Growth properties of the Fourier transform

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Abstract. In a recent paper by the authors, growth properties of the Fourier transform on Euclidean space and the Helgason Fourier transform on rank one symmetric spaces of non-compact type were proved and expressed in terms of a modulus of continuity based on spherical means. The methodology employed first proved the result on Euclidean space and then, via a comparison estimate for spherical functions on rank one symmetric spaces to those on Euclidean space, we obtained the results on symmetric spaces. In this note, an analytically simple, yet overlooked refinement of our estimates for spherical Bessel functions is presented which provides significant improvement in the growth property estimates.

In memory of Časlav V. Stanojević as teacher, mentor and friend

1. Euclidean space

In a paper by the authors [1], we proved the following result providing a growth property of the Euclidean Fourier transform.

Theorem 1.1. *Let $1 \leq p \leq 2$ and $n \geq 2$. Then there is a constant $C = C(p, n)$ such that the following hold:*

1. *If $f \in L^p(\mathbb{R}^n)$ and $1 < p \leq 2$, then*

$$\left(\int_{\mathbb{R}^n} \min\left\{ 1, \left(\frac{|\xi|}{r} \right)^{2q} \right\} |\widehat{f}(\xi)|^q d\xi \right)^{1/q} \leq C\, \Omega_p[f]\left(\frac{1}{r} \right).$$

2. *If $f \in L^1(\mathbb{R}^n)$, then*

$$\sup_{\xi} \left[\min\left\{ 1, \left(\frac{|\xi|}{r} \right)^2 \right\} |\widehat{f}(\xi)| \right] \leq C\, \Omega_1[f]\left(\frac{1}{r} \right).$$

Here the modulus of continuity was defined as

$$\Omega_p[f](r) = \sup_{0 \leq t \leq r} \| M^t f - f \|_p,$$

and $M^t f$ is the usual spherical mean of f,

$$M^t f(x) = \frac{1}{\omega_{n-1}} \int_{S^{n-1}} f(x + t\omega) \, d\omega,$$

S^{n-1} is the unit sphere in $\mathbb{R}^n$, ω_{n-1} its total surface measure with respect to the usual induced measure $d\omega$. In essence, the proof of this theorem is based on three things: (1) the Fourier transform identity $\widehat{M^t f}(\xi) = \widehat{f}(\xi) \, j_{\frac{n-2}{2}}(t|\xi|)$, where $j_\alpha(r)$ is normalized spherical Bessel function of order α,

$$j_\alpha(r) = 2^\alpha \Gamma(\alpha + 1) r^{-\alpha} J_\alpha(r),$$

(2) the Hausdorff-Young theorem, and (3) a careful estimate of (see [1], Lemma 6) $I(\lambda/r, z) = 1 - j_\alpha(\lambda z/r)$ of the form,

$$C_{1,\alpha} \min\left\{1, \left(\tfrac{\lambda}{r}\right)^2\right\} \le \int_0^1 I\left(\tfrac{\lambda}{r}, z\right) dz \le \sup_{0 \le z \le 1} I\left(\tfrac{\lambda}{r}, z\right) \le C_{2,\alpha} \min\left\{1, \left(\tfrac{\lambda}{r}\right)^2\right\}. \tag{1}$$

Here, $C_{k,\alpha}$ are positive constants.

A technically simple refinement of this estimate leads to the following generalization; a result of the same form as Theorem 1.1, yet dispenses with the need for the supremum in the spherical modulus of continuity.

Theorem 1.2. *Let $1 \le p \le 2$ and $n \ge 2$. Then there exists a constant $C = C(p, n)$ such that the following hold.*

1. *If $1 < p < 2$ and $f \in L^p(\mathbb{R}^n)$, then*

$$\left(\int_{\mathbb{R}^n} \min\left\{1, (t|\xi|)^{2q}\right\} |\widehat{f}(\xi)|^q d\xi\right)^{1/q} \le C \, \|M^t f(\cdot) - f(\cdot)\|_p.$$

2. *If $f \in L^1(\mathbb{R}^n)$, then*

$$\sup_\xi \left[\min\left\{1, (t|\xi|)^2\right\}\right] |\widehat{f}(\xi)| \le C \, \|M^t f(\cdot) - f(\cdot)\|_1.$$

3. *If $f \in L^2(\mathbb{R}^n)$, then the result takes sharper form:*

$$\left(\int_{\mathbb{R}^n} \min\{1, (t|\xi|)^4\} |\widehat{f}(\xi)|^2 d\xi\right)^{1/2} \asymp \|M^t f(\cdot) - f(\cdot)\|_2.$$

($r(t) \asymp s(t)$ means the left hand side is bounded above and below by positive constants times the right hand side)

The sharper form in the the case $p = 2$ is because the inequality in the Hausdorff-Young theorem for $p < 2$ becomes equality in the Plancherel theorem. The proof of this result follows the same method as for Theorem 1.1, with the estimate for spherical Bessel functions given above (Lemma 6 in [1]) replaced by the following.

Lemma 1.3. *Let $\alpha > -\frac{1}{2}$. Then there are positive constants $c_{1,\alpha}$ and $c_{2,\alpha}$ such that*

$$c_{1,\alpha} \min\{1, (\lambda t)^2\} \le 1 - j_\alpha(\lambda t) \le c_{2,\alpha} \min\{1, (\lambda t)^2\} \tag{2}$$

for all $\lambda > 0$.

Proof. The proof makes use the the Mehler formula for the spherical Bessel function given by

$$j_\alpha(\lambda t) = \frac{2\Gamma(\alpha + 1)}{\sqrt{\pi}\Gamma(\alpha + \frac{1}{2})} \int_0^1 (1 - y^2)^{\alpha - \frac{1}{2}} \cos(\lambda t y) \, dy.$$

It follows that

$$1 - j_\alpha(\lambda t) = \frac{4\Gamma(\alpha + 1)}{\sqrt{\pi}\Gamma(\alpha + \frac{1}{2})} \int_0^1 (1 - y^2)^{\alpha - \frac{1}{2}} \sin^2\left(\frac{\lambda t y}{2}\right) dy.$$

Since, $\sin \frac{\lambda t y}{2} \geq \frac{\lambda t y}{\pi}$, provided $\lambda t \leq \pi$, it follows that

$$1 - j_\alpha(\lambda t) \geq \frac{4\Gamma(\alpha + 1)}{\pi^{3/2}\Gamma(\alpha + \frac{1}{2})} (\lambda t)^2 \int_0^1 (1 - y^2)^{\alpha - \frac{1}{2}} y^2 dy = \frac{(\lambda t)^2}{\pi(\alpha + 1)},$$

the last step by evaluating the beta integral and simplifying the resulting gamma functions. For all $\lambda t > 0$, $|j_\alpha(\lambda t)| < 1$. Hence, for $\lambda t \geq \pi$, there is a constant $c > 0$ such that $1 - j_\alpha(\lambda t) \geq c$. Combining the estimates gives the left hand side of (2). The right hand side follows by similar technique. $\square$

The following corollary represents a quantified Riemann-Lebesgue lemma and is an extension/variant of results in one dimension given in Titchmarsh [11, page 117].

Corollary 1.4. *Let $1 \leq p < 2$ and $n \geq 2$. Then there is a positive constant $C = C(p, n)$ such that the following hold.*

1. *If $1 < p < 2$ and $f \in L^p(\mathbb{R}^n)$, then*

$$\left(\int_{|\xi| > 1/t} |\widehat{f}(\xi)|^q d\xi\right)^{1/q} \leq C \, \|M^t f(\cdot) - f(\cdot)\|_p.$$

2. *If $f \in L^1(\mathbb{R}^n)$, then*

$$\sup_{|\xi| > 1/t} |\widehat{f}(\xi)| \leq C \, \|M^t f(\cdot) - f(\cdot)\|_1.$$

3. *If $f \in L^2(\mathbb{R}^n)$, then*

$$\left(\int_{|\xi| > 1/t} |\widehat{f}(\xi)|^2 d\xi\right)^{1/2} \approx \|M^t f(\cdot) - f(\cdot)\|_2.$$

Remark 1.5. Theorem 1.2 was also obtained by Ditzian [2] as a consequence of a rather technical result in approximation theory. Our proof lies completely within the framework of harmonic analysis and lends its self to the extensions described below.

2. Rank one symmetric spaces

In this section we follow the notation given in [1]; basic references for the background material are Helgason's books [5, 6] and Koornwinder's survey paper on Jacobi functions [7]. Let $X = G/K$ where G is a connected non-compact semisimple Lie group with finite center and real rank one and K is a maximal compact subgroup. At the Lie algebra level, the Cartan decomposition has form $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$, where $\mathfrak{k}$ is the Lie algebra of K. The Iwasawa decomposition takes the form $\mathfrak{g} = \mathfrak{k} + \mathfrak{a} + \mathfrak{n}$, where $\mathfrak{a}$ is a maximal abelian subalgebra of $\mathfrak{p}$ and $\mathfrak{n}$ is a nilpotent subalgebra of $\mathfrak{g}$. The rank one condition is that dim $\mathfrak{a} = 1$. The nilpotent subalgebra $\mathfrak{n}$ has root space decomposition $\mathfrak{n} = \mathfrak{n}_\gamma + \mathfrak{n}_{2\gamma}$, where γ and 2γ are the positive roots. Let m_γ and $m_{2\gamma}$ be the respective root space dimensions and set $\rho = \frac{1}{2}(m_\gamma + 2m_{2\gamma})$. Choose $H_0 \in \mathfrak{a}$ such that $\gamma(H_0) = 1$. This allows identifying $\mathfrak{a}$ with $\mathbb{R}$ by the map $\mathbb{R} \ni t \to tH_0 \in \mathfrak{a}$, and on the dual side, $\mathfrak{a}_\mathbb{C}^*$ with $\mathbb{C}$. At the group level, the Iwasawa decomposition has form $G = KAN$, and we write $G \ni g = k \exp(H(g)) n$, where $H(g) \in \mathfrak{a}$ and exp is the exponential function. Because of the above identification, we often write $a_t = \exp(H(g))$, $t \in \mathbb{R}$ being identified with $H(g)$.

The polar decomposition of G takes the form $G = KA^+K$, where $A^+ = \{a_t : t \geq 0\}$. Following standard practice, functions f on X are identified with right K−invariant functions on G and write $f(x) = f(g)$, where

$x = gK$. In terms of this decomposition, the invariant measure dx on X has the form $dx = \Delta(t)\, dt dk$, where $\Delta(t) = \Delta_{(\alpha,\beta)}(t) = (2\sinh t)^{2\alpha+1}(2\cosh t)^{2\beta+1}$, $\alpha = (m_\gamma + m_{2\gamma} - 1)/2$ and $\beta = (m_{2\gamma} - 1)/2$, and dk is normalized Haar measure on K. The Laplacian on X is denoted L and its radial part is given by

$$L_r = \frac{d^2}{dt^2} + \frac{\Delta'(t)}{\Delta(t)}\frac{d}{dt}.$$

The spherical function on X is the unique radial solution to the equation

$$Lu = -(\lambda^2 + \rho^2)u$$

which is one at the origin of X. Let M be the centralizer of A in K and set $B = K/M$. For $x = gK \in X$ and $b = kM \in B$, let $A(x,b) = -H(g^{-1}k)$ (called the horocycle distance function). Then the Harish-Chandra formula for the spherical function is

$$\phi_\lambda(x) = \int_B e^{(i\lambda+\rho)A(x,b)}db,$$

where db is normalized measure on B. If we write $x = ka_t K$, then it is well known that $\phi_\lambda(x) = \phi_\lambda^{(\alpha,\beta)}(t)$, where $\phi_\lambda^{(\alpha,\beta)}(t)$ is Jacobi function of the first kind. Key properties of Jacobi functions are given in the following three bullet items [7].

- $|\phi_{\mu+i\eta}^{(\alpha,\beta)}(t)| \leq \phi_{i\eta}^{(\alpha,\beta)}(t) \leq 1$ for $\mu \in \mathbb{R}$ and $|\eta| \leq \rho$.

- $|\phi_{\mu+i\eta}^{(\alpha,\beta)}(t)| \leq e^{|\eta|t}\phi_0^{(\alpha,\beta)}(t) \leq C(1 + t)e^{(|\eta|-\rho)t}$.

- Let $1 < p < 2$, and define $D_p = \{\lambda = \mu + i\eta : |\eta| < (\frac{2}{p} - 1)\rho\}$. Then

$$\lambda \in D_p \implies \phi_\lambda^{(\alpha,\beta)} \in L^q(\mathbb{R}^+, \Delta_{(\alpha,\beta)}(t)dt),$$

  where q is the Hölder conjugate index: $\frac{1}{p} + \frac{1}{q} = 1$.

In sharp contrast to the Euclidean space setting, the third property written out for symmetric space states: for $\lambda \in D_p$, the spherical function $\phi_\lambda \in L^q(X)$.

In [1], we proved the following result.

Lemma 2.1. *Let $\alpha > -1/2$, $-1/2 \leq \beta \leq \alpha$, and let $t_0 > 0$. Then for $|\eta| \leq \rho$, there exists a positive constant $C = C(\alpha, \beta, t_0)$ such that*

$$|1 - \phi_{\mu+i\eta}^{(\alpha,\beta)}(t)| \geq C\,[1 - j_\alpha(\mu t)],$$

for all $0 \leq t \leq t_0$.

In the symmetric space realm, the above gives a local estimate involving the spherical function on X with that for the spherical function on the Euclidean tangent space to X at the origin and is the technical heart of the extension of Theorem 1.2 to symmetric spaces. The following example illustrates the essential ideas underlying this estimate.

Example 2.2. Consider the case $\alpha = \frac{1}{2}$, $\beta = -\frac{1}{2}$. Then the Jacobi function is elementary

$$\phi_\lambda(t) = \phi_\lambda^{(1/2,-1/2)}(t) = \frac{\sin \lambda t}{\lambda \sinh t},$$

and gives the spherical function on three dimensional real hyperbolic space $SO_e(1,3)/SO(3)$. Using the fundamental theorem of calculus

$$1 - \phi_\lambda(t) = \frac{1}{\sinh t}\int_0^t [\cosh s - \cos \lambda s]ds.$$

Substituting $\lambda = \mu + i\eta$, applying the addition theorem for the cosine function, and the fact that modulus dominates the real part, we obtain

$$\left|1 - \phi_\lambda(t)\right| \geq \frac{1}{\sinh t} \int_0^t [\cosh s - \cosh \eta s \cos \mu s]\, ds.$$

It follows that

$$
\begin{aligned}
\left|1 - \phi_\lambda(t)\right| &\geq \frac{1}{\sinh t} \int_0^t \left[1 - \frac{\cosh \eta s}{\cosh s} \cos \mu s\right] ds \\
&\geq \frac{t}{\sinh t} \frac{1}{t} \int_0^t [1 - \cos \mu s]\, ds \\
&= \frac{t}{\sinh t} \left[1 - j_{1/2}(\mu t)\right].
\end{aligned}
$$

For fixed $t_0 > 0$, the ratio $\frac{t}{\sinh t} \geq C$ for all $0 \leq t \leq t_0$, which gives the proof of the lemma for this example. The proof in the general case is based on more elaborate estimates applied to the Mehler identity for the Jacobi functions (the Mehler identity for the example is straightforward via the fundamental theorem of calculus).

As a consequence of the second bullet item above,

$$\lim_{t \to \infty} \phi_\lambda^{(\alpha,\beta)}(t) = 0$$

uniformly on any strip of the form $\{\lambda = \mu + i\eta : \mu \in \mathbb{R}, |\eta| \leq \eta_0 < \rho\}$. Combining the above lemma and this fact with (2) gives the following estimate.

Lemma 2.3. *Let $\alpha > -1/2$, $-1/2 \leq \beta \leq \alpha$, and let $0 < \eta_0 < \rho$. Then there exists a positive constant $C = C(\alpha, \beta, \eta_0)$ such that*

$$|1 - \phi_{\mu+i\eta}^{(\alpha,\beta)}(t)| \geq C \, \min\{1, (\mu t)^2\}$$

for all $\mu \in \mathbb{R}$, $|\eta| \leq \eta_0$, and $t > 0$.

The Helgason Fourier transform for functions defined on X is given by

$$\widehat{f}(\lambda, b) = \int_X f(x)\, e^{(-i\lambda + \rho)A(x,b)}\, dx.$$

The following estimate due to Sarkar and Sitaram [10] for this transform provides one avenue for developing an analog of Theorem 1.2 in symmetric spaces; it is a direct consequence of the aforementioned integrability property of Jacobi functions.

Lemma 2.4. *Let $1 \leq p < 2$ and $f \in L^p(X)$. Then for $\lambda = \mu + i\eta \in D_p$, $\widehat{f}(\lambda, b)$ is defined a.e. (b) and*

$$\int_B |\widehat{f}(\mu + i\eta, b)|\, db \leq c_p(|\eta|)\, \|f\|_p, \tag{3}$$

where $c_p(\cdot)$ is a positive function defined on $[0, (\frac{2}{p} - 1)\rho)$.

Group theoretically, the spherical mean of a function f on X is given by

$$M^t f(g) = \int_K f(gka_t)\, dk.$$

The main result generalizing Theorem 12 of [1] is the following.

Theorem 2.5. *Let $1 \leq p < 2$ and $f \in L^p(X)$. Then for $|\eta| < (2/p - 1)$ and $t > 0$, there exists a positive function $c_p(|\eta|)$ such that*

$$\sup_\mu \left[\min\{1, (t\mu)^2\} \int_B |\widehat{f}(\mu + i\eta, b)| \, db \right] \leq c_p(|\eta|) \, \|M^t f(\cdot) - f(\cdot)\|_p.$$

Proof. For completeness, we sketch the proof. From the operational property

$$\widehat{M^t f}(\lambda, b) = \phi_\lambda(a_t)\widehat{f}(\lambda, b)$$

and Lemma 2.4 we have

$$\left| 1 - \phi_{\mu+i\eta}(a_t) \right| \int_B \left| \widehat{f}(\mu + i\eta, b) \right| db \leq c_p(|\eta|)\|M^t f(\cdot) - f(\cdot)\|_p.$$

The result then follows by applying Lemma 2.3. $\quad\square$

Corollary 2.6. *Let $1 \leq p < 2$ and $f \in L^p(X)$. Then for $|\eta| < (2/p - 1)\rho$ and $t > 0$, there exists a positive function $c_p(|\eta|)$ such that*

$$\sup_{|\mu|>1/t} \int_B |\widehat{f}(\mu + i\eta, b)| \, db \leq c_p(|\eta|) \, \|M^t f(\cdot) - f(\cdot)\|_p.$$

In the case $p = 2$, the inequalities above breakdown. However, one can resort to the known Plancherel theorem for the Helgason Fourier transform and obtain a direct analog of Theorem 1.2. The following result generalizes Theorem 14 of [1].

Theorem 2.7. *Let $f \in L^2(X)$. Then there exists a positive constant C such that*

$$\left(\int_\mathbb{R} \min\{1, (\lambda t)^4\} \int_B |\widehat{f}(\lambda, b)|^2 db \, |c_X(\lambda)|^{-2} d\lambda \right)^{1/2} \leq C \, \|M^t f(\cdot) - f(\cdot)\|_2,$$

where $c_X(\lambda)$ is the Harish-Chandra $c-$function for X.

Remark 2.8. At the time of writing [1], a Hausdorff-Young inequality for the Helgason Fourier transform was known only in the case of radial functions. Hence the analog of Theorem 11 for the case $f \in L^p(X)$, $1 \leq p < 2$ was left as conjecture. In the same time frame as [1], Ray and Sarkar [9] proved the Hausdorff-Young theorem and provided nice generalizations to the inequality (3) in the context of Lorentz spaces using complex interpolation techniques. This has been applied in [4] to obtain corresponding extensions/refinements of Theorems 2.5 and 2.7 in the context of harmonic $NA-$groups. The latter include the rank one symmetric spaces as special cases.

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