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Semi-compatible and reciprocally continuous maps in weak non-Archimedean Menger PM-spaces

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Abstract. In this paper, we introduce semi-compatible maps and reciprocally continuous maps in weak non-Archimedean PM-spaces and establish a common fixed point theorem for such maps. Moreover, we show that, in the context of reciprocal continuity, the notions of compatibility and semi-compatibility of maps become equivalent. Our result generalizes several fixed point theorems in the sense that all maps involved in the theorem can be discontinuous even at the common fixed point.

1. Introduction

In this section, we give a short survey of the study of finding weaker forms of commutativity to have a common fixed point. In fact, this problem seems to be of vital interest and was initiated by Jungck [7] with the introduction of the concept of commuting maps. In 1982, Sessa [15] introduced the notion of weakly commutativity as a generalization of commutativity and this was a turning point in the development of Fixed Point Theory and its applications in various branches of mathematical sciences. To be precise, Sessa [15] defined the concept of weakly commuting maps by calling self-maps $\mathcal{A}$ and $\mathcal{B}$ of a metric space $(\mathcal{X}, d)$ a weakly commuting pair if and only if

$$d(\mathcal{A}\mathcal{B}x, \mathcal{B}\mathcal{A}x) \leq d(\mathcal{A}x, \mathcal{B}x),$$

for all $x \in \mathcal{X}$. Further to this, other authors gave some common fixed point theorems for weakly commuting maps [1, 4, 7]. Note that commuting maps are weakly commuting, but the converse is not true.

In 1986, Jungck [8] introduced the new notion of compatibility of maps as a generalization of weak commutativity. Thereafter, a flood of common fixed point theorems was produced by using the improved notion of compatibility of maps. In fact, every weak commutative pair of maps is compatible but the converse is not true. Later on, Jungck [9] introduced the concept of compatible maps of type ($\mathcal{A}$) or of

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type (α), Pathak et al. [11–13] introduced compatible maps of type ($\mathcal{B}$) or of type (β), type (C) and type ($\mathcal{P}$) in metric spaces and using these concepts, several researchers and mathematicians have proved common fixed point theorems. Recently, Cho [3] introduced the notion of compatible maps of type ($\mathcal{A}$) in non-Archimedean Menger PM-spaces and proved some interesting results. In this direction, a weaker notion of compatible maps, called semi-compatible maps, was introduced in fuzzy metric spaces by Singh et al. [16]. In particular, they proved that the concept of semi-compatible maps is equivalent to the concept of compatible maps and compatible maps of type (α) and of type (β) under some conditions on the maps.
In this paper, attempts have been made to introduce semi-compatible and reciprocally continuous maps in weak non-Archimedean Menger PM-spaces and it was also shown that in the context of reciprocal continuity, the notions of compatibility and semi-compatibility become equivalent. Here, we also present the concepts of compatible maps of type ($\mathcal{A}-1$) and ($\mathcal{A}-2$) and give the comparative study of these with semi-compatible maps.

2. Preliminaries and notations

For the sake of convenience, we gather some basic definitions and set out the terminology needed in the sequel.

Definition 2.1. ([14]) *A triangular norm* $*$ (shortly t-norm) is a binary operation on the unit interval $[0,1]$ such that for all $a,b,c,d \in [0,1]$, the following conditions are satisfied:

(1) $a * 1 = a$;
(2) $a * b = b * a$;
(3) $a * b \leq c * d$ whenever $a \leq c$ and $b \leq d$;
(4) $a * (b * c) = (a * b) * c$.

Definition 2.2. ([14]) *A distribution function* is a function $\mathcal{F} : (-\infty, +\infty) \to [0,1]$ that is left continuous on $\mathbb{R}$, non-decreasing and such that $\mathcal{F}(-\infty) = 0$, $\mathcal{F}(+\infty) = 1$.

Let Δ be the set of all distribution functions and denote by $\mathcal{H}(t)$ the function defined as

$$\mathcal{H}(t) = \begin{cases} 0 & \text{if } t \leq 0, \\ 1 & \text{if } t > 0. \end{cases} \tag{1}$$

Definition 2.3. ([14]) If $\mathcal{X}$ is a non empty set, $\mathcal{F} : \mathcal{X} \times \mathcal{X} \to \Delta$ is called a probabilistic distance on $\mathcal{X}$ and $\mathcal{F}(x,y)$ is usually denoted by $\mathcal{F}_{x,y}$.

Definition 2.4. ([5]) The ordered pair $(\mathcal{X}, \mathcal{F})$ is called a *non-Archimedean probabilistic metric space* (shortly NAPM-space) if $\mathcal{X}$ is a non-empty set and $\mathcal{F}$ is a probabilistic distance satisfying, for all $x,y,z \in \mathcal{X}$ and $t,s \geq 0$, the following conditions:

(PM-1) $\mathcal{F}_{x,y}(t) = 1$, $t > 0 \Leftrightarrow x = y$;
(PM-2) $\mathcal{F}_{x,y} = \mathcal{F}_{y,x}$;
(PM-3) $\mathcal{F}_{x,y}(0) = 0$;
(PM-4) $\mathcal{F}_{x,y}(t) = 1, \mathcal{F}_{y,z}(s) = 1 \Rightarrow \mathcal{F}_{x,z}(\max\{t,s\}) = 1$.

Remark 2.5. Every metric space $(\mathcal{X}, d)$ can always be realized as a PM-space by considering $\mathcal{F} : \mathcal{X} \times \mathcal{X} \to \Delta$ defined by $\mathcal{F}_{x,y}(t) = \mathcal{H}(t - d(x,y))$, for all $x, y \in \mathcal{X}$ and for all $t > 0$. So PM-spaces offer a wider framework than that of metric spaces and are general enough to cover even wider statistical situations.

The ordered triple $(\mathcal{X}, \mathcal{F}, *)$ is called a non-Archimedean Menger probabilistic metric space (shortly Menger NAPM-space) if $(\mathcal{X}, \mathcal{F})$ is a NAPM-space, $*$ is a t-norm and the following condition is also satisfied:

(PM − 5) $\mathcal{F}_{x,z}(\max\{t,s\}) \geq \mathcal{F}_{x,y}(t) * \mathcal{F}_{y,z}(s)$, for all $x, y, z \in \mathcal{X}$ and $t, s > 0$.

If the triangular inequality (PM-5) is replaced by the following:

(WNA) $\mathcal{F}_{x,z}(t) \geq \max\{\mathcal{F}_{x,y}(t) * \mathcal{F}_{y,z}(t/2), \mathcal{F}_{x,y}(t/2) * \mathcal{F}_{y,z}(t)\}$, for all $x, y, z \in \mathcal{X}$ and $t > 0$,

then the triple $(\mathcal{X}, \mathcal{F}, *)$ is called a weak non-Archimedean Menger probabilistic metric space (shortly Menger WNAPM-space). Obviously every Menger NAPM-space is itself a Menger WNA-space (see Vetro [18] for the same concept in fuzzy metric spaces).

Remark 2.6. Condition (WNA) does not imply that $\mathcal{F}_{y,z}(t)$ is nondecreasing and thus a Menger WNAPM-space is not necessarily a Menger PM-space. If $\mathcal{F}_{y,z}(t)$ is nondecreasing, then a Menger WNA-space is a Menger PM-space.

Remark 2.7. Recall that a Menger space is also a fuzzy metric space, for more details see Hadzic [6].

Example 2.8. Let $\mathcal{X} = [0, +\infty)$, $a * b = ab$ for every $a, b \in [0, 1]$. Define $\mathcal{F}_{x,y}(t)$ by: $\mathcal{F}_{x,y}(0) = 0$, $\mathcal{F}_{x,x}(t) = 1$ for all $t > 0$, $\mathcal{F}_{x,y}(t) = t$ for $x \neq y$ and $0 < t \leq 1$, $\mathcal{F}_{x,y}(t) = t/2$ for $x \neq y$ and $1 < t \leq 2$, $\mathcal{F}_{x,y}(t) = 1$ for $x \neq y$ and $t > 2$. Then $(\mathcal{X}, \mathcal{F}, *)$ is a Menger WNAPM-space, but it is not a PM-space.

We recall that the concept of neighborhood in Menger PM-spaces was introduced by Schweizer and Sklar [14] as follows.
If $x \in \mathcal{X}$, $\epsilon > 0$ and $\lambda \in (0, 1)$, then an (ϵ, λ)-neighborhood of x, $\mathcal{U}_x(\epsilon, \lambda)$ is defined by

$$\mathcal{U}_x(\epsilon, \lambda) = \{y \in \mathcal{X} : \mathcal{F}_{x,y}(\epsilon) > 1 - \lambda\}.$$

If the t-norm $*$ is continuous and strictly increasing, then $(\mathcal{X}, \mathcal{F}, *)$ is a Hausdorff space in the topology induced by the family $\{\mathcal{U}_x(\epsilon, \lambda) : x \in \mathcal{X}, \epsilon > 0, \lambda \in (0, 1)\}$ of neighborhoods. We refer the reader also to [17].

Let $\Omega = \{g$ such that $g : [0, 1] \to [0, +\infty)$ is continuous, strictly decreasing, $g(1) = 0$ and $g(0) < +\infty\}$.

Definition 2.9. ([3]) A PM-space $(\mathcal{X}, \mathcal{F})$ is said to be of type $(\mathcal{C})_g$ if there exists a $g \in \Omega$ such that, for all $x, y, z \in \mathcal{X}$ and $t \geq 0$, we have

$$g(\mathcal{F}_{x,y}(t)) \leq g(\mathcal{F}_{x,z}(t)) + g(\mathcal{F}_{z,y}(t)).$$

Definition 2.10. ([3]) A PM-space $(\mathcal{X}, \mathcal{F})$ is said to be of type $(\mathcal{D})_g$ if there exists a $g \in \Omega$ such that, for all $s, t \in [0, 1]$, we have

$$g(t * s) \leq g(t) + g(s).$$

Remark 2.11. If a Menger WNAPM-space $(\mathcal{X}, \mathcal{F}, *)$ is of type $(\mathcal{D})_g$ then $(\mathcal{X}, \mathcal{F}, *)$ is of type $(\mathcal{C})_g$. On the other hand if $(\mathcal{X}, \mathcal{F}, *)$ is a WNAPM-space such that $a * b \geq \max\{a + b - 1, 0\}$ for all $a, b \in [0, 1]$, then $(\mathcal{X}, \mathcal{F}, *)$ is of type $(\mathcal{D})_g$ for $g \in \Omega$ defined by $g(t) = 1 - t,\ t \geq 0$.

Throughout this paper, even when not specified, $(\mathcal{X}, \mathcal{F}, *)$ will be a complete Menger WNAPM-space of type $(\mathcal{D})_g$ with a continuous strictly increasing t-norm $*$.
Let $\phi : [0, +\infty) \to [0, +\infty)$ be a function satisfying the condition:

(Φ) ϕ is upper-semicontinuous from the right and $\phi(t) < t$, for all $t > 0$.

Lemma 2.12. ([2]) *If a function* $\phi : [0, +\infty) \to [0, +\infty)$ *satisfies the condition* (Φ), *then*

(a) $\lim_{n\to+\infty} \phi^n(t) = 0$ *for all* $t \geq 0$, *where* $\phi^n(t)$ *is the* n^{th} *iteration of* $\phi(t)$;
(b) if $\{t_n\}$ *is a non-increasing sequence of real numbers and* $t_{n+1} \leq \phi(t_n)$, $n = 1, 2, \ldots$, *then* $\lim_{n\to+\infty} t_n = 0$.

Lemma 2.13. ([3]) *Let $\{y_n\}$ be a sequence in $\mathcal{X}$ such that for all $t > 0$, $\lim_{n\to+\infty} \mathcal{F}_{y_n,y_{n+1}}(t) = 1$. If $\{y_n\}$ is not a Cauchy sequence in $\mathcal{X}$, then there exist $\epsilon_0 > 0$, $t_0 > 0$ and two sequences $\{m_i\}$, $\{n_i\}$ of positive integers such that*

(a) $m_i > n_{i+1}$ and $n_i \to +\infty$ as $i \to +\infty$;

(b) $\mathcal{F}_{y_{m_i},y_{n_i}}(t_0) < 1 - \epsilon_0$ and $\mathcal{F}_{y_{m_i-1},y_{n_i}}(t_0) \geq 1 - \epsilon_0$, $i = 1, 2, \ldots$.

Definition 2.14. ([3]) Two self-maps $\mathcal{A}$ and $\mathcal{B}$ of a Menger WNAPM-space $(\mathcal{X}, \mathcal{F}, *)$ are said to be compatible if $g(\mathcal{F}_{\mathcal{A}\mathcal{B}x_n,\mathcal{B}\mathcal{A}x_n}(t)) \to 0$ for all $t > 0$, whenever $\{x_n\}$ is a sequence in $\mathcal{X}$ such that $\mathcal{A}x_n, \mathcal{B}x_n \to z$ for some z in $\mathcal{X}$ as $n \to +\infty$.

Now, we introduce the concept of semi-compatible maps in Menger WNAPM-spaces and in the next proposition, we show that if the pair of maps is reciprocally continuous, then, the semi-compatibility of maps is equivalent to the compatibility of maps.

Definition 2.15. Two self-maps $\mathcal{A}$ and $\mathcal{B}$ of a Menger WNAPM-space $(\mathcal{X}, \mathcal{F}, *)$ are said to be semi-compatible if $g(\mathcal{F}_{\mathcal{A}\mathcal{B}x_n,\mathcal{B}z}(t)) \to 0$ for all $t > 0$, whenever $\{x_n\}$ is a sequence in $\mathcal{X}$ such that $\mathcal{A}x_n, \mathcal{B}x_n \to z$ for some z in $\mathcal{X}$ as $n \to +\infty$.

The notion of reciprocal continuity was defined by Pant [10] in ordinary metric space. Now, following the same line, we introduce reciprocally continuous maps in Menger WNAPM-spaces.

Definition 2.16. A pair of self-maps $(\mathcal{A}, \mathcal{B})$ of a Menger WNAPM-space $(\mathcal{X}, \mathcal{F}, *)$ is said to be *reciprocally continuous* if $g(\mathcal{F}_{\mathcal{A}\mathcal{B}x_n,\mathcal{A}z}(t)) \to 0$ and $g(\mathcal{F}_{\mathcal{B}\mathcal{A}x_n,\mathcal{B}z}(t)) \to 0$ for all $t > 0$, whenever there exists a sequence $\{x_n\}$ in $\mathcal{X}$ such that $\mathcal{A}x_n \to z, \mathcal{B}x_n \to z$ for some z in $\mathcal{X}$ as $n \to +\infty$.

If $\mathcal{A}$ and $\mathcal{S}$ are both continuous, then, they are obviously reciprocally continuous but the converse generally is not true.

Proposition 2.17. *Let $\mathcal{A}$ and $\mathcal{B}$ be two self-maps of a Menger WNAPM-space $(\mathcal{X}, \mathcal{F}, *)$. Assume that $(\mathcal{A}, \mathcal{B})$ is reciprocally continuous, then $(\mathcal{A}, \mathcal{B})$ is semi-compatible if and only if $(\mathcal{A}, \mathcal{B})$ is compatible.*

Proof. Let $\{x_n\}$ be a sequence in $\mathcal{X}$ such that $\mathcal{A}x_n \to z$ and $\mathcal{B}x_n \to z$ since the pair of maps $(\mathcal{A}, \mathcal{B})$ is reciprocally continuous, then for all $t > 0$, we have

$$\lim_{n\to+\infty} g(\mathcal{F}_{\mathcal{A}\mathcal{B}x_n,\mathcal{A}z}(t)) = 0 \text{ and } \lim_{n\to+\infty} g(\mathcal{F}_{\mathcal{B}\mathcal{A}x_n,\mathcal{B}z}(t)) = 0. \tag{2}$$

Suppose that $(\mathcal{A}, \mathcal{B})$ is semi-compatible. Then, we also get

$$\lim_{n\to+\infty} g(\mathcal{F}_{\mathcal{A}\mathcal{B}x_n,\mathcal{B}z}(t)) = 0, \text{ for all } t > 0. \tag{3}$$

Now, we have

$$g(\mathcal{F}_{\mathcal{A}\mathcal{B}x_n,\mathcal{B}\mathcal{A}x_n}(t)) \leq g(\mathcal{F}_{\mathcal{A}\mathcal{B}x_n,\mathcal{B}z}(t)) + g(\mathcal{F}_{\mathcal{B}z,\mathcal{B}\mathcal{A}x_n}(t)),$$

and letting $n \to +\infty$, we get

$$\lim_{n\to+\infty} g(\mathcal{F}_{\mathcal{A}\mathcal{B}x_n,\mathcal{B}\mathcal{A}x_n}(t)) = 0.$$

Thus, $\mathcal{A}$ and $\mathcal{B}$ are compatible maps.

Conversely, suppose that $(\mathcal{A}, \mathcal{B})$ is compatible and reciprocally continuous, then, for $t > 0$, we have

$$\lim_{n\to+\infty} g(\mathcal{F}_{\mathcal{A}\mathcal{B}x_n,\mathcal{B}\mathcal{A}x_n}(t)) = 0, \text{ for all } x_n \in \mathcal{X}. \tag{4}$$

Then, we get

$$g(\mathcal{F}_{\mathcal{A}\mathcal{B}x_n,\mathcal{B}z}(t)) \quad \leq \quad g(\mathcal{F}_{\mathcal{A}\mathcal{B}x_n,\mathcal{B}\mathcal{A}x_n}(t)) + g(\mathcal{F}_{\mathcal{B}\mathcal{A}x_n,\mathcal{B}z}(t)),$$

and so, letting $n \to +\infty$, we obtain

$$\lim_{n\to+\infty} g(\mathcal{F}_{\mathcal{A}\mathcal{B}x_n,\mathcal{B}z}(t)) = 0.$$

Thus, $\mathcal{A}$ and $\mathcal{B}$ are semi-compatible. This completes the proof. □

Naturally, we can define the concept of compatible mappings of type $(\mathcal{A}-1)$ and type $(\mathcal{A}-2)$ in Menger WNAPM-spaces as follows.

Definition 2.18. Two self-maps $\mathcal{A}$ and $\mathcal{B}$ of a Menger WNAPM-space $(\mathcal{X}, \mathcal{F}, *)$ are said to be compatible of type $(\mathcal{A}-1)$ if, for all $t > 0$, $\lim_{n\to+\infty} g(\mathcal{F}_{\mathcal{AB}x_n,\mathcal{BB}x_n}(t)) = 0$, whenever $\{x_n\}$ is a sequence in $\mathcal{X}$ such that $\mathcal{A}x_n, \mathcal{B}x_n \to z$ for some $z \in \mathcal{X}$ as $n \to +\infty$.

Definition 2.19. Two self-maps $\mathcal{A}$ and $\mathcal{B}$ of a Menger WNAPM-space $(\mathcal{X}, \mathcal{F}, *)$ are said to be compatible of type $(\mathcal{A}-2)$ if, for all $t > 0$, $\lim_{n\to+\infty} g(\mathcal{F}_{\mathcal{BA}x_n,\mathcal{AA}x_n}(t)) = 0$, whenever $\{x_n\}$ is a sequence in $\mathcal{X}$ such that $\mathcal{A}x_n, \mathcal{B}x_n \to z$ for some $z \in \mathcal{X}$ as $n \to +\infty$.

In the following proposition, it is shown that the concept of compatible maps of type $(\mathcal{A}-1)$, type $(\mathcal{A}-2)$ and semi-compatible maps are equivalent under given conditions.

Proposition 2.20. *Let $\mathcal{A}$ and $\mathcal{B}$ be two self-maps of a Menger WNAPM-space $(\mathcal{X}, \mathcal{F}, *)$. The following conditions hold:*

(a) If $\mathcal{B}$ is continuous, then the pair $(\mathcal{A}, \mathcal{B})$ is compatible of type $(\mathcal{A}-1)$ if and only if $(\mathcal{A}, \mathcal{B})$ is semi-compatible;
(b) if $\mathcal{A}$ is continuous, then the pair $(\mathcal{A}, \mathcal{B})$ is compatible of type $(\mathcal{A}-2)$ if and only if $(\mathcal{A}, \mathcal{B})$ is semi-compatible.

Proof. To prove condition (*a*), let $\{x_n\}$ be a sequence in $\mathcal{X}$ such that $\mathcal{A}x_n, \mathcal{B}x_n \to z$ for some $z \in \mathcal{X}$, as $n \to +\infty$ and let the pair $(\mathcal{A}, \mathcal{B})$ be compatible of type $(\mathcal{A}-1)$. The continuity of $\mathcal{B}$ gives

$$\lim_{n\to+\infty} \mathcal{BB}x_n = \mathcal{B}z \quad \text{and} \quad \lim_{n\to+\infty} \mathcal{BA}x_n = \mathcal{B}z.$$

From

$$g(\mathcal{F}_{\mathcal{AB}x_n,\mathcal{B}z}(t)) \le g(\mathcal{F}_{\mathcal{AB}x_n,\mathcal{BB}x_n}(t)) + g(\mathcal{F}_{\mathcal{BB}x_n,\mathcal{B}z}(t)),$$

letting $n \to +\infty$, we get $g(\mathcal{F}_{\mathcal{AB}x_n,\mathcal{B}z}(t)) \to 0$ and hence the pair $(\mathcal{A}, \mathcal{B})$ is semi-compatible. Now, since $\mathcal{B}$ is continuous, we can show easily that

$$g(\mathcal{F}_{\mathcal{AB}x_n,\mathcal{BB}x_n}(t)) \le g(\mathcal{F}_{\mathcal{AB}x_n,\mathcal{B}z}(t)) + g(\mathcal{F}_{\mathcal{B}z,\mathcal{BB}x_n}(t)).$$

On letting $n \to +\infty$, we get $g(\mathcal{F}_{\mathcal{AB}x_n,\mathcal{BB}x_n}(t)) \to 0$ and hence the pair $(\mathcal{A}, \mathcal{B})$ is compatible of type $(\mathcal{A}-1)$.
The proof of part (*b*) is analogous and so we omit it. □

Using similar arguments as above, the reader can easily prove the following result.

Proposition 2.21. *Let $\mathcal{A}$ and $\mathcal{B}$ be two self-maps of a Menger WNAPM-space $(\mathcal{X}, \mathcal{F}, *)$. If the pair $(\mathcal{A}, \mathcal{B})$ is semi-compatible and reciprocally continuous and $\{x_n\}$ is a sequence in $\mathcal{X}$ such that $\mathcal{A}x_n, \mathcal{B}x_n \to z$ for some $z \in \mathcal{X}$ as $n \to +\infty$, then $\mathcal{A}z = \mathcal{B}z$.*

Before proving our main theorem, we need the following lemma.

Lemma 2.22. *Let $\mathcal{A}, \mathcal{B}, \mathcal{L}, \mathcal{M}, \mathcal{S}$ and $\mathcal{T}$ be self-maps of a complete Menger WNAPM-space $(\mathcal{X}, \mathcal{F}, *)$ of type $(\mathcal{D})_g$, satisfying the following conditions:*

(i) $\mathcal{L}(\mathcal{X}) \subseteq \mathcal{ST}(\mathcal{X})$, $\mathcal{M}(\mathcal{X}) \subseteq \mathcal{AB}(\mathcal{X})$;
(ii) for all $x, y \in \mathcal{X}$ and $t > 0$,

$$g(\mathcal{F}_{\mathcal{L}x,\mathcal{M}y}(t)) \le \phi(\max\{g(\mathcal{F}_{\mathcal{AB}x,\mathcal{ST}y}(t)), g(\mathcal{F}_{\mathcal{L}x,\mathcal{AB}x}(t)), g(\mathcal{F}_{\mathcal{M}y,\mathcal{ST}y}(t)), \frac{1}{2}[g(\mathcal{F}_{\mathcal{AB}x,\mathcal{M}y}(t)) + g(\mathcal{F}_{\mathcal{L}x,\mathcal{ST}y}(t))]\}),$$

where the function $\phi : [0, +\infty) \to [0, +\infty)$ satisfies the condition (Φ).

Fix $x_0 \in \mathcal{X}$, then the sequence $\{y_n\}$ defined, for all $n = 0, 1, 2, ...$, by

$$y_{2n} = \mathcal{L}x_{2n} = \mathcal{ST}x_{2n+1}, \quad y_{2n+1} = \mathcal{AB}x_{2n+2} = \mathcal{M}x_{2n+1},$$

is a Cauchy sequence in $\mathcal{X}$ provided that $\lim_{n\to+\infty} g(\mathcal{F}y_n, y_{n+1}(t)) = 0$, for all $t > 0$.

Proof. Note that the sequence $\{y_n\}$ can be defined by the virtue of (*i*). Now, since $g \in \Omega$, it follows that, for all $t > 0$, $\lim_{n\to+\infty} \mathcal{F}_{y_n,y_{n+1}}(t) = 1$ if and only if $\lim_{n\to+\infty} g(\mathcal{F}_{y_n,y_{n+1}}(t)) = 0$. By Lemma 2.13, if $\{y_n\}$ is not a Cauchy sequence in $\mathcal{X}$, there exist $\epsilon_0 > 0$, $t_0 > 0$ and two sequences $\{m_i\}, \{n_i\}$ of positive integers such that

(a) $m_i > n_i + 1$ and $n_i \to +\infty$ as $i \to +\infty$;
(b) $\mathcal{F}_{y_{m_i},y_{n_i}}(t_0) < 1 - \epsilon_0$ and $\mathcal{F}_{y_{m_i-1},y_{n_i}}(t_0) \geq 1 - \epsilon_0$, i = 1,2,3,. . . .

Thus, we have

$$\begin{aligned} g(1-\epsilon_0) < g(\mathcal{F}_{y_{m_i},y_{n_i}}(t_0)) &\leq g(\mathcal{F}_{y_{m_i},y_{m_i-1}}(t_0)) + g(\mathcal{F}_{y_{m_i-1},y_{n_i}}(t_0)) \\ &\leq g(\mathcal{F}_{y_{m_i},y_{m_i-1}}(t_0)) + g(1-\epsilon_0) \end{aligned}$$

and letting $i \to +\infty$, we get

$$\lim_{i\to+\infty} g(\mathcal{F}_{y_{m_i},y_{n_i}}(t_0)) = g(1-\epsilon_0). \tag{5}$$

On the other hand, we have

$$g(1-\epsilon_0) < g(\mathcal{F}_{y_{m_i},y_{n_i}}(t_0)) \leq g(\mathcal{F}_{y_{m_i},y_{n_i+1}}(t_0)) + g(\mathcal{F}_{y_{n_i+1},y_{n_i}}(t_0)). \tag{6}$$

Let us assume that both m_i and n_i are even. By contractive condition (*ii*), we get

$$\begin{aligned} g(\mathcal{F}_{y_{m_i},y_{n_i+1}}(t_0)) &= g(\mathcal{F}_{\mathcal{L}x_{m_i},\mathcal{M}x_{n_i+1}}(t_0)) \\ &\leq \phi(\max\{g(\mathcal{F}_{\mathcal{AB}x_{m_i},\mathcal{ST}x_{n_i+1}}(t_0)), g(\mathcal{F}_{\mathcal{AB}x_{m_i},\mathcal{L}x_{m_i}}(t_0)), g(\mathcal{F}_{\mathcal{ST}x_{n_i+1},\mathcal{M}x_{n_i+1}}(t_0)), \\ &\quad \frac{1}{2}[g(\mathcal{F}_{\mathcal{AB}x_{m_i},\mathcal{M}x_{n_i+1}}(t_0)) + g(\mathcal{F}_{\mathcal{ST}x_{n_i+1},\mathcal{L}x_{m_i}}(t_0))]\}) \\ &\leq \phi(\max\{g(\mathcal{F}_{y_{m_i-1},y_{n_i}}(t_0)), g(\mathcal{F}_{y_{m_i-1},y_{m_i}}(t_0)), g(\mathcal{F}_{y_{n_i},y_{n_i+1}}(t_0)), \frac{1}{2}[g(\mathcal{F}_{y_{m_i-1},y_{n_i+1}}(t_0)) + g(\mathcal{F}_{y_{n_i},y_{m_i}}(t_0))]\}), \end{aligned}$$

that is

$$g(\mathcal{F}_{y_{m_i},y_{n_i+1}}(t_0)) \leq \phi(\max\{g(1-\epsilon_0), g(\mathcal{F}_{y_{m_i-1},y_{m_i}}(t_0)), g(\mathcal{F}_{y_{n_i+1},y_{n_i}}(t_0)), \frac{1}{2}[g(1-\epsilon_0) + g(\mathcal{F}_{y_{n_i},y_{n_i+1}}(t_0)) + g(\mathcal{F}_{y_{n_i},y_{m_i}}(t_0))]\}).$$

Putting this value in (6), using (5) and letting $i \to +\infty$, we get

$$g(1-\epsilon_0) \leq \phi(\max\{g(1-\epsilon_0), 0, 0, g(1-\epsilon_0)\}) = \phi(g(1-\epsilon_0)) < g(1-\epsilon_0),$$

a contradiction. Hence $\{y_n\}$ is a Cauchy sequence in $\mathcal{X}$. □

3. Main Result

Now, we are ready to give our main result.

Theorem 3.1. *Let $\mathcal{A}, \mathcal{B}, \mathcal{L}, \mathcal{M}, \mathcal{S}$ and $\mathcal{T}$ be self-maps of a complete Menger WNAPM-space $(\mathcal{X}, \mathcal{F}, *)$ of type $(\mathcal{D})_g$, satisfying (i) and (ii) of Lemma 2.22. Assume also that the following conditions hold:*

(i) $\mathcal{AB} = \mathcal{BA}$, $\mathcal{ST} = \mathcal{TS}$, $\mathcal{LB} = \mathcal{BL}$, $\mathcal{MT} = \mathcal{TM}$;*
(ii) the pair $(\mathcal{M}, \mathcal{ST})$ is weakly compatible.*

If, the pair $(\mathcal{L}, \mathcal{AB})$ is semi-compatible and reciprocally continuous, then $\mathcal{A}, \mathcal{B}, \mathcal{L}, \mathcal{M}, \mathcal{S}$ and $\mathcal{T}$ have a unique common fixed point.

Proof. Fix $x_0 \in \mathcal{X}$, then by (*i*) of Lemma 2.22, there exist $x_1, x_2 \in \mathcal{X}$ such that $\mathcal{L}x_0 = \mathcal{ST}x_1 = y_0$ and $\mathcal{M}x_1 = \mathcal{AB}x_2 = y_1$. Inductively, we can construct sequences $\{x_n\}$ and $\{y_n\}$ in $\mathcal{X}$ such that

$$\mathcal{L}x_{2n} = \mathcal{ST}x_{2n+1} = y_{2n} \text{ and } \mathcal{M}x_{2n+1} = \mathcal{AB}x_{2n+2} = y_{2n+1},$$

for $n = 0, 1, 2, \ldots$. Now, since $\mathcal{L}x_{2n} = \mathcal{ST}x_{2n+1}$, if we prove that, for all $t > 0$, $\lim_{n\to+\infty} g(\mathcal{F}_{y_n,y_{n+1}}(t)) = 0$, then by Lemma 2.22, we can conclude that the sequence $\{y_n\}$ is a Cauchy sequence in $\mathcal{X}$. To this aim, by (*ii*) of Lemma 2.22, we have

$$\begin{aligned}
g(\mathcal{F}_{y_{2n},y_{2n+1}}(t)) &= g(\mathcal{F}_{\mathcal{L}x_{2n},\mathcal{M}x_{2n+1}}(t)) \\
&\leq \phi(\max\{g(\mathcal{F}_{\mathcal{AB}x_{2n},\mathcal{ST}x_{2n+1}}(t)), g(\mathcal{F}_{\mathcal{AB}x_{2n},\mathcal{L}x_{2n}}(t)), g(\mathcal{F}_{\mathcal{ST}x_{2n+1},\mathcal{M}x_{2n+1}}(t)), \\
&\quad \frac{1}{2}[g(\mathcal{F}_{\mathcal{AB}x_{2n},\mathcal{M}x_{2n+1}}(t)) + g(\mathcal{F}_{\mathcal{ST}x_{2n+1},\mathcal{L}x_{2n}}(t))]\}) \\
&= \phi(\max\{g(\mathcal{F}_{y_{2n-1},y_{2n}}(t)), g(\mathcal{F}_{y_{2n-1},y_{2n}}(t)), g(\mathcal{F}_{y_{2n},y_{2n+1}}(t)), \\
&\quad \frac{1}{2}[g(\mathcal{F}_{y_{2n-1},y_{2n+1}}(t)) + g(\mathcal{F}_{y_{2n},y_{2n}}(t))]\}) \\
&\leq \phi(\max\{g(\mathcal{F}_{y_{2n-1},y_{2n}}(t)), g(\mathcal{F}_{y_{2n-1},y_{2n}}(t)), g(\mathcal{F}_{y_{2n},y_{2n+1}}(t)), \\
&\quad \frac{1}{2}[g(\mathcal{F}_{y_{2n-1},y_{2n}}(t)) + g(\mathcal{F}_{y_{2n},y_{2n+1}}(t))]\}).
\end{aligned}$$

If $g(\mathcal{F}_{y_{2n-1},y_{2n}}(t)) \leq g(\mathcal{F}_{y_{2n},y_{2n+1}}(t))$ for all $t > 0$, it follows

$$g(\mathcal{F}_{y_{2n},y_{2n+1}}(t)) \leq \phi(g(\mathcal{F}_{y_{2n},y_{2n+1}}(t))),$$

which, by Lemma 1, means that $g(\mathcal{F}_{y_{2n},y_{2n+1}}(t)) = 0$, for all $t > 0$. Similarly, we get $g(\mathcal{F}_{y_{2n+1},y_{2n+2}}(t)) = 0$, for all $t > 0$. Thus, for all $t > 0$, we have

$$\lim_{n\to+\infty} g(\mathcal{F}_{y_n,y_{n+1}}(t)) = 0.$$

On the other hand, if $g(\mathcal{F}_{y_{2n-1},y_{2n}}(t)) \geq g(\mathcal{F}_{y_{2n},y_{2n+1}}(t))$, then by the above contractive condition, for all $t > 0$, we have

$$g(\mathcal{F}_{y_{2n},y_{2n+1}}) \leq \phi(g(\mathcal{F}_{y_{2n-1},y_{2n}}(t))).$$

Similarly, we obtain that $g(\mathcal{F}_{y_{2n+1},y_{2n+2}}(t)) \leq \phi(g(\mathcal{F}_{y_{2n},y_{2n+1}}(t)))$, for all $t > 0$. Thus, we get

$$g(\mathcal{F}_{y_n,y_{n+1}}(t)) \leq \phi(g(\mathcal{F}_{y_{n-1},y_n}(t)))$$

for all $t > 0$ and $n = 1, 2, 3, \ldots$. Therefore, by Lemma 2.12, for all $t > 0$,

$$\lim_{n\to+\infty} g(\mathcal{F}_{y_n,y_{n+1}}(t)) = 0,$$

which implies that $\{y_n\}$ is a Cauchy sequence in $\mathcal{X}$ by Lemma 2.22. Since $\mathcal{X}$ is complete, therefore $\{y_n\}$ is convergent to a point z (say) in $\mathcal{X}$. Also, its subsequences $\{\mathcal{L}x_{2n}\}$, $\{\mathcal{AB}x_{2n}\}$, $\{\mathcal{M}x_{2n+1}\}$ and $\{\mathcal{ST}x_{2n+1}\}$ converge to the same point z in $\mathcal{X}$.

Now, since the pair of maps $(\mathcal{L}, \mathcal{AB})$ is reciprocally continuous, therefore, we have $g(\mathcal{F}_{\mathcal{LAB}x_{2n},\mathcal{L}z}(t)) \to 0$ and $g(\mathcal{F}_{\mathcal{ABL}x_{2n},\mathcal{AB}z}(t)) \to 0$ as $n \to +\infty$. By the semi-compatibility of $(\mathcal{L}, \mathcal{AB})$, we get $g(\mathcal{F}_{\mathcal{LAB}x_{2n},\mathcal{AB}z}(t)) \to 0$, that is, $\mathcal{AB}z = \mathcal{L}z$.

Now, taking $x = z$ and $y = x_{2n+1}$ in (*ii*), we have

$$\begin{aligned}
g(\mathcal{F}_{\mathcal{L}z,\mathcal{M}x_{2n+1}}(t)) &\leq \phi(\max\{g(\mathcal{F}_{\mathcal{AB}z,\mathcal{ST}x_{2n+1}}(t)), g(\mathcal{F}_{\mathcal{AB}z,\mathcal{L}z}(t)), g(\mathcal{F}_{\mathcal{ST}x_{2n+1},\mathcal{M}x_{2n+1}}(t)), \\
&\quad \frac{1}{2}[g(\mathcal{F}_{\mathcal{AB}z,\mathcal{M}x_{2n+1}}(t)) + g(\mathcal{F}_{\mathcal{ST}x_{2n+1},\mathcal{L}z})]\}).
\end{aligned}$$

On letting $n \to +\infty$, we get

$$\begin{aligned} g(\mathcal{F}_{\mathcal{L}z,z}(t)) &\leq \phi(\max\{g(\mathcal{F}_{\mathcal{AB}z,z}(t)), g(\mathcal{F}_{\mathcal{AB}z,\mathcal{L}z}(t)), g(\mathcal{F}_{z,z}(t)), \frac{1}{2}[g(\mathcal{F}_{\mathcal{AB}z,z}(t)) + g(\mathcal{F}_{z,\mathcal{L}z})]\}) \\ &= \phi(g(\mathcal{F}_{\mathcal{L}z,z}(t))). \end{aligned}$$

Thus, we get $z = \mathcal{AB}z = \mathcal{L}z$. Again taking $x = \mathcal{B}z, y = x_{2n+1}$ in (ii), we have

$$\begin{aligned} g(\mathcal{F}_{\mathcal{LB}z,\mathcal{M}x_{2n+1}}(t)) \leq\ & \phi(\max\{g(\mathcal{F}_{(\mathcal{AB})\mathcal{B}z,\mathcal{ST}x_{2n+1}}(t)), g(\mathcal{F}_{(\mathcal{AB})\mathcal{B}z,\mathcal{LB}z}(t)), g(\mathcal{F}_{\mathcal{ST}x_{2n+1},\mathcal{M}x_{2n+1}}(t)), \\ & \frac{1}{2}[g(\mathcal{F}_{(\mathcal{AB})\mathcal{B}z,\mathcal{M}x_{2n+1}}(t)) + g(\mathcal{F}_{\mathcal{ST}x_{2n+1},\mathcal{LB}z})]\}), \end{aligned}$$

on letting $n \to +\infty$ and using (i^*), we get

$$\begin{aligned} g(\mathcal{F}_{\mathcal{B}z,z}(t)) &\leq \phi(\max\{g(\mathcal{F}_{\mathcal{B}z,z}(t), g(\mathcal{F}_{\mathcal{B}z,\mathcal{B}z}(t)), g(\mathcal{F}_{z,z}(t)), \frac{1}{2}[g(\mathcal{F}_{\mathcal{B}z,z}(t)) + g(\mathcal{F}_{z,\mathcal{B}z}(t))]\}) \\ &= g(\mathcal{F}_{\mathcal{B}z,z}(t)). \end{aligned}$$

Thus, $z = \mathcal{AB}z = \mathcal{L}z = \mathcal{B}z \Rightarrow z = \mathcal{A}z = \mathcal{L}z = \mathcal{B}z$. Now, since $\mathcal{L}(\mathcal{X}) \subseteq \mathcal{ST}(\mathcal{X})$, there exists $w \in \mathcal{X}$ such that $z = \mathcal{L}z = \mathcal{ST}w$.

Let $x = x_{2n}$ and $y = w$ in (ii), we have

$$\begin{aligned} g(\mathcal{F}_{\mathcal{L}x_{2n},\mathcal{M}w}(t)) \leq\ & \phi(\max\{g(\mathcal{F}_{\mathcal{AB}x_{2n},\mathcal{ST}w}(t)), g(\mathcal{F}_{\mathcal{AB}x_{2n},\mathcal{L}x_{2n}}(t)), g(\mathcal{F}_{\mathcal{ST}w,\mathcal{M}w}(t)), \\ & \frac{1}{2}[g(\mathcal{F}_{\mathcal{AB}x_{2n},\mathcal{M}w}(t)) + g(\mathcal{F}_{\mathcal{ST}w,\mathcal{L}x_{2n}}(t))]\}), \end{aligned}$$

on letting $n \to +\infty$, we get

$$\begin{aligned} g(\mathcal{F}_{z,\mathcal{M}w}(t)) &\leq \phi(\max\{g(\mathcal{F}_{z,z}(t)), g(\mathcal{F}_{z,z}(t)), g(\mathcal{F}_{z,\mathcal{M}w}(t)), \frac{1}{2}[g(\mathcal{F}_{z,\mathcal{M}w}(t)) + g(\mathcal{F}_{z,z}(t))]\}) \\ &= \phi(g(\mathcal{F}_{z,\mathcal{M}w})), \end{aligned}$$

which implies that $z = \mathcal{M}w$. Hence, $\mathcal{ST}w = z = \mathcal{M}w$. Since $(\mathcal{M}, \mathcal{ST})$ is weakly compatible, therefore, $\mathcal{STM}w = \mathcal{MST}w \Rightarrow \mathcal{ST}z = \mathcal{M}z$. Now, we want to show that $\mathcal{ST}z = \mathcal{M}z = z$ and so, taking $x = x_{2n}$ and $y = z$ in (ii), we have

$$\begin{aligned} g(\mathcal{F}_{\mathcal{L}x_{2n},\mathcal{M}z}(t)) \leq\ & \phi(\max\{g(\mathcal{F}_{\mathcal{AB}x_{2n},\mathcal{ST}z}(t)), g(\mathcal{F}_{\mathcal{AB}x_{2n},\mathcal{L}x_{2n}}(t)), g(\mathcal{F}_{\mathcal{ST}z,\mathcal{M}z}(t)), \\ & \frac{1}{2}[g(\mathcal{F}_{\mathcal{AB}x_{2n},\mathcal{M}z}(t)) + g(\mathcal{F}_{\mathcal{ST}z,\mathcal{L}x_{2n}}(t))]\}). \end{aligned}$$

On letting $n \to +\infty$, we get

$$\begin{aligned} g(\mathcal{F}_{z,\mathcal{M}z}(t)) &\leq \phi(\max\{g(\mathcal{F}_{z,\mathcal{M}z}(t), g(\mathcal{F}_{z,z}(t)), g(\mathcal{F}_{\mathcal{M}z,\mathcal{M}z}(t)), \frac{1}{2}[g(\mathcal{F}_{z,\mathcal{M}z}(t)) + g(\mathcal{F}_{\mathcal{M}z,z}(t))]\}) \\ &= \phi(g(\mathcal{F}_{z,\mathcal{M}z})), \end{aligned}$$

which implies that $\mathcal{ST}z = \mathcal{M}z = z$. Hence, $z = \mathcal{A}z = \mathcal{B}z = \mathcal{L}z = \mathcal{M}z = \mathcal{ST}z$. Thus, z is a common fixed point of $\mathcal{A}, \mathcal{B}, \mathcal{L}, \mathcal{M}, \mathcal{S}$ and $\mathcal{T}$.

Uniqueness can be seen easily by using again contractive condition (ii) and so we omit it. This completes the proof. □

Finally, we give some examples to illustrate Theorem 3.1.

Example 3.2. Consider $\mathcal{X} = [0, 2]$ equipped, for every $x, y \in X$, with the usual metric $d(x, y) =| x - y |$. Let $(\mathcal{X}, \mathcal{F}, *)$ be the induced Menger WNAPM-space with $g(\alpha) = 1 - \alpha$ and $\mathcal{F}_{x,y}(t) = \mathcal{H}(t - d(x, y))$, for all $x, y \in \mathcal{X}$ and for all $t > 0$. Assume $\phi(x) = x/2$ and define self-maps $\mathcal{A}, \mathcal{B}, \mathcal{L}, \mathcal{M}, \mathcal{S}$ and $\mathcal{T}$ on $\mathcal{X}$ by:

$$\mathcal{A}x = \mathcal{B}x = \begin{cases} 1 & \text{if } x \text{ is rational,} \\ 1/3 & \text{if } x \text{ is irrational,} \end{cases} \quad \mathcal{L}x = \mathcal{M}x = 1 \text{ for all } x \in \mathcal{X}, \ \mathcal{S}x = \mathcal{T}x = \begin{cases} 1 & \text{if } x \text{ is rational,} \\ 2/3 & \text{if } x \text{ is irrational.} \end{cases}$$

Taking $\{x_n\} = \{1/n\}$, one can easily show that all the conditions of Lemma 2.22 and Theorem 3.1 hold. Thus, $x = 1$ is the unique common fixed point of $\mathcal{A}, \mathcal{B}, \mathcal{L}, \mathcal{M}, \mathcal{S}$ and $\mathcal{T}$.

In the next example, we consider a pair $(\mathcal{M}, \mathcal{ST})$ of non compatible maps.

Example 3.3. Let $(\mathcal{X}, \mathcal{F}, *)$, g and $\phi(x)$ as in the example above. Define self-maps $\mathcal{A}, \mathcal{B}, \mathcal{L}, \mathcal{M}, \mathcal{S}$ and $\mathcal{T}$ on $\mathcal{X}$ by:

$$\mathcal{A}x = \mathcal{B}x = \begin{cases} 2x & \text{if } x \in [0,1], \\ 0 & \text{if } x \in (1,2], \end{cases} \quad \mathcal{L}x = 2 \text{ for all } x \in \mathcal{X},$$

$$\mathcal{M}x = \begin{cases} 2-x & \text{if } x \in [0,1), \\ 2 & \text{if } x \in [1,2], \end{cases} \quad \mathcal{S}x = \mathcal{T}x = \begin{cases} x & \text{if } x \in [0,1), \\ 2 & \text{if } x \in [1,2]. \end{cases}$$

Taking $\{x_n\} = \{1 - 1/n\}$, from $\mathcal{F}_{Mx_n,1}(t) = \mathcal{H}(t + 1/n)$, we have:

1. $\lim_{n\to+\infty} \mathcal{F}_{\mathcal{M}x_n,1}(t) = 1$. Hence, $\mathcal{M}x_n \to 1$ as $n \to +\infty$. Similarly, $\mathcal{S}x_n \to 1$ as $n \to +\infty$;
2. $\mathcal{F}_{\mathcal{MS}x_n,\mathcal{SM}x_n}(t) = \mathcal{H}(t - 1 + 1/n)$, $\lim_{n\to+\infty} \mathcal{F}_{\mathcal{MS}x_n,\mathcal{SM}x_n}(t) \neq 1$ for all $t > 0$. Hence, $(\mathcal{M}, \mathcal{S})$ is non compatible. The coincidence points of $\mathcal{M}$ and $\mathcal{S}$ are $[1,2]$. Now, for all $x \in [1,2]$, $\mathcal{M}x = \mathcal{S}x = 2$ and $\mathcal{MS}x = \mathcal{M}2 = 2 = \mathcal{S}2 = \mathcal{SM}x$.

Thus $\mathcal{M}$ and $\mathcal{ST}$ are weakly compatible but non compatible. After routine calculations, one can show that the condition (*ii*) of Lemma 2.22 is also satisfied. Thus, all the conditions of Theorem 3.1 hold and $x = 2$ is the unique common fixed point of $\mathcal{A}, \mathcal{B}, \mathcal{L}, \mathcal{M}, \mathcal{S}$ and $\mathcal{T}$.

In the last example, we consider a pair $(\mathcal{L}, \mathcal{AB})$ that is discontinuous at the common fixed point.

Example 3.4. Let $\mathcal{F}$, g and $\phi(x)$ as in the examples above and $\mathcal{X} = [0,3]$. Define self-maps $\mathcal{A}, \mathcal{B}, \mathcal{L}, \mathcal{M}, \mathcal{S}$ and $\mathcal{T}$ on $\mathcal{X}$ by:

$$\mathcal{A}x = \mathcal{B}x = \begin{cases} 0 & \text{if } x \in [0,1) \cup (1,2) \cup (2,3], \\ 1 & \text{if } x = 1, \\ 3 & \text{if } x = 2, \end{cases} \qquad \mathcal{S}x = \mathcal{T}x = \begin{cases} 1 & \text{if } x \in [0,2) \cup (2,3], \\ 2 & \text{if } x = 2, \\ 0 & \text{if } x = 3, \end{cases}$$

$$\mathcal{L}x = \begin{cases} 1 & \text{if } x \in [0,2) \cup (2,3], \\ 2 & \text{if } x = 2, \end{cases} \quad \mathcal{M}x = \begin{cases} 0 & \text{if } x \in [0,1) \cup (1,2) \cup (2,3], \\ 1 & \text{if } x \in \{1,2\}. \end{cases}$$

Now, if $\lim_{n\to+\infty} x_n = 1$, where $\{x_n\}$ is a sequence in $\mathcal{X}$ such that $\lim_{n\to+\infty} \mathcal{L}x_n = \lim_{n\to+\infty} \mathcal{AB}x_n = 1$ for some $1 \in \mathcal{X}$, then

$$\lim_{n\to+\infty} g(\mathcal{F}_{\mathcal{LAB}x_n,\mathcal{AB}z}(t)) = \lim_{n\to+\infty} g(\mathcal{F}_{\mathcal{LAB}x_n,\mathcal{L}z}(t)) = \lim_{n\to+\infty} g(\mathcal{F}_{\mathcal{ABL}x_n,\mathcal{AB}z}(t)) = g(\mathcal{H}(t)) = 0.$$

Hence, $(\mathcal{L}, \mathcal{AB})$ is reciprocally semi-compatible but neither $\mathcal{L}$ nor $\mathcal{AB}$ is continuous even at the common fixed point $(z = 1)$. Clearly, $(\mathcal{M}, \mathcal{ST})$ is weakly compatible. After routine calculations, one can show also that the condition (*ii*) of Lemma 2.22 is satisfied. Thus, all the conditions of Theorem 3.1 hold and $x = 1$ is the unique common fixed point of $\mathcal{A}, \mathcal{B}, \mathcal{L}, \mathcal{M}, \mathcal{S}$ and $\mathcal{T}$.

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