Filomat 26:3 (2012), 519–528
DOI 10.2298/FIL1203519K

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Power series of fuzzy numbers with real or fuzzy coefficients

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Abstract. Following Talo and Başar [Determination of the duals of classical sets of sequences of fuzzy numbers and related matrix transformations, Comput. Math. Appl. 58 (2009) 717–733], we essentially deal with the power series of fuzzy numbers. We give two examples on the geometric series of fuzzy numbers and by using the four different cases of u^n of n^{th} power of a fuzzy number u, we interest in the convergence of a power series of fuzzy numbers. Finally, we introduce the concept of a power series of fuzzy numbers with fuzzy coefficients.

1. Introduction

By $\mathbb{N}$, we denote the set of natural numbers. We begin with some required definitions and consequences on the sequences and series of fuzzy numbers.

Definition 1.1. ([11, Definition 2.7]) A sequence $u = (u_k)$ of fuzzy numbers is a function u from the set $\mathbb{N}$ into the set $L(\mathbb{R})$ of fuzzy numbers. *The fuzzy number* u_k denotes the value of the function at $k \in \mathbb{N}$ and is called as the k^{th} *term of the sequence*. By $\omega(F)$, we denote the set of all sequences of fuzzy numbers.

Definition 1.2. ([11, Definition 2.11]) A sequence $(u_k) \in \omega(F)$ is called *bounded* if and only if the set of fuzzy numbers consisting of the terms of the sequence (u_k) is a bounded set. That is to say that a sequence $(u_k) \in \omega(F)$ is said to be *bounded* if and only if there exist two fuzzy numbers m and M such that $m \preceq u_k \preceq M$ for all $k \in \mathbb{N}$. This means that $m^-(\lambda) \leq u_k^-(\lambda) \leq M^-(\lambda)$ and $m^+(\lambda) \leq u_k^+(\lambda) \leq M^+(\lambda)$ for all $\lambda \in [0,1]$.

Definition 1.3. ([11, Definition 2.12]) Let $(u_k) \in \omega(F)$. Then the expression $\sum_{k=0}^{\infty} u_k$ is called a *series of fuzzy numbers*. If the sequence $(s_n) = (\sum_{k=0}^{n} u_k)_{n\in\mathbb{N}}$ converges to an $u \in L(\mathbb{R})$, then we say that the series $\sum_{k=0}^{\infty} u_k$ of fuzzy numbers converges to u and write $\sum_{k=0}^{\infty} u_k = u$ which implies as $n \to \infty$ that

$$\sum_{k=0}^{n} u_k^-(\lambda) \to u^-(\lambda) \text{ and } \sum_{k=0}^{n} u_k^+(\lambda) \to u^+(\lambda),$$

uniformly in $\lambda \in [0,1]$. Conversely, if the fuzzy numbers $u_k = \{(u_k^-(\lambda), u_k^+(\lambda)) : \lambda \in [0,1]\}$, $\sum_{k=0}^{\infty} u_k^-(\lambda) = u^-(\lambda)$ and $\sum_{k=0}^{\infty} u_k^+(\lambda) = u^+(\lambda)$ converge uniformly in λ, then $u = \{(u^-(\lambda), u^+(\lambda)) : \lambda \in [0,1]\}$ defines a fuzzy number such that $u = \sum_{k=0}^{\infty} u_k$.

2010 *Mathematics Subject Classification*. Primary: 46S40; Secondary: 03E72, 46A45, 40A05
Keywords. Fuzzy numbers, level set, level convergence, series of fuzzy sets, power series of fuzzy numbers
Received: 11 November 2010; Revised: 25 March 2011; Accepted: 21 September 2011
Communicated by Dragan S. Djordjević
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Definition 1.4. ([11, Definition 2.14]) Let $\{f_k(\lambda)\}$ be a sequence of functions defined on $[a,b]$ and $\lambda_0 \in]a,b]$. Then, $\{f_k(\lambda)\}$ is said to be eventually equi-left-continuous at λ_0 if for any $\varepsilon > 0$ there exist $n_0 \in \mathbb{N}$ and $\delta > 0$ such that $|f_k(\lambda) - f_k(\lambda_0)| < \varepsilon$ whenever $\lambda \in]\lambda_0 - \delta, \lambda_0]$ and $k \geq n_0$.

Similarly, eventually equi-right-continuity at $\lambda_0 \in [a,b[$ of $\{f_k(\lambda)\}$ can be defined.

Theorem 1.5. ([11, Theorem 2.15]) *Let (u_k) be a sequence of fuzzy numbers such that $\lim_{k\to\infty} u_k^-(\lambda) = u^-(\lambda)$ and $\lim_{k\to\infty} u_k^+(\lambda) = u^+(\lambda)$ for each $\lambda \in [0,1]$. Then the pair of functions u^- and u^+ determine a fuzzy number if and only if the sequences of functions $\{u_k^-(\lambda)\}$ and $\{u_k^+(\lambda)\}$ are eventually equi-left-continuous at each $\lambda \in]0,1]$ and eventually equi-right-continuous at $\lambda = 0$.*

Thus, it is deduced that the series $\sum_{k=0}^{\infty} u_k^-(\lambda) = u^-(\lambda)$ and $\sum_{k=0}^{\infty} u_k^+(\lambda) = u^+(\lambda)$ define a fuzzy number if the sequences

$$\{s_n^-(\lambda)\} = \left\{\sum_{k=0}^{n} u_k^-(\lambda)\right\} \text{ and } \{s_n^+(\lambda)\} = \left\{\sum_{k=0}^{n} u_k^+(\lambda)\right\}$$

satisfy the conditions of Theorem 1.5. Of course, this is a weaker condition than the uniform convergence.

The main purpose of the present paper is to present the beginning concepts on the power series of fuzzy numbers with real and fuzzy coefficients. Prior to giving a corollary and two examples on the geometric series of fuzzy numbers, four different cases of u^n of n^{th} power of a fuzzy number u is examined in a basic lemma. By using these situations, the convergence of a power series of fuzzy numbers is investigated. Finally, the concept of a power series of fuzzy numbers with fuzzy coefficients is introduced.

2. Preliminaries, background and notation

In this section, we begin with giving some required definitions and statements of theorems, propositions, and lemmas.

A *fuzzy number* is a fuzzy set on the real axis, i.e., a mapping $u : \mathbb{R} \to [0,1]$ which satisfies the following four conditions:

(i) u is normal, i.e., there exists an $x_0 \in \mathbb{R}$ such that $u(x_0) = 1$.

(ii) u is fuzzy convex, i.e., $u[\lambda x + (1-\lambda)y] \geq \min\{u(x), u(y)\}$ for all $x, y \in \mathbb{R}$ and for all $\lambda \in [0,1]$.

(iii) u is upper semi-continuous.

(iv) The set $[u]_0 = \overline{\{x \in \mathbb{R} : u(x) > 0\}}$ is compact, (cf. Zadeh [13]), where $\overline{\{x \in \mathbb{R} : u(x) > 0\}}$ denotes the closure of the set $\{x \in \mathbb{R} : u(x) > 0\}$ in the usual topology of $\mathbb{R}$.

We denote the set of all fuzzy numbers on $\mathbb{R}$ by $L(\mathbb{R})$ and called it as *the space of fuzzy numbers*. *λ-level set* $[u]_\lambda$ of $u \in L(\mathbb{R})$ is defined by

$$[u]_\lambda := \begin{cases} \{t \in \mathbb{R} : u(t) \geq \lambda\} & , \quad (0 < \lambda \leq 1), \\ \overline{\{t \in \mathbb{R} : u(t) > \lambda\}} & , \quad (\lambda = 0). \end{cases}$$

The set $[u]_\lambda$ is closed, bounded and non-empty interval for each $\lambda \in [0,1]$ which is defined by $[u]_\lambda := [u^-(\lambda), u^+(\lambda)]$. $\mathbb{R}$ can be embedded in $L(\mathbb{R})$, since each $r \in \mathbb{R}$ can be regarded as a fuzzy number $\overline{r}$ defined by

$$\overline{r}(x) := \begin{cases} 1 & , \quad (x = r), \\ 0 & , \quad (x \neq r). \end{cases}$$

Representation Theorem 1. (cf. [2]) *Let $[u]_\lambda = [u^-(\lambda), u^+(\lambda)]$ for $u \in L(\mathbb{R})$ and for each $\lambda \in [0,1]$. Then the following statements hold:*

(i) *u^- is a bounded and non-decreasing left continuous function on* $]0,1]$.

(ii) *u^+ is a bounded and non-increasing left continuous function on* $]0,1]$.

(iii) *The functions u^- and u^+ are right continuous at the point $\lambda = 0$.*

(iv) $u^-(1) \leq u^+(1)$.

Conversely, if the pair of functions u^- and u^+ satisfies the conditions (i)-(iv), then there exists a unique $u \in L(\mathbb{R})$ such that $[u]_\lambda := [u^-(\lambda), u^+(\lambda)]$ for each $\lambda \in [0,1]$. The fuzzy number u corresponding to the pair of functions u^- and u^+ is defined by $u : \mathbb{R} \to [0,1]$, $u(x) := \sup\{\lambda : u^-(\lambda) \leq x \leq u^+(\lambda)\}$.

Let $u, v, w \in L(\mathbb{R})$ and $\alpha \in \mathbb{R}$. Then the operations addition, scalar multiplication and product defined on $L(\mathbb{R})$ by

$$\begin{aligned} u + v = w \quad &\Leftrightarrow \quad [w]_\lambda = [u]_\lambda + [v]_\lambda \text{ for all } \lambda \in [0,1] \\ &\Leftrightarrow \quad w^-(\lambda) = u^-(\lambda) + v^-(\lambda) \text{ and } w^+(\lambda) = u^+(\lambda) + v^+(\lambda) \text{ for all } \lambda \in [0,1], \end{aligned}$$

$$[\alpha u]_\lambda = \alpha[u]_\lambda \text{ for all } \lambda \in [0,1]$$

and

$$uv = w \Leftrightarrow [w]_\lambda = [u]_\lambda [v]_\lambda \text{ for all } \lambda \in [0,1],$$

where it is immediate that

$$w^-(\lambda) = \min\{u^-(\lambda)v^-(\lambda), u^-(\lambda)v^+(\lambda), u^+(\lambda)v^-(\lambda), u^+(\lambda)v^+(\lambda)\}$$

and

$$w^+(\lambda) = \max\{u^-(\lambda)v^-(\lambda), u^-(\lambda)v^+(\lambda), u^+(\lambda)v^-(\lambda), u^+(\lambda)v^+(\lambda)\}$$

for all $\lambda \in [0,1]$. Let W be the set of all closed bounded intervals A of real numbers with endpoints $\underline{A}$ and $\overline{A}$, i.e., $A := [\underline{A}, \overline{A}]$. Define the relation d on W by

$$d(A, B) := \max\{|\underline{A} - \underline{B}|, |\overline{A} - \overline{B}|\}.$$

Then it can easily be observed that d is a metric on W (cf. Diamond and Kloeden [1]) and (W, d) is a complete metric space, (cf. Nanda [7]). Now, we may define the metric D on $L(\mathbb{R})$ by means of the Hausdorff metric d as

$$D(u, v) := \sup_{\lambda \in [0,1]} d([u]_\lambda, [v]_\lambda) := \sup_{\lambda \in [0,1]} \max\{|u^-(\lambda) - v^-(\lambda)|, |u^+(\lambda) - v^+(\lambda)|\}.$$

One can extend the natural order relation on the real line to intervals as follows:

$$A \leq B \text{ if and only if } \underline{A} \leq \underline{B} \text{ and } \overline{A} \leq \overline{B}.$$

The partial ordering relation on $L(\mathbb{R})$ is defined as follows:

$$u \leq v \Leftrightarrow [u]_\lambda \leq [v]_\lambda \Leftrightarrow u^-(\lambda) \leq v^-(\lambda) \text{ and } u^+(\lambda) \leq v^+(\lambda) \text{ for all } \lambda \in [0,1].$$

Definition 2.1. ([11, Definition 2.1]) $u \in L(\mathbb{R})$ is said to be a *non-negative fuzzy number* if and only if $u(x) = 0$ for all $x < 0$. It is immediate that $u \geq \overline{0}$ if u is a non-negative fuzzy number.

One can see that

$$D(u,\overline{0}) = \sup_{\lambda\in[0,1]} \max\{|u^-(\lambda)|, |u^+(\lambda)|\} = \max\{|u^-(0)|, |u^+(0)|\}.$$

Now, we may give:

Proposition 2.2. ([11, Proposition 2.2]) *Let $u, v, w, z \in L(\mathbb{R})$ and $\alpha \in \mathbb{R}$. Then,*

(i) $(L(\mathbb{R}), D)$ *is a complete metric space, (cf. Puri and Ralescu [8]).*

(ii) $D(\alpha u, \alpha v) = |\alpha| D(u, v)$.

(iii) $D(u + v, w + v) = D(u, w)$.

(iv) $D(u + v, w + z) \leq D(u, w) + D(v, z)$.

(v) $|D(u,\overline{0}) - D(v,\overline{0})| \leq D(u, v) \leq D(u,\overline{0}) + D(v,\overline{0})$.

Definition 2.3. ([11, Definition 2.3]) An absolute value $|u|$ of a fuzzy number u is defined by

$$|u|(t) := \begin{cases} \max\{u(t), u(-t)\} & , \quad (t \geq 0), \\ 0 & , \quad (t < 0). \end{cases}$$

λ-level set $[|u|]_\lambda$ of the absolute value of $u \in L(\mathbb{R})$ is in the form $[|u|]_\lambda := [|u|^-(\lambda), |u|^+(\lambda)]$, where

$$\begin{aligned} |u|^-(\lambda) &:= \max\{0, u^-(\lambda), -u^+(\lambda)\}, \\ |u|^+(\lambda) &:= \max\{|u^-(\lambda)|, |u^+(\lambda)|\}. \end{aligned}$$

Proposition 2.4. ([3, Proposition 2.4]) *Let $u, v, m \in L(\mathbb{R})$ with $m \geq \overline{0}$ and $\alpha \in \mathbb{R}$. Then,*

(i) $|u| = \begin{cases} u & , \quad (u \geq \overline{0}), \\ -u & , \quad (u < \overline{0}). \end{cases}$

(ii) $|u + v| \leq |u| + |v|$.

(iii) $|\alpha u| = |\alpha||u|$.

(iv) $|u| = \overline{0}$ *if and only if* $u = \overline{0}$.

(v) $|u| \leq m$ *if and only if* $-m \leq u \leq m$.

3. Generalized Hukuhara difference

Let $\mathcal{K}$ be the space of non-empty compact and convex sets in the n-dimensional Euclidean space $\mathbb{R}^n$. If $n = 1$, denote the set of closed bounded intervals of the real line by I. Given two elements $A, B \in \mathcal{K}$ and $\alpha \in \mathbb{R}$, the usual interval arithmetic operations, i.e., Minkowski addition and scalar multiplication, are defined by $A + B = \{a + b : a \in A, b \in B\}$ and $\alpha A = \{\alpha a : a \in A\}$. It is well known that addition is associative and commutative, and with neutral element $\{0\}$. If $\alpha = -1$, scalar multiplication gives the opposite $-A = (-1)A = \{-a : a \in A\}$, but, in general, $A + (-A) \neq 0$, i.e., the opposite of A is not the inverse of A in Minkowski addition unless A is a singleton. A first consequence of this fact is that, in general, additive simplification is not valid.

To partially overcome this situation, the *Hukuhara difference*, H-difference for short, has been introduced as a set C for which $A \ominus B = C \Leftrightarrow A = B + C$ and an important property of $\ominus$ is that $A \ominus A = \{0\}$ for all $A \in \mathcal{K}$ and $(A + B) \ominus B = A$ for all $A, B \in \mathcal{K}$. The H-difference is unique, but it does not always exist. A necessary condition for $A \ominus B$ to exist is that A contains a translate $\{c\} + B$ of B.

A generalization of the Hukuhara difference proposed in [9] aims to overcome this situation.

Definition 3.1. ([9, Definition 1]) The generalized Hukuhara difference of two sets $A, B \in \mathcal{K}$ is defined as follows;

$$A \ominus B = C \Longleftrightarrow \begin{cases} A = B + C, \\ B = A + (-1)C. \end{cases} \tag{3.1}$$

Proposition 3.2. ([9, Proposition 3]) *Let $A, B \in \mathcal{K}$ be two compact convex sets. Then, we have:*

(i) If the H-difference exists, it is unique and is a generalization of the usual Hukuhara difference since $A \ominus B = A - B$, whenever $A \ominus B$ exists.

(ii) $A + (-A) \neq 0$.

(iii) $(A + B) \ominus B = A$.

(iv) $A \ominus B = B \ominus A = C \Leftrightarrow C = \{0\}$ and $A = B$.

Proposition 3.3. ([9, Proposition 4]) *The H-difference $C = A \ominus B$ of two intervals $A = [a^-, a^+]$ and $B = [b^-, b^+]$ always exists, and*

$$[a^-, a^+] \ominus [b^-, b^+] = [c^-, c^+],$$

where

$$\begin{aligned} c^- &= \min\{a^- - b^-, a^+ - b^+\}, \\ c^+ &= \max\{a^- - b^-, a^+ - b^+\} \end{aligned}$$

conditions in (3.1) are satisfied simultaneously if and only if the two intervals have the same length and $c^- = c^+$.

Proposition 3.4. ([10, Proposition 6]) *If A and B are two compact convex sets, then $d(A, B) = d(A \ominus B, \{0\})$.*

Proof. We observe by Proposition 3.3 that $d(A \ominus B, \{0\}) = max\{|c^-|, |c^+|\}$. Then we obtain $d(A \ominus B, \{0\}) = max\{|a^- - b^-|, |a^+ - b^+|\} = d(A, B)$. □

Proposition 3.5. ([10, Proposition 7]) *Let $u : [a, b] \to I$ be such that $u(x) = [u^-(x), u^+(x)]$. Then we have*

$$\begin{aligned} \lim_{x\to x_0} u(x) = \ell \quad &\Leftrightarrow \quad \lim_{x\to x_0}(u(x) \ominus \ell) = \{0\}, \\ \lim_{x\to x_0} u(x) = u(x_0) \quad &\Leftrightarrow \quad \lim_{x\to x_0}(u(x) \ominus u(x_0)) = \{0\}, \end{aligned}$$

where the limits are in the Hausdorff metric d for intervals.

4. Power series of fuzzy numbers

In this section, we deal with the power series with real and fuzzy coefficients.

We begin with a basic definition. Following Mares [5], we define u^n for $u > \overline{0}$ and $n \in \mathbb{R}/\{0\}$ as follows:

Definition 4.1. Let u be any non-negative fuzzy number. We define u^n for non-zero real number n by

$$u^n(x) = \begin{cases} u(x^{1/n}) &, \quad (x > 0), \\ 0 &, \quad (x \leq 0). \end{cases}$$

The λ-level set of the fuzzy number u^n with $[u]_\lambda = [u_\lambda^-, u_\lambda^+]$ is determined as follows: Since $[u^n]_\lambda = \{x : u^n(x) \geq \lambda\} = \{x : u(x^{1/n}) \geq \lambda\}$, we have

$$\begin{aligned} x \in [u]_\lambda^n \quad &\Leftrightarrow \quad u(x^{1/n}) \geq \lambda \Leftrightarrow x^{1/n} \in [u]_\lambda \Leftrightarrow u_\lambda^- \leq x^{1/n} \leq u_\lambda^+ \\ &\Leftrightarrow \quad (u_\lambda^-)^n \leq x \leq (u_\lambda^+)^n \Leftrightarrow x \in [(u_\lambda^-)^n, (u_\lambda^+)^n] \Leftrightarrow x \in [u]_\lambda^n \end{aligned}$$

In the case $n = 0$, we define u^0 by

$$u^0(x) = \begin{cases} 1 &, \quad (x > 0), \\ 0 &, \quad (x \leq 0). \end{cases}$$

Now, we can give four different forms of u^n respect to the Definition 4.1.

Basic Lemma. Suppose that $u \in L(\mathbb{R})$ and $n \in \mathbb{N}$. Then, the following four statements hold for u^n:

(i) If $u_\lambda^- > 0$ and $u_\lambda^+ > 0$, then $[u]_\lambda^n = [(u_\lambda^-)^n, (u_\lambda^+)^n]$.

(ii) If $u_\lambda^- < 0$ and $u_\lambda^+ > 0$, then $[u]_\lambda^n = [(u_\lambda^+)^{n-1}u_\lambda^-, (u_\lambda^+)^n]$.

(iii) If $u_\lambda^- < 0$ and $u_\lambda^+ < 0$, then $[u]_\lambda^n = \begin{cases} [(u_\lambda^+)^n, (u_\lambda^-)^n] & , \quad (n \text{ even}), \\ [(u_\lambda^-)^n, (u_\lambda^+)^n] & , \quad (n \text{ odd}). \end{cases}$

(iv) If $u_\lambda^- < 0$ and $u_\lambda^+ = 0$, then $[u]_\lambda^n = \begin{cases} [0, (u_\lambda^-)^n] & , \quad (n \text{ even}), \\ [(u_\lambda^-)^n, 0] & , \quad (n \text{ odd}). \end{cases}$

Proof. Since the cases (ii)-(iv) can be established by the similar way, we consider only the case (i). We prove (i) by mathematical induction.

(a) The statement is true for $n = 1$, since

$$([u]_\lambda)^1 = [u]_\lambda = [u_\lambda^-, u_\lambda^+].$$

(b) Assume that the statement true for $n = m$, i.e., $([u]_\lambda)^m = [(u_\lambda^-)^m, (u_\lambda^+)^m]$.

(c) For $n = m + 1$, one can see by taking into account (b) that

$$\begin{aligned} ([u]_\lambda)^{m+1} &= ([u]_\lambda)^m [u]_\lambda \\ &= [(u_\lambda^-)^m, (u_\lambda^+)^m][(u_\lambda^-), (u_\lambda^+)] \\ &= [(u_\lambda^-)^{m+1}, (u_\lambda^+)^{m+1}]. \end{aligned}$$

This shows that the statement is also true for $n = m + 1$, if it is true for $n = m$ which completes the proof. □

Now, we give the result on the geometric series of non-negative fuzzy numbers.

Corollary 4.2. *For a given geometric series* $\sum_{n=1}^{\infty} u^n(x)$ *as in the Definition 4.1,*

$$\sum_{n=1}^{\infty} [u]_\lambda^n = \left[\frac{u_\lambda^-}{1 - u_\lambda^-}, \frac{u_\lambda^+}{1 - u_\lambda^+} \right] \tag{4.1}$$

holds, where $\overline{0} \preceq u \prec \overline{1}$.

Proof. Taking into account the part (i) of Basic Lemma, define the sequence (s_p) by

$$s_p = \sum_{n=1}^{p} [u]_\lambda^n = \sum_{n=1}^{p} \left[(u_\lambda^-)^n, (u_\lambda^+)^n \right] = \left[\sum_{n=1}^{p} (u_\lambda^-)^n, \sum_{n=1}^{p} (u_\lambda^+)^n \right]$$

for all $p \in \mathbb{N}$. Then, we have

$$s_p = \frac{[u]_\lambda \left\{ \overline{1} - ([u]_\lambda)^p \right\}}{\overline{1} - [u]_\lambda} \quad \text{for all } p \in \mathbb{N}. \tag{4.2}$$

At this stage, since u is a non-negative fuzzy number for every $\varepsilon > 0$ there exists a $n_0 = n_0(\varepsilon) \in \mathbb{N}$ such that

$$D(u^p, \overline{0}) = \sup_{\lambda \in [0,1]} \max \left\{ \left| (u_\lambda^-)^p \right|, \left| (u_\lambda^+)^p \right| \right\} = \max \left\{ \left| (u_0^-)^p \right|, \left| (u_0^+)^p \right| \right\} < \varepsilon.$$

Therefore, taking into account this fact, we obtain by letting $p \to \infty$ in (4.2) that

$$\lim_{p \to \infty} s_p = \frac{[u]_\lambda}{\overline{1} - [u]_\lambda},$$

which means that (4.1) holds. □

Example 4.3. Consider the geometric series $\sum_{n=1}^{\infty} u^n$ of fuzzy numbers, where

$$u(\lambda) := \begin{cases} 1 & , \quad (1/3 \le \lambda \le 1/2), \\ 0 & , \quad (\lambda < 1/3 \text{ or } \lambda > 1/2). \end{cases}$$

Then, since $\overline{0} \le u \prec \overline{1}$ with $u_\lambda^- = 1/3$ and $u_\lambda^+ = 1/2$, it is immediate by (4.1) that $\sum_{n=1}^{\infty} [u]_\lambda^n = [1/2, 1]$.

Example 4.4. Consider the geometric series $\sum_{n=1}^{\infty} u^n$ of fuzzy numbers, where

$$u(\lambda) := \begin{cases} 4\lambda - 1 & , \quad (1/4 \le \lambda < 1/2), \\ -4\lambda + 3 & , \quad (1/2 \le \lambda \le 3/4), \\ 0 & , \quad (\lambda < 1/4 \text{ or } \lambda > 3/4). \end{cases}$$

Then, since $\overline{0} \le u \prec \overline{1}$ with $u_\lambda^- = (\lambda + 1)/4$ and $u_\lambda^+ = (3 - \lambda)/4$, we derive from here that

$$\sum_{n=1}^{\infty} [u]_\lambda^n = \left[\frac{(\lambda+1)/4}{1 - \frac{\lambda+1}{4}}, \frac{(3-\lambda)/4}{1 - \frac{3-\lambda}{4}} \right] = \left[\frac{\lambda+1}{3-\lambda}, \frac{3-\lambda}{1+\lambda} \right].$$

Definition 4.5. Let u be any element and u_0 be a fixed element in the space $L(\mathbb{R})$ of fuzzy numbers such that $u - u_0 \succeq \overline{0}$. Then, the power series of fuzzy numbers with real coefficients a_n is in the form

$$\sum_{n=1}^{\infty} a_n (u - u_0)^n = a_1(u - u_0) + a_2(u - u_0)^2 + \cdots + a_n(u - u_0)^n + \cdots$$

For simplicity in notation, here and after, we write w instead of the fuzzy number $u - u_0$. Therefore, we have $[u - u_0]_\lambda = [w]_\lambda = [w_\lambda^-, w_\lambda^+]$ with $w_\lambda^- = u^-(\lambda) - u_0^-(\lambda)$ and $w_\lambda^+ = u^+(\lambda) - u_0^+(\lambda)$ from Hukuhara difference given by Proposition 3.3. Then, the given power series is reduced to

$$\begin{aligned} \sum_{n=1}^{\infty} a_n [w]_\lambda^n &= a_1[w]_\lambda + a_2[w]_\lambda^2 + \cdots + a_n[w]_\lambda^n + \cdots \\ &= a_1[w_\lambda^-, w_\lambda^+] + a_2[(w_\lambda^-)^2, (w_\lambda^+)^2] + \cdots + a_n[(w_\lambda^-)^n, (w_\lambda^+)^n] + \cdots . \end{aligned}$$

Now, we can give the result concerning with the four choices of $w \in L(\mathbb{R})$ in the power series, as follows:

Proposition 4.6. *Consider the power series of fuzzy numbers $\sum_{n=1}^{\infty} a_n[w]_\lambda^n$. Therefore,*

(i) If $w_\lambda^- < 0$ and $w_\lambda^+ > 0$,

$$\sum_{n=1}^{\infty} a_n [w]_\lambda^n = \left[\sum_{n=1}^{\infty} a_n [(w_\lambda^+)^{n-1}](w_\lambda^-), \sum_{n=1}^{\infty} a_n (w_\lambda^+)^n \right].$$

(ii) If $w_\lambda^- < 0$ and $w_\lambda^+ < 0$,

a) If n is even,

$$\sum_{n=1}^{\infty} a_{2n} [w]_\lambda^{2n} = \left[\sum_{n=1}^{\infty} a_{2n} (w_\lambda^+)^{2n}, \sum_{n=1}^{\infty} a_{2n} (w_\lambda^-)^{2n} \right].$$

b) If n is odd,

$$\sum_{n=1}^{\infty} a_{2n-1} [w]_\lambda^{2n-1} = \left[\sum_{n=1}^{\infty} a_{2n-1} (w_\lambda^-)^{2n-1}, \sum_{n=1}^{\infty} a_{2n-1} (w_\lambda^+)^{2n-1} \right].$$

(iii) If $w_\lambda^- < 0$ and $w_\lambda^+ = 0$,
a) If n is even,

$$\sum_{n=1}^{\infty} a_{2n}[w]_\lambda^{2n} = \left[0, \sum_{n=1}^{\infty} a_{2n}(w_\lambda^-)^{2n}\right].$$

b) If n is odd,

$$\sum_{n=1}^{\infty} a_{2n-1}[w]_\lambda^{2n-1} = \left[\sum_{n=1}^{\infty} a_{2n-1}(w_\lambda^-)^{2n-1}, 0\right].$$

(iv) If $w_\lambda^- > 0$ and $w_\lambda^+ > 0$, then we have

$$\sum_{n=1}^{\infty} a_n[w]_\lambda^n = \left[\sum_{n=1}^{\infty} a_n(w_\lambda^-)^n, \sum_{n=1}^{\infty} a_n(w_\lambda^+)^n\right].$$

Proof. (i) Let us consider the power series of fuzzy numbers with $w_\lambda^- < 0$ and $w_\lambda^+ > 0$. Then, the straightforward calculation leads us to the consequence that

$$\begin{aligned}\sum_{n=1}^{\infty} a_n[w]_\lambda^n &= a_1[w]_\lambda + a_2[w]_\lambda^2 + \cdots + a_n[w]_\lambda^n + \cdots \\ &= a_1[w_\lambda^-, w_\lambda^+] + a_2[w_\lambda^+ w_\lambda^-, (w_\lambda^+)^2] + \cdots + a_n[(w_\lambda^+)^{n-1}w_\lambda^-, (w_\lambda^+)^n] + \cdots \\ &= \left[\sum_{n=1}^{\infty} a_n(w_\lambda^+)^{n-1}(w_\lambda^-), \sum_{n=1}^{\infty} a_n(w_\lambda^+)^n\right].\end{aligned}$$

(ii) Given the power series of fuzzy numbers with $w_\lambda^- < 0$ and $w_\lambda^+ < 0$. Then, one can immediately see that

$$\begin{aligned}\sum_{n=1}^{\infty} a_n[w]_\lambda^n &= a_1[w]_\lambda + a_2[w]_\lambda^2 + \cdots + a_n[w]_\lambda^n + \cdots \\ &= \begin{cases} \sum_{n=1}^{\infty} a_{2n}[(w_\lambda^+)^{2n}, (w_\lambda^-)^{2n}] & , \quad (n \text{ even}), \\ \sum_{n=1}^{\infty} a_{2n-1}[(w_\lambda^-)^{2n-1}, (w_\lambda^+)^{2n-1}] & , \quad (n \text{ odd}). \end{cases}\end{aligned}$$

a) If n is even, then we have

$$\sum_{n=1}^{\infty} a_{2n}[w]_\lambda^{2n} = \left[\sum_{n=1}^{\infty} a_{2n}(w_\lambda^+)^{2n}, \sum_{n=1}^{\infty} a_{2n}(w_\lambda^-)^{2n}\right].$$

b) If n is odd, then we have

$$\sum_{n=1}^{\infty} a_{2n-1}[w]_\lambda^{2n-1} = \left[\sum_{n=1}^{\infty} a_{2n-1}(w_\lambda^-)^{2n-1}, \sum_{n=1}^{\infty} a_{2n-1}(w_\lambda^+)^{2n-1}\right].$$

(iii) Given the power series of fuzzy numbers with $w_\lambda^- < 0$ and $w_\lambda^+ = 0$. Then, we conclude by the routine verification that

$$\begin{aligned}\sum_{n=1}^{\infty} a_n[w]_\lambda^n &= a_1[w]_\lambda + a_2[w]_\lambda^2 + \cdots + a_n[w]_\lambda^n + \cdots \\ &= \begin{cases} \sum_{n=1}^{\infty} a_{2n}[0, (w_\lambda^-)^{2n}] & , \quad (n \text{ even}), \\ \sum_{n=1}^{\infty} a_{2n-1}[(w_\lambda^-)^{2n-1}, 0] & , \quad (n \text{ odd}). \end{cases}\end{aligned}$$

a) If n is even, then we have

$$\sum_{n=1}^{\infty} a_{2n}[w]_{\lambda}^{2n} = \left[0, \sum_{n=1}^{\infty} a_{2n}(w_{\lambda}^{-})^{2n}\right].$$

b) If n is odd, then we have

$$\sum_{n=1}^{\infty} a_{2n-1}[w]_{\lambda}^{2n-1} = \left[\sum_{n=1}^{\infty} a_{2n-1}(w_{\lambda}^{-})^{2n-1}, 0\right].$$

(iv) Let us consider the power series $\sum_{n=1}^{\infty} a_n[w]_{\lambda}^{n}$ of fuzzy numbers with $w_{\lambda}^{-} > 0$ and $w_{\lambda}^{+} > 0$. Then, one can easily derive that

$$\begin{aligned}\sum_{n=1}^{\infty} a_n[w]_{\lambda}^{n} &= a_1[w]_{\lambda} + a_2[w]_{\lambda}^{2} + \cdots + a_n[w]_{\lambda}^{n} + \cdots \\ &= a_1[w_{\lambda}^{-}, w_{\lambda}^{+}] + a_2[(w_{\lambda}^{-})^2, (w_{\lambda}^{+})^2] + \cdots + a_n[(w_{\lambda}^{-})^n, (w_{\lambda}^{+})^n] + \cdots \\ &= \left[\sum_{n=1}^{\infty} a_n(w_{\lambda}^{-})^n, \sum_{n=1}^{\infty} a_n(w_{\lambda}^{+})^n\right].\end{aligned}$$

□

Now, we can give the definition of the power series of non-negative fuzzy numbers with the non-negative fuzzy coefficients.

Definition 4.7. u be any element and u_0 also be a fixed element in the space $L(\mathbb{R})$ of fuzzy numbers such that $u - u_0$ is non-negative fuzzy number, and (v_n) be a sequence of non-negative fuzzy numbers. Therefore, we have $[u - u_0]_{\lambda} = [w]_{\lambda} = [w^{-}(\lambda), w^{+}(\lambda)]$ with $w^{-}(\lambda) = u^{-}(\lambda) - u_0^{-}(\lambda)$ and $w^{+}(\lambda) = u^{+}(\lambda) - u_0^{+}(\lambda)$, and $[v]_{\lambda} = [v^{-}(\lambda), v^{+}(\lambda)]$. Then, the power series of fuzzy numbers with the coefficients v_n is given by $\sum_{n=1}^{\infty} v_n(u - u_0)^n$ which can be expressed in terms of λ-level sets as follows:

$$\begin{aligned}\sum_{n=1}^{\infty} v_n[w]_{\lambda}^{n} &= v_1[w]_{\lambda} + v_2[w]_{\lambda}^{2} + \cdots + v_n[w]_{\lambda}^{n} + \cdots \\ &= [v_1^{-}(\lambda), v_1^{+}(\lambda)][w^{-}(\lambda), w^{+}(\lambda)] + [v_2^{-}(\lambda), v_2^{+}(\lambda)][(w^{-}(\lambda))^2, (w^{+}(\lambda)^2)] + \cdots \\ &= \left[\sum_{n=1}^{\infty} v_n^{-}(\lambda)[w^{-}(\lambda)]^n, \sum_{n=1}^{\infty} v_n^{+}(\lambda)[w^{+}(\lambda)]^n\right].\end{aligned}$$

5. Conclusion

By using the sum of the series of λ-level sets, Talo and Başar [11] have recently determined the $\alpha-$, $\beta-$ and $\gamma-$duals of the classical sets of sequences of fuzzy numbers and given the necessary and sufficient conditions on an infinite matrix of fuzzy numbers transforming one of the classical sets to the another one together with certain basic results related to the convergence and absolute convergence of series of fuzzy numbers.

The present paper is a natural continuation of Talo and Başar [11]. We have studied the fuzzy analogues of the beginning results related to the power series of real or complex numbers. We should record from now on that the main results given in Section 4 of the present paper will base on examining the alternating and binomial series of fuzzy.

Acknowledgement

We have benefited a lot from discussions with Professor Özer Talo on this work. We would like to express our gratitude for his many valuable comments and advices.

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