\}$.
4. Describe harmonic qc mapping of $\mathbb{H}$ onto a rectangle $(0, a) \times (0, 1)$, $a > 0$.
   Let $Q = \{(x, y, z) : x, y, z > 0\}$ and denote by $HQC_0(Q, \mathbb{H}^3)$ (respectively $HQC_0^k(Q, \mathbb{H}^3)$) the set of all qc (respectively k-qc) harmonic mappings h of Q onto $\mathbb{H}^3 = \{(x, y, z) : x, y \in R, z > 0\}$ for which $h(\infty) = \infty$.
5. Describe $HQC_0(Q, \mathbb{H}^3)$ and the corresponding problem in 2-dimension?

Theorem 5.14. *Let $h : \overline{\mathbb{D}} \to \mathbb{R}^n$, $n \geq 2$, be a homeomorphism, which is vector harmonic on the unit disk and let γ_r be curves defined by $h(re^{it})$, $0 \leq t \leq 2\pi$ and $\gamma = \gamma_1$.*
Then $h_t'(re^{it}) \in h^1(\mathbb{D})$ if and only if γ is rectifiable. In this setting,
$|\gamma_r| \to |\gamma|$ if $r \to 1$.

Further results of this type will appear in a joint work of the author, D. Kalaj and M. Marković; see also [28].
Note that there is the difference between harmonic and holomorphic version of Theorem 5.14 concerning the property of absolute continuity; see Proposition 2.1 [52]. Under the hypothesis of Theorem 5.14, in general, γ is not absolutely continuous. In particular, if $n = 2$ and h is holomorphic, then γ is absolutely continuous (Smirnov theorem).
We also announce a version of the isoperimetric inequality for harmonic functions:

Theorem 5.15. *Let $f : \mathbb{D} \to \mathbb{C}$ be harmonic, continuous on $\overline{\mathbb{D}}$ and γ curve defined by $f(e^{it})$, $0 \leq t \leq 2\pi$. Suppose that $f = g + \overline{h}$, where g and h are holomorphic function on $\mathbb{D}$. If γ is rectifiable curve of length L, then*

$$A(\gamma) = \int_{\mathbb{D}} (|g'|^2 - |h'|^2) dxdy = \int_{\mathbb{C}} n_\gamma(w) du\, dv \leq \frac{L^2}{4\pi}, \tag{5.2}$$

where $n_\gamma(w)$ is the index γ with respect to w.

6. Acknowledgment

The author would like to thank Professors K. Sakan and M. Vuorinen for careful reading the manuscript and valuable advice.

References

[1] L. Ahlfors, Lectures on quasiconformal mappings, Van Nostrand Princeton, N.J., 1966.
[2] G. Alessandrini, V. Nesi, Inverttible Harmonic mappinngs, beyond Kneser, arXiv:0712.3840v2 [math.AP] 8 Apr 2008.
[3] G. Alessandrini, V. Nesi, Invertible harmonic mappings, beyond Kneser, Ann. Scuola Norm. Sup. Pisa, Cl. Sci. (5) VIII (2009) 451-468.
[4] M. Arsenović, V. Manojlović, M. Mateljević, Lipschitz-type spaces and harmonic mappings in the space, Ann. Acad. Sci. Fenn. 35 (2010) 379-387.
[5] K. Astala, T. Iwaniec, G.J. Martin, J. Onninen, Extremal mappings of finite distortion, Proc. London Math. Soc. 91 (2005) 655-702.
[6] K. Astala, T. Iwaniec, G.J. Martin, Elliptic partial differential equations and quasiconformal mappings in the plane, Princeton University Press, 2009. MR 2472875 (2010j:30040)
[7] S. Axler, P. Bourdon, W. Ramey, Harmonic function theory, Springer-Verlag, New York 1992.
[8] C.A. Berenstein, R. Gay, Complex Variables. An Introduction New York, Springer-Verlag, 1991.
[9] A. Beurling, L. Ahlfors, The boundary correspondence under quasiconformal mappings, Acta Math. 96 (1956) 125-142.
[10] V. Božin, M. Mateljević, Some counterexamples related to the theory of HQC mappings, Filomat 24 (2010) 25-34.
[11] V. Božin, M. Mateljević, Quasiconformal harmonic mapping between $C^{1,\alpha}$ Jordan domains, to appear.
[12] X. Chen, A. Fang, A SchwarzPick inequality for harmonic quasiconformal mappings and its applications, J. Math. Anal. Appl. (2010), DOI:10.1016/j.jmaa.2010.02.031.
[13] X. Chen, A. Fang, A class of harmonic quasiconformal mappings with strongly hyperbolically convex ranges, Complex Variables and Elliptic Equations 2011, DOI: 10.1080/17476933.2011.622044, to appear.
[14] P.L. Duren, Theory of H_p Spaces, New York, Academic Press, 1970.
[15] J. Garnett, Bounded Analytic Functions, New York, Academic Press, 1981.

[16] F.W. Gehring, B. Osgood, Uniform domains and the quasi-hyperbolic metric, Journal d'Analyse Mathmatique 36(1) (1979) 50-74.
[17] D. Gilbarg, N. Trudinger, Elliptic Partial Differential Equations of Second Order. Vol. 224, 2 Edition, Springer 1977, 1983.
[18] D. Gilbarg, L. Hörmander, Intermediate Schauder estimates. Arch. Rational Mech. Anal. 74(4) (1980), 297–318.
[19] G.M. Goluzin, Geometric function theory, Nauka Moskva, Russian translation, 1966.
[20] E. Heinz, On certain nonlinear elliptic differential equations and univalent mappings, J. d' Anal. 5, 1956/57, 197-272.
[21] X.Z. Huang, Harmonic quasiconformal mappings on the upper half-plane, Complex Variables and Elliptic Equations, to appear.
[22] D. Kalaj, Univalent harmonic mappings between Jordan domains, Publ. Inst. Math. 69(83) (2001) 108-112 .
[23] D. Kalaj, On harmonic diffeomorphisms of the unit disc onto a convex domain, Complex Variables, Theory Appl. 48(2) (2003) 175-187.
[24] D. Kalaj, Harmonic functions and harmonic quasiconformal mappings between convex domains, Thesis, Beograd, 2002.
[25] D. Kalaj, Quasiconformal harmonic functions between convex domains, Publ. Inst. Math. 76(90) (2004) 3-20.
[26] D. Kalaj, Quasiconformal harmonic mapping between Jordan domains, Math. Z. Volume 260(2) (2008) 237-252.
[27] D. Kalaj, Lipschitz spaces and harmonic mappings, Ann. Acad. Sci. Fenn. Math. 34(2) (2009) 475-485
[28] D. Kalaj, M. Marković, On isoperimetric and Fejér-Riesz inequality for harmonic surfaces, manuscript, arXiv:1105.0273 (May 2011)
[29] D. Kalaj, Invertible harmonic mappings beyond the Kneser theorem and quasiconformal harmonic mappings. (English) Stud. Math. 207(2) (2011) 117-136.
[30] D. Kalaj, M. Mateljević, Inner estimate and quasiconformal harmonic maps between smooth domains, Journal d'Analise Math. 100 (2006) 117-132.
[31] D. Kalaj, M. Mateljević, Quasiconformal and harmonic mappings between Jordan smooth domains, Novi Sad J. Math. Vol. 38(3) (2008) 147-156.
[32] D. Kalaj, On quasiconformal harmonic maps between surfaces, arXiv:0905.0699v4 (Submitted on 5 May 2009 (v1), last revised 2 Aug 2011 (this version, v4))
[33] D. Kalaj, Radial extension of a bi-Lipschitz parametrization of a starlike Jordan curve, Dec 2010, to appear.
[34] D. Kalaj, M. Mateljević, On quasiconformal harmonic surfaces with rectifiable boundary, Complex Analysis and Operator Theory 5(3) (2011) 633-646, DOI: 10.1007/s11785-010-0062-9.
[35] D. Kalaj, M. Mateljević, On certain nonlinear elliptic PDE and quasiconfomal mapps between Euclidean surfaces, Potential Analysis 34(1) (2011) 13-22, DOI: 10.1007/s11118-010-9177-x.
[36] D. Kalaj, M. Mateljević, (K, K')-quasiconformal harmonic mappings, Potential analysis, (ISSN:0926-2601), Volume 34, Issue 4, May 2011, Page(s) 21 pages [11051].
[37] D. Kalaj, M. Mateljević, Quasiconformal Harmonic Mappings and Generalizations, J. Analysis Volume 18 (2010), 239- 260, Proceedings of the ICM2010 Satellite Conference International Workshop on Harmonic and Quasiconformal Mappings (HQM2010), Editors: D. Minda, S. Ponnusamy, N. Shanmugalingam.
[38] D. Kalaj, M. Pavlović, Boundary correspondence under harmonic quasiconformal homeomorfisms of a half-plane Ann. Acad. Sci. Fenn., Math. 30(1) (2005) 159-165.
[39] O. Kellogg, On the derivatives of harmonic functions on the boundary, Trans. Amer. Math. Soc. 33 (1931) 689-692.
[40] T. Kilpeläinen, H. Shahgholian, X. Zhong, Growth estimates through scaling for quasilinear partial differential equations, Ann. Acad. Sci. Fenn. Math. 32 (2007) 595–599.
[41] M. Knežević, M. Mateljević, On the quasi-isometries of harmonic quasiconformal mappings, J. Math. Anal. Appl. 334(1) (2007) 404-413.
[42] P. Koosis, Introduction to H_p Spaces, 2nd ed., Cambridge, England, Cambridge University Press, 1998.
[43] O. Lehto, K.I. Virtanen, Quasiconformal Mappings in the Plane, Springer-Verlag, 1973.
[44] F.D. Lesley, S.E. Warschawski, Boundary behavior of the Riemann mapping function of asymptotically conformal curves, Math. Z. 179 (1982) 299-323.
[45] H. Lewy, On the non-vanishing of the Jacobian in certain in one-to-one mappings, Bull. Amer. Math. Soc. 42. (1936) 689-692.
[46] G.J. Martin, The Teichmüller problem for mean distortion, Ann. Acad. Sci. Fenn. 34 (2009) 233–247.
[47] O. Martio, On harmonic quasiconformal mappings, Ann. Acad. Sci. Fenn., Ser. A I 425 (1968) 3-10.
[48] M. Mateljević, Estimates for the modulus of the derivatives of harmonic univalent mappings, Proceedings of International Conference on Complex Analysis and Related Topics (IXth Romanian-Finnish Seminar, 2001), Rev Roum Math Pures Appliq (Romanian Journal of Pure and Applied mathematics) 47 (5-6) (2002) 709-711.
[49] M. Mateljević, Distortion of harmonic functions and harmonic quasiconformal quasi-isometry, Revue Roum. Math. Pures Appl. 51 (5-6) (2006) 711-722.
[50] M. Mateljević, Dirichlet's principle, uniqueness of harmonic maps and extremal qc mappings, Zbornik radova 10(18), Two topics in mathematics, editor B. Stanković, 2004, 41-91; ISBN 86-80593-36-2.
[51] M. Mateljević, On quasiconformal harmonic mappings, unpublished manuscript, 2000, 2006.
[52] M. Mateljević, V. Božin, M. Knežević, Quasiconformality of harmonic mappings between Jordan domains, Filomat 24:3 (2010) 111-124.
[53] M. Mateljević, M. Vuorinen, On uniform continuity of harmonic maps, to appear in Journal of Inequalities and Applications, Volume 2010, Article ID 178732, 19 pages DOI:10.1155/2010/1787; arXiv: 0709.4546v1.
[54] D. Partyka, K. Sakan, Quasiconformality of harmonic extensions, J. Comput. Appl. Math. 105(1-2) (1999) 425–436.
[55] D. Partyka, K. Sakan, On bi-Lipschitz type inequalities for quasiconformal harmonic mappings, Ann. Acad. Sci. Fenn. Math. 32 (2007) 579-594.
[56] M. Pavlović, Boundary correspondence under harmonic quasiconformal homeomorfisms of the unit disc, Ann. Acad. Sci. Fenn. 27 (2002) 365-372.
[57] C. Pommerenke, Univalent functions, Vanderhoeck & Riprecht, 1975.

[58] C. Pommerenke, Boundary behavour of conformal maps, Springer-Verlag, New York, 1991.
[59] W. Rudin, Real and complex analysis, McGraw-Hill, 1966.
[60] M. Vuorinen: Conformal geometry and quasiregular mappings, Lecture Notes in Math. **V 1319**, Springer-Verlag, 1988.
[61] S.E. Warschawski, On the higher derivatives at the boundary in conformal mapping, Trans. Amer. Math. Soc. 38(2) (1935) 310-340.
[62] A. Zygmund, Trigonometric Series I, Cambrige University Press, 1958.