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On Carlsson type orthogonality and characterization of inner product spaces

Eder Kikianty[a,b,c]**, Sever S. Dragomir**[a,c]

[a] *School of Computational & Applied Mathematics, University of the Witwatersrand, Private Bag-3, Wits-2050, Johannesburg, South Africa*
[b] *National ICT Australia, The University of Melbourne, Parkville, VIC 3010, Australia.*
[c] *School of Engineering and Science, Victoria University, PO Box 14428, Melbourne City, VIC 8001, Australia*

Abstract. In an inner product space, two vectors are orthogonal if their inner product is zero. In a normed space, numerous notions of orthogonality have been introduced via equivalent propositions to the usual orthogonality, e.g. orthogonal vectors satisfy the Pythagorean law. In 2010, Kikianty and Dragomir [9] introduced the p-HH-norms ($1 \leq p < \infty$) on the Cartesian square of a normed space. Some notions of orthogonality have been introduced by utilizing the 2-HH-norm [10]. These notions of orthogonality are closely related to the classical Pythagorean orthogonality and Isosceles orthogonality. In this paper, a Carlsson type orthogonality in terms of the 2-HH-norm is considered, which generalizes the previous definitions. The main properties of this orthogonality are studied and some useful consequences are obtained. These consequences include characterizations of inner product space.

1. Introduction

In an inner product space $(\mathbf{X}, \langle \cdot, \cdot \rangle)$, a vector $x \in \mathbf{X}$ is said to be orthogonal to $y \in \mathbf{X}$ (denoted by $x \perp y$) if the inner product $\langle x, y \rangle$ is zero. In the general setting of normed spaces, numerous notions of orthogonality have been introduced via equivalent propositions to the usual orthogonality in inner product spaces, e.g. orthogonal vectors satisfy the Pythagorean law. For more results on other notions of orthogonality, their main properties, and the implications as well as equivalent statements amongst them, we refer to the survey papers by Alonso and Benitez [1, 2].

The following are the main properties of orthogonality in inner product spaces (we refer to the works by Alonso and Benitez [1], James [6], and Partington [12] for references). In the study of orthogonality in normed spaces, these properties are investigated to see how "close" the definition is to the usual orthogonality. Suppose that $(\mathbf{X}, \langle \cdot, \cdot \rangle)$ is an inner product space and $x, y, z \in \mathbf{X}$. Then,

1. if $x \perp x$, then $x = 0$ (*nondegeneracy*);
2. if $x \perp y$, then $\lambda x \perp \lambda y$ for all $\lambda \in \mathbb{R}$ (*simplification*);
3. if $(x_n), (y_n) \subset \mathbf{X}$ such that $x_n \perp y_n$ for every $n \in \mathbb{N}$, $x_n \to x$ and $y_n \to y$, then $x \perp y$ (*continuity*);
4. if $x \perp y$, then $\lambda x \perp \mu y$ for all $\lambda, \mu \in \mathbb{R}$ (*homogeneity*);
5. if $x \perp y$, then $y \perp x$ (*symmetry*);

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Communicated by Dragan S. Djordjević
Email addresses: eder.kikianty@research.vu.edu.au (Eder Kikianty[a,b,c]), sever.dragomir@vu.edu.au (Sever S. Dragomir[a,c])

6. if $x \perp y$ and $x \perp z$, then $x \perp (y+z)$ (*additivity*);
7. if $x \neq 0$, then there exists $\lambda \in \mathbb{R}$ such that $x \perp (\lambda x + y)$ (*existence*);
8. the above λ is unique (*uniqueness*).

Any pair of vectors in a normed space $(\mathbf{X}, \|\cdot\|)$ can be viewed as an element of the Cartesian square $\mathbf{X}^2$. The space $\mathbf{X}^2$ is again a normed space, when it is equipped with any of the well known p-norms. In 2008, Kikianty and Dragomir [9] introduced the p-HH-norms ($1 \leq p < \infty$) on $\mathbf{X}^2$ as follows:

$$\|(x,y)\|_{p-HH} := \left(\int_0^1 \|(1-t)x + ty\|^p dt \right)^{\frac{1}{p}},$$

for any $(x, y) \in \mathbf{X}$. These norms are equivalent to the p-norms. However, unlike the p-norms, they do not depend only on the norms of the two elements in the pair, but also reflect the relative position of the two elements within the original space $\mathbf{X}$.

Some new notions of orthogonality have been introduced by using the 2-HH-norm [10]. These notions of orthogonality are closely related to the Pythagorean and Isosceles orthogonalities (cf. James [6]). The results are summarized as follows.

Let $(\mathbf{X}, \|\cdot\|)$ be a normed space.

1. A vector $x \in \mathbf{X}$ is *HH-P-orthogonal* to $y \in \mathbf{X}$ (denoted by $x \perp_{HH-P} y$) iff

$$\int_0^1 \|(1-t)x + ty\|^2 \, dt = \frac{1}{3}(\|x\|^2 + \|y\|^2); \tag{1.1}$$

2. A vector $x \in \mathbf{X}$ is *HH-I-orthogonal* to $y \in \mathbf{X}$ (denoted by $x \perp_{HH-I} y$) iff

$$\int_0^1 \|(1-t)x + ty\|^2 dt = \int_0^1 \|(1-t)x - ty\|^2 dt; \tag{1.2}$$

3. The homogeneity (or additivity) of the HH-P-(and HH-I-) orthogonality characterizes inner product space.

The Pythagorean and Isosceles orthogonalities have been generalized by Carlsson in 1962 [4]. In a normed space, x is said to be *C-orthogonal* to y (denoted by $x \perp y$ (C)) if and only if

$$\sum_{i=1}^{m} \alpha_i \|\beta_i x + \gamma_i y\|^2 = 0,$$

where α_i, β_i, γ_i are real numbers such that

$$\sum_{i=1}^{m} \alpha_i \beta_i^2 = \sum_{i=1}^{m} \alpha_i \gamma_i^2 = 0 \quad \text{and} \quad \sum_{i=1}^{m} \alpha_i \beta_i \gamma_i = 1$$

for some $m \in \mathbb{N}$. Carlsson's orthogonality satisfies the following properties (cf. Alonso and Benitez, [1]; and Carlsson [4]):

1. C-orthogonality satisfies nondegeneracy, simplification, and continuity;
2. C-orthogonality is symmetric in some cases (e.g. Pythagorean and Isosceles orthogonalities) and not symmetric in other cases (e.g., $x \perp y$ (C) when $\|x + 2y\| = \|x - 2y\|$);
3. C-orthogonality is either homogeneous or additive to the left iff the underlying normed space is an inner product space;
4. C-orthogonality is existent to the right and to the left;
5. with regards to uniqueness, C-orthogonality is non-unique when the space is non-rotund; in particular, P-orthogonality is unique, and I-orthogonality is unique iff the underlying normed space is rotund.

In this paper, we consider a notion of Carlsson's orthogonality in HH-sense (which will be called HH-C-orthogonality), which also generalizes HH-I- and HH-P-orthogonalities. We discuss its main properties in Section 3. Some characterizations of inner product spaces are provided in Section 3. Our approach follows that of Carlsson's [4], considering a condition which is weaker than homogeneity and additivity of the orthogonality. It will be shown that this condition implies that the norm is induced by an inner product. Consequently, the homogeneity (and additivity) of this orthogonality characterizes inner product spaces.

2. HH-C-orthogonality

Motivated by the relation between P-orthogonality and HH-P-orthogonality (also, those of I-orthogonality and HH-I-orthogonality) as stated in Section 1, we consider a Carlsson type orthogonality in terms of the 2-HH-norm. Let x and y be two vectors in $\mathbf{X}$ and $t \in [0,1]$. Suppose that $(1-t)x \perp ty$ (C), almost everywhere on $[0,1]$, i.e.

$$\sum_{i=1}^{m} \alpha_i \|(1-t)\beta_i x + t\gamma_i y\|^2 = 0,$$

for some $m \in \mathbb{N}$ and real numbers α_i, β_i, γ_i such that

$$\sum_{i=1}^{m} \alpha_i \beta_i^2 = \sum_{i=1}^{m} \alpha_i \gamma_i^2 = 0 \quad \text{and} \quad \sum_{i=1}^{m} \alpha_i \beta_i \gamma_i = 1. \tag{2.1}$$

Then,

$$\sum_{i=1}^{m} \alpha_i \int_0^1 \|(1-t)\beta_i x + t\gamma_i y\|^2 \, dt = 0. \tag{2.2}$$

Definition 2.1. In a normed space $(\mathbf{X}, \|\cdot\|)$, $x \in \mathbf{X}$ is said to be *HH-C-orthogonal* to $y \in \mathbf{X}$ (we denote it by $x \perp_{HH-C} y$) iff x and y satisfy (2.2), with the conditions (2.1).

It can be shown that the HH-C-orthogonality is equivalent to the usual orthogonality in any inner product space. The proof is omitted.

HH-P-orthogonality is a particular case of HH-C-orthogonality, which is obtained by choosing $m = 3$, $\alpha_1 = -1$, $\alpha_2 = \alpha_3 = 1$, $\beta_1 = \beta_2 = 1$, $\beta_3 = 0$, $\gamma_1 = \gamma_3 = 1$, and $\gamma_2 = 0$. Similarly, HH-I-orthogonality is also a particular case of HH-C-orthogonality, which is obtained by choosing $m = 2$, $\alpha_1 = \frac{1}{2}$, $\alpha_2 = -\frac{1}{2}$, $\beta_1 = \beta_2 = 1$, $\gamma_1 = 1$, and $\gamma_2 = -1$.

We now discuss the main properties of HH-C-orthogonality. The following proposition follows by the definition of HH-C-orthogonality; and we omit the proof.

Proposition 2.2. *HH-C-orthogonality satisfies the nondegeneracy, simplification, and continuity.*

With regards to symmetry, HH-C-orthogonality is symmetric in some cases, for example, HH-P- and HH-I-orthogonalities are symmetric [10]. The following provides an example of a nonsymmetric HH-C-orthogonality.

Example 2.3. HH-C-orthogonality is not symmetric.

Proof. Define $x \perp_{HH-C_2} y$ to be

$$\int_0^1 \|(1-t)x + 2ty\|^2 \, dt = \int_0^1 \|(1-t)x - 2ty\|^2 \, dt.$$

In $\mathbb{R}^2$ with ℓ^1-norm, $x = (2,1)$ is HH-C$_2$-orthogonal to $y = (\frac{1}{2}, -1)$ but $y \not\perp_{HH-C_2} x$. $\square$

Therefore, it is important to distinguish the existence (as well as additivity) to the left and to the right.

Since HH-P- and HH-I-orthogonalities are neither additive nor homogeneous [10], we conclude that HH-C-orthogonality is neither additive nor homogeneous. We will discuss these properties further in Section 3 with regards to some characterizations of inner product spaces.

The following lemma is due to Carlsson [4, p. 299]; and it will be used in proving the existence of HH-C-orthogonality.

Lemma 2.4. ([4]) *Let $x, y \in \mathbf{X}$. Then,*

$$\lim_{\lambda\to\pm\infty} \lambda^{-1}\left[\|(\lambda+a)x+y\|^2 - \|\lambda x+y\|^2\right] = 2a\|x\|^2.$$

Theorem 2.5. *Let $(\mathbf{X}, \|\cdot\|)$ be a normed space. Then, HH-C-orthogonality is existent.*

Proof. The proof follows a similar idea to that of Carlsson [4, p. 301]. We only prove for the existence to the right, since the other case follows analogously. Let g be a function on $\mathbb{R}$ defined by

$$g(\lambda) := \sum_{i=1}^{m} \alpha_i \int_0^1 \|(1-t)\beta_i x + t\gamma_i(\lambda x+y)\|^2\,dt,$$

where α_i, β_i, and γ_i are real numbers that satisfy (2.1). Note that our domain of integration is on $(0,1)$ (we exclude the extremities) to ensure that we can employ Lemma 2.4. Therefore, for any $\lambda \neq 0$,

$$\begin{aligned} \lambda^{-1}g(\lambda) &= \lambda^{-1}\sum_{i=1}^{m} \alpha_i \int_0^1 \|(1-t)\beta_i x + t\gamma_i(\lambda x+y)\|^2\,dt \\ &= \lambda^{-1}\sum_{i=1}^{m} \alpha_i \int_0^1 \left[\|(1-t)\beta_i x + t\gamma_i(\lambda x+y)\|^2 - \|t\gamma_i(\lambda x+y)\|^2\right]dt. \end{aligned} \tag{2.3}$$

Note the use of $\sum_{i=1}^{m} \alpha_i\gamma_i^2 = 0$. Therefore, (2.3) becomes

$$\begin{aligned} \lambda^{-1}g(\lambda) &= \lambda^{-1}\left[\sum_{\gamma_i\neq 0} \alpha_i \int_0^1 \|[t\lambda + (1-t)\beta_i\gamma_i^{-1}]\gamma_i x + \gamma_i t y\|^2 - \|t\lambda\gamma_i x + t\gamma_i y\|^2\,dt + \sum_{\gamma_i=0} \alpha_i \int_0^1 \|(1-t)\beta_i x\|^2\,dt\right] \\ &= \lambda^{-1}\left[\sum_{\gamma_i\neq 0} \alpha_i \int_0^1 \|[t\lambda + (1-t)\beta_i\gamma_i^{-1}]\gamma_i x + \gamma_i t y\|^2 - \|t\lambda\gamma_i x + t\gamma_i y\|^2\,dt + \frac{1}{3}\sum_{\gamma_i=0} \alpha_i\beta_i^2\|x\|^2\right]. \end{aligned}$$

Note that

$$\lim_{\lambda\to\pm\infty} \frac{1}{3}\lambda^{-1}\sum_{\gamma_i=0} \alpha_i\beta_i^2\|x\|^2 = 0.$$

By using Lemma 2.4, we obtain

$$\lim_{\lambda\to\pm\infty} \lambda^{-1}g(\lambda) = \sum_{\gamma_i\neq 0} 2\alpha_i \int_0^1 t(1-t)\beta_i\gamma_i^{-1}\,\|\gamma_i x\|^2\,dt = \frac{1}{3}\sum_{\gamma_i\neq 0} \alpha_i\beta_i\gamma_i\|x\|^2 = \frac{1}{3}\|x\|^2,$$

since $\sum_{i=1}^{m} \alpha_i\beta_i\gamma_i = 1$. It follows that $g(\lambda)$ is positive for sufficiently large positive number λ, and negative for sufficiently large negative number λ. By the continuity of g, we conclude that there exists an λ_0 such that $g(\lambda_0) = 0$, as required. $\square$

3. Characterization of inner product spaces

The main result of this section is a characterization of inner product spaces via the homogeneity (or additivity to the left) of HH-C-orthogonality.

Theorem 3.1. *Let* $(\mathbf{X}, \|\cdot\|)$ *be a normed space in which HH-C-orthogonality is homogeneous (or additive to the left). Then,* $\mathbf{X}$ *is an inner product space.*

Our approach follows that of Carlsson [4]. The proof of this theorem is described in this section in two separate cases: the case for normed spaces of dimension 3 and higher, and the 2-dimensional case. In both cases, we consider a property introduced by Carlsson [4, p. 301], which is weaker than homogeneity and additivity of the orthogonality. The following is a 'modified' definition of the property.

Definition 3.2. HH-C-orthogonality is said to have property (H) in a normed space $\mathbf{X}$, if $x \perp_{HH-C} y$ implies that

$$\lim_{n\to\infty} n^{-1} \int_0^1 \sum_{i=1}^{m} \alpha_i \|n\beta_i(1-t)x + \gamma_i t y\|^2 \, dt = 0. \tag{3.1}$$

It is obvious that

1. If HH-C-orthogonality is homogeneous (or additive to the left) in $\mathbf{X}$, then it has property (H);
2. If $\mathbf{X}$ is an inner product space, then HH-C-orthogonality is homogeneous (or additive) and therefore, it has property (H).

Thus, in order to prove Theorem 3.1, it is sufficient to show that if the HH-C-orthogonality has property (H) in $\mathbf{X}$, then $\mathbf{X}$ is an inner product space.

3.1. The case of dimension 3 and higher

Before stating the proof, recall that two vectors x, y in a normed space $(\mathbf{X}, \|\cdot\|)$ is said to be orthogonal in the sense of Birkhoff (B-orthogonal) if and only if $\|x\| \le \|x + \lambda y\|$ for any $\lambda \in \mathbb{R}$. Birkhoff's orthogonality has a close connection to the smoothness of the given normed space. Let us recall the definition of smoothness. In any normed space $\mathbf{X}$, the *Gâteaux lateral derivatives* of the norm $\|\cdot\|$ at a point $x \in \mathbf{X} \setminus \{0\}$, i.e. the following limits

$$\tau_+(x, y) := \lim_{t\to 0^+} \frac{\|x + ty\| - \|x\|}{t} \text{ and } \tau_-(x, y) := \lim_{t\to^-} \frac{\|x + ty\| - \|x\|}{t}$$

exist for all $y \in \mathbf{X}$ [11, p. 483–485]. The norm $\|\cdot\|$ is *Gâteaux differentiable* at $x \in \mathbf{X} \setminus \{0\}$ if and only if $\tau_+(x, y) = \tau_-(x, y)$, for all $y \in \mathbf{X}$. A normed linear space $(\mathbf{X}, \|\cdot\|)$ is said to be *smooth* if and only if the norm $\|\cdot\|$ is Gâteaux differentiable on $\mathbf{X} \setminus \{0\}$.

Lemma 3.3. ([8]) *Let* $(\mathbf{X}, \|\cdot\|)$ *be a normed space. If the norm* $\|\cdot\|$ *is Gâteaux differentiable, then* x *is B-orthogonal to* y *iff* $\tau(x, y) = 0$.

In general, B-orthogonality is homogeneous. However, it is not always symmetric. In normed spaces of dimension 3 and higher, the symmetry of B-orthogonality characterize inner product spaces [1, p. 5]. Therefore, in proving Theorem 3.1, it is sufficient to show that the property (H) of HH-C-orthogonality implies that this orthogonality is symmetric and equivalent to Birkhoff orthogonality in normed spaces of dimension 3 and higher.

The following propositions will be used in proving the theorem (see Lemma 2.6. and Lemma 2.7. of Carlsson [4] for the proof).

Proposition 3.4. ([4]) *Let* $(\mathbf{X}, \|\cdot\|)$ *be a normed space. Recall the following notation*

$$\tau_{\pm}(x, y) := \lim_{t\to 0^{+(-)}} \frac{\|x + ty\| - \|x\|}{t}$$

i.e. the right-(left-)Gâteaux differentiable at $x \in \mathbf{X} \setminus \{0\}$. *For* $\lambda\mu > 0$ *we have*

$$\tau_+(\lambda x, \mu y) = |\mu|\tau_+(x, y) \quad \text{and} \quad \tau_-(\lambda x, \mu y) = |\mu|\tau_-(x, y);$$

and for $\lambda\mu < 0$

$$\tau_+(\lambda x, \mu y) = -|\mu|\tau_-(x, y) \quad \text{and} \quad \tau_-(\lambda x, \mu y) = -|\mu|\tau_+(x, y).$$

Proposition 3.5. ([4]) *If* $(\mathbf{X}, \|\cdot\|)$ *is a normed linear space and there exist two real numbers* λ *and* μ *with* $\lambda + \mu \neq 0$, *such that* $\lambda\tau_+(x, y) + \mu\tau_-(x, y)$ *is a continuous function of* $x, y \in \mathbf{X}$, *then the norm* $\|\cdot\|$ *is Gâteaux differentiable.*

We will start with the following lemma, which also gives us the uniqueness of the HH-C-orthogonality.

Lemma 3.6. *Let* $(\mathbf{X}, \|\cdot\|)$ *be a normed space where HH-C-orthogonality has property (H). Suppose that for any* $x, y \in \mathbf{X}$, *there exists* $\lambda \in \mathbb{R}$ *such that* $x \perp_{HH-C} (\lambda x + y)$. *Then*

$$\lambda = -\|x\|^{-1}\left[\sum_{\beta_i\gamma_i>0} \alpha_i\beta_i\gamma_i\tau_+(x, y) + \sum_{\beta_i\gamma_i<0} \alpha_i\beta_i\gamma_i\tau_-(x, y)\right].$$

Proof. By assumption, we have

$$\lim_{n\to\infty} n^{-1} \int_0^1 \sum_{i=1}^m \alpha_i\|n\beta_i(1-t)x + \gamma_i t(\lambda x + y)\|^2\, dt = 0. \tag{3.2}$$

Note that by Lemma 2.4, we have the following for any i, and $t \in (0, 1)$ (again, note that we exclude the extremities to ensure that we can employ Lemma 2.4)

$$n^{-1}\|[n\beta_i(1-t) + \gamma_i t\lambda]x + \gamma_i ty\|^2 = n^{-1}\|n\beta_i(1-t)x + \gamma_i ty\|^2 + 2\beta_i(1-t)\gamma_i t\lambda\|x\|^2 + \varepsilon_i(n), \tag{3.3}$$

where $\varepsilon_i(n) \to 0$ when $n \to 0$. Now, we multiply (3.3) by α_i and integrate it over $(0, 1)$, to get

$$\begin{aligned} & n^{-1}\alpha_i \int_0^1 \|[n\beta_i(1-t) + \gamma_i t\lambda]x + \gamma_i ty\|^2\, dt \qquad (3.4)\\ = \;& n^{-1}\alpha_i \int_0^1 \|n\beta_i(1-t)x + \gamma_i ty\|^2\, dt + 2\alpha_i\beta_i\gamma_i\lambda\|x\|^2 \int_0^1 (1-t)t\, dt + \varepsilon_i(n). \end{aligned}$$

Take the sum and let $n \to \infty$ to get

$$0 = \lim_{n\to\infty} n^{-1} \sum_{i=1}^m \alpha_i \int_0^1 \|n\beta_i(1-t)x + \gamma_i ty\|^2 dt + \frac{1}{3}\lambda\|x\|^2 \tag{3.5}$$

(note the use of (3.4) and $\sum_{i=1}^m \alpha_i\beta_i\gamma_i = 1$). Now, note that $\sum_{i=1}^m \alpha_i\beta_i^2 = 0$, and therefore

$$\begin{aligned} & n^{-1}\sum_{i=1}^m \alpha_i \int_0^1 \|n\beta_i(1-t)x + \gamma_i ty\|^2 dt \\ = \;& n^{-1}\sum_{i=1}^m \alpha_i \int_0^1 [\|n\beta_i(1-t)x + \gamma_i ty\|^2 - \|n\beta_i(1-t)x\|^2] dt \\ = \;& \sum_{i=1}^m \alpha_i \int_0^1 [\|n\beta_i(1-t)x + \gamma_i ty\| - \|n\beta_i(1-t)x\|] n^{-1}[\|n\beta_i(1-t)x + \gamma_i ty\| + \|n\beta_i(1-t)x\|] dt. \end{aligned}$$

Rewrite $\|n\beta_i(1-t)x+\gamma_i ty\|-\|n\beta_i(1-t)x\|$ as $n\left(\|\beta_i(1-t)x+\frac{1}{n}\gamma_i ty\|-\|\beta_i(1-t)x\|\right)$, to obtain

$$\begin{aligned}\lim_{n\to\infty} n\left(\|\beta_i(1-t)x+\tfrac{1}{n}\gamma_i ty\|-\|\beta_i(1-t)x\|\right) &= \lim_{s\to 0^+}\frac{\|\beta_i(1-t)x+s\gamma_i ty\|-\|\beta_i(1-t)x\|}{s}\\ &= \tau_+(\beta_i(1-t)x,\gamma_i ty).\end{aligned}$$

Note also that

$$\begin{aligned}\lim_{n\to\infty} n^{-1}[\|n\beta_i(1-t)x+\gamma_i ty\|+\|n\beta_i(1-t)x\|] &= \lim_{n\to\infty}[\|\beta_i(1-t)x+n^{-1}\gamma_i ty\|+\|\beta_i(1-t)x\|]\\ &= 2\|\beta_i(1-t)x\|.\end{aligned}$$

Thus,

$$\lim_{n\to\infty} n^{-1}\sum_{i=1}^m \alpha_i\int_0^1 \|n\beta_i(1-t)x+\gamma_i ty\|^2 dt = 2\sum_{i=1}^m \alpha_i\int_0^1 \tau_+(\beta_i(1-t)x,\gamma_i ty)\|\beta_i(1-t)x\|dt.$$

Therefore,

$$\begin{aligned}\lambda &= -3\|x\|^{-2}\sum_{i=1}^m \alpha_i\int_0^1 \tau_+(\beta_i(1-t)x,\gamma_i ty)2\|\beta_i(1-t)x\|\,dt\\ &= -6\|x\|^{-1}\sum_{i=1}^m \alpha_i|\beta_i|\int_0^1 (1-t)\tau_+(\beta_i(1-t)x,\gamma_i ty)\,dt.\end{aligned}$$

By Proposition 3.4, (3.5) gives us

$$\begin{aligned}\lambda &= -6\|x\|^{-1}\left[\sum_{\beta_i\gamma_i>0}\alpha_i\beta_i\gamma_i\tau_+(x,y)+\sum_{\beta_i\gamma_i<0}\alpha_i\beta_i\gamma_i\tau_-(x,y)\right]\int_0^1(1-t)t\,dt\\ &= -\|x\|^{-1}\left[\sum_{\beta_i\gamma_i>0}\alpha_i\beta_i\gamma_i\tau_+(x,y)+\sum_{\beta_i\gamma_i<0}\alpha_i\beta_i\gamma_i\tau_-(x,y)\right],\end{aligned}$$

and the proof is completed. □

Now, we have a unique λ for any $x,y\in\mathbf{X}$ such that $x\perp_{HH-C}\lambda x+y$. As a function of x and y, $\lambda=\lambda(x,y)$ is a continuous function [4, p. 303]. Thus,

$$\sum_{\beta_i\gamma_i>0}\alpha_i\beta_i\gamma_i\tau_+(x,y)+\sum_{\beta_i\gamma_i<0}\alpha_i\beta_i\gamma_i\tau_-(x,y)$$

is also a continuous function in $x,y\in\mathbf{X}$. By Proposition 3.5, the norm $\|\cdot\|$ is Gâteaux differentiable. Together with Lemma 3.3, we have the following consequence.

Corollary 3.7. *If HH-C-orthogonality has property (H), then the norm of $\mathbf{X}$ is Gâteaux differentiable and $x\perp_{HH-C} y$ holds if and only if $\tau(x,y)=0$, i.e. $x\perp y$ (B).*

Remark 3.8. We note that the function $\tau(\cdot,\cdot)$ is also continuous as a function of x and y.

Let us assume that x is HH-C-anti-orthogonal to y if and only if $y\perp_{HH-C} x$. We have shown that when HH-C-orthogonality has property (H), then it is equivalent to B-orthogonality and therefore is homogeneous, since B-orthogonality is homogeneous (in any case). This fact implies that HH-C-anti-orthogonality has property (H) as well. Therefore, the above results also hold for HH-C-anti-orthogonality. In particular, $\tau(x,y)=0$ implies that $x\perp_{HH-C} y$, i.e., y is HH-C-anti-orthogonal to x; hence $\tau(y,x)=0$. Thus, B-orthogonality is symmetric, and we obtain the following consequence.

Corollary 3.9. *If HH-C-orthogonality has property (H), then it is symmetric and equivalent to B-orthogonality.*

3.2. The 2-dimensional case

Previously, we have defined that $x \perp_{HH-C} y$, when x and y satisfy the following:

$$\sum_{i=1}^{m} \alpha_i \int_0^1 \|(1-t)\beta_i x + t\gamma_i y\|^2 \, dt = 0,$$

where

$$\sum_{i=1}^{m} \alpha_i \beta_i^2 = \sum_{i=1}^{m} \alpha_i \gamma_i^2 = 0 \quad \text{and} \quad \sum_{i=1}^{m} \alpha_i \beta_i \gamma_i = 1.$$

In this subsection, we use a slightly different notation, in order to resolve the 2-dimensional problem. Note that

$$\sum_{i=1}^{m} \alpha_i \int_0^1 \|(1-t)\beta_i x + t\gamma_i y\|^2 \, dt \tag{3.6}$$

$$= \sum_{\beta_i \neq 0, \gamma_i \neq 0} \alpha_i \beta_i^2 \int_0^1 \left\|(1-t)x + t\frac{\gamma_i}{\beta_1} y\right\|^2 \, dt + \frac{1}{3} \sum_{\beta_i \neq 0, \gamma_i = 0} \alpha_i \beta_i^2 \|x\|^2 + \frac{1}{3} \sum_{\beta_i = 0, \gamma_i \neq 0} \alpha_i \gamma_i^2 \|y\|^2. \tag{3.7}$$

Since $\sum_{i=1}^{m} \alpha_i \beta_i^2 = \sum_{i=1}^{m} \alpha_i \gamma_i^2 = 0$, then we may rewrite (3.6) as

$$\sum_{i=1}^{m} \alpha_i \int_0^1 \|(1-t)\beta_i x + t\gamma_i y\|^2 \, dt = \sum_{\beta_i \neq 0, \gamma_i \neq 0} \alpha_i \beta_i^2 \int_0^1 \left\|(1-t)x + t\frac{\gamma_i}{\beta_1} y\right\|^2 \, dt - \frac{1}{3}\left(\sum_{\beta_i \neq 0, \gamma_i \neq 0} \alpha_i \beta_i^2 \|x\|^2 + \sum_{\beta_i \neq 0, \gamma_i \neq 0} \alpha_i \gamma_i^2 \|y\|^2 \right).$$

We set $p_i = \alpha_i \beta_i^2$ and $q_i = \gamma_i / \beta_i$, and rearrange the indices,

$$\sum_{i=1}^{m} \alpha_i \int_0^1 \|(1-t)\beta_i x + t\gamma_i y\|^2 \, dt \tag{3.8}$$

$$= \sum_{k=1}^{r} p_k \int_0^1 \left\|(1-t)x + tq_k y\right\|^2 \, dt - \frac{1}{3} \sum_{k=1}^{r} p_k \|x\|^2 - \frac{1}{3} \sum_{k=1}^{r} p_k q_k^2 \|y\|^2.$$

Assume that HH-C-orthogonality has property (H). Then, it is equivalent to B-orthogonality, and therefore is homogeneous. Denote $S_{\mathbf{X}}$ to be the unit circle of $\mathbf{X}$ and let $x, y \in S_{\mathbf{X}}$ such that $x \perp_{HH-C} y$. Then, (3.8) gives us

$$3 \sum_{k=1}^{r} p_k \int_0^1 \left\|(1-t)x + tq_k \alpha y\right\|^2 \, dt = C_1 + C_2 \alpha^2,$$

where $C_1 = \sum_{k=1}^{r} p_k$ and $C_2 = \sum_{k=1}^{r} p_k q_k^2$. We may conclude that the function

$$\phi(\alpha) = 3 \int_0^1 \left\|(1-t)x + t\alpha y\right\|^2 \, dt$$

is the solution of the functional equation

$$\sum_{k=1}^{r} p_k F(q_k \alpha) = C_1 + C_2 \alpha^2, \quad -\infty < \alpha < \infty, \tag{3.9}$$

where

$$\sum_{k=1}^{r} p_k = C_1, \; \sum_{k=1}^{r} p_k q_k^2 = C_2, \; \sum_{k=1}^{r} p_k q_k = 1, \; q_k \neq 0, \; k = 1, \ldots, r. \tag{3.10}$$

We note that the function ϕ is continuously differentiable from Corollary 3.7 and Remark 3.8.

In the following results by Carlsson [4], it is shown that the behaviour of ϕ for large and small values of $|\alpha|$ gives us an explicit formula for ϕ.

Definition 3.10. Given a functional equation

$$\sum_{k=1}^{r} p_k F(q_k\alpha) = C_1 + C_2\alpha^2, \quad -\infty < \alpha < \infty,$$

for some real numbers C_1, C_2, p_k and q_k. We say that the equation is symmetrical if it can be written in the form

$$\sum_{k=1}^{s} m_k F(n_k\alpha) - \sum_{k=1}^{s} m_k F(-n_k\alpha) = C_1 + C_2\alpha^2$$

for some real numbers C_1, C_2, m_k and n_k; otherwise, it is non-symmetrical.

Lemma 3.11. ([4]) *Let $\phi(\alpha)$ be a continuously differentiable solution of the functional equation (3.9) satisfying (3.10) and*

$$\begin{aligned} \phi(\alpha) &= 1 + O(\alpha^2) \quad \text{when } \alpha \to 0 \\ \phi(\alpha) &= \alpha^2 + O(\alpha) \quad \text{when } \alpha \to \pm\infty. \end{aligned}$$

If (3.9) is non-symmetrical, then $\phi(\alpha) = 1 + \alpha^2$ for $-\infty < \alpha < \infty$. If (3.9) is (non-trivially) symmetrical, then $\phi(\alpha) = \phi(-\alpha)$ for $-\infty < \alpha < \infty$.

A 2-dimensional normed space has certain properties that enable us to work on a smaller subset. One of the useful properties is stated in Lemma 3.12. Before stating the lemma, let us recall that the norm $\|\cdot\| : \mathbf{X} \to \mathbb{R}$ is said to be *Fréchet differentiable* at $x \in \mathbf{X}$ if and only if there exists a continuous linear functional φ'_x on $\mathbf{X}$ such that

$$\lim_{\|z\|\to 0} \frac{|\|x+z\| - \|x\| - \varphi'_x(z)|}{\|z\|} = 0.$$

It is said to be *twice (Fréchet) differentiable* at $x \in \mathbf{X}$ if and only if there exists a continuous bilinear functional φ''_x on $\mathbf{X}^2$ such that

$$\lim_{\|z\|\to 0} \frac{|\|x+z\| - \|x\| - \varphi'_x(z) - \varphi''_x(z,z)|}{\|z\|^2} = 0.$$

Lemma 3.12. ([3]) *If $(\mathbf{X}, \|\cdot\|)$ is a 2-dimensional normed space, then the norm is twice differentiable almost everywhere in the unit circle $S_{\mathbf{X}} = \{u \in \mathbf{X} : \|u\| = 1\}$.*

This result follows by the fact that the direction of the left-side tangent is a monotone function, and therefore, by Lebesgue's theorem, is differentiable almost everywhere [3, p. 22].

For us to prove Theorem 3.1 for the 2-dimensional normed spaces, we work on the assumption that the normed space has property (H). Thus, we only need to consider the unit vectors, as the results will hold for all vectors due to homogeneity. Furthermore, the previous proposition enables us to consider the vectors in a dense subset of the unit circle.

The following lemma will be employed in the proof of Theorem 3.1.

Lemma 3.13. ([2]) *Let $(\mathbf{X}, \|\cdot\|)$ be a normed space. If an existing orthogonality implies Roberts orthogonality, that is, $x \in \mathbf{X}$ is R-orthogonal to $y \in \mathbf{X}$ iff*

$$\|x + \lambda y\| = \|x - \lambda y\|, \text{ for all } \lambda \in \mathbb{R},$$

then $\mathbf{X}$ is an inner product space.

Proof of Theorem 3.1 for 2-dimensional case

Since HH-C-orthogonality has property (H), it is equivalent to B-orthogonality, and therefore is homogeneous. Since $\dim(\mathbf{X}) = 2$, the norm $\|\cdot\|$ is twice differentiable for almost every $u \in S_{\mathbf{X}}$. Let D be the subset of $S_{\mathbf{X}}$ consists of all points where the norm $\|\cdot\|$ is twice differentiable. Let $x \in D$ and $x \perp_{HH-C} y$ (or, equivalently $x \perp y\ (B)$) with $\|y\| = 1$. Then, the function

$$\phi(\alpha) = 3\int_0^1 \left\|(1-t)x + t\alpha y\right\|^2\, dt$$

is a continuously differentiable solution of the functional equation (3.9) satisfying (3.10).

Claim 3.14. The function ϕ satisfies

$$\begin{aligned} \phi(\alpha) &= 1 + O(\alpha^2) \quad \text{when } \alpha \to 0 \\ \phi(\alpha) &= \alpha^2 + O(\alpha) \quad \text{when } \alpha \to \pm\infty. \end{aligned}$$

The proof of claim will be stated in the end of this section as Lemmas 3.15 and 3.16.

Case 1: Equation (3.10) is non-symmetrical. It follows from Lemma 3.11 that $\phi(\alpha) = 1 + \alpha^2$. If we choose x and y as the unit vectors of a coordinate system in the plane $\mathbf{X}$ and write $w = \alpha x + \beta y$, we see that $\|w\| = 1$ iff $\alpha^2 + \beta^2 = 1$. This means that the unit circle has the equation $\alpha^2 + \beta^2 = 1$, i.e. an Euclidean circle. Therefore, $\mathbf{X}$ is an inner product space.

Case 2: Equation (3.10) is symmetrical. It follows from Lemma 3.11 that $\phi(\alpha) = \phi(-\alpha)$ for all α, i.e.

$$\int_0^1 \left\|(1-t)x + t\alpha y\right\|^2\, dt = \int_0^1 \left\|(1-t)x - t\alpha y\right\|^2\, dt \tag{3.11}$$

holds for any $\alpha \in \mathbb{R}$, $x, y \in \mathbf{X}$ where $x \in D$ and $x \perp_{HH-C} y$. Since D is a dense subset of C and HH-C-orthogonality is homogeneous, we conclude that (3.11) also holds for any $x \in \mathbf{X}$ where $x \perp_{HH-C} y$.

Let $t \in (0,1)$, and choose $\alpha = \frac{(1-t)}{t}\beta$, then (3.11) gives us

$$\int_0^1 \left\|(1-t)x + (1-t)\beta y\right\|^2\, dt = \int_0^1 \left\|(1-t)x - (1-t)\beta y\right\|^2\, dt,$$

or equivalently

$$\|x + \beta y\| = \|x - \beta y\|,$$

i.e. $x \perp y\ (R)$. We conclude that HH-C-orthogonality implies R-orthogonality. Since HH-C-orthogonality is existent, $\mathbf{X}$ is an inner product space.

The proof of claim is stated as the following lemmas:

Lemma 3.15. *Let $(\mathbf{X}, \|\cdot\|)$ be a 2-dimensional normed space, and denote its unit circle by $S_{\mathbf{X}}$. Let $u, v \in S_{\mathbf{X}}$. Then, the function*

$$\phi(\alpha) = 3\int_0^1 \|(1-t)u + t\alpha v\|^2\, dt$$

satisfies the condition

$$\phi(\alpha) = \alpha^2 + O(\alpha) \quad \textit{when } \alpha \to \pm\infty.$$

Proof. For any $u, v \in S_{\mathbf{X}}$ and $\alpha \in \mathbb{R}$, we have

$$\begin{aligned} |\phi(\alpha) - \alpha^2| &= \left| 3\left(\int_0^1 \|(1-t)u + t\alpha v\|^2 \, dt - \frac{1}{3}\alpha^2 \right) \right| \\ &\leq 3 \int_0^1 \left| \|(1-t)u + t\alpha v\|^2 - \|t\alpha v\|^2 \right| dt \\ &= 3 \int_0^1 \left| \|(1-t)u + t\alpha v\| - \|t\alpha v\| \right| \Big(\|(1-t)u + t\alpha v\| + \|t\alpha v\| \Big) \, dt \\ &\leq 3 \int_0^1 (1-t)\,\|u\| \Big((1-t)\|u\| + 2t\|\alpha v\| \Big) \, dt \\ &= 3 \int_0^1 ((1-t)^2 + 2t(1-t)|\alpha|) \, dt = 1 + |\alpha|. \end{aligned}$$

Thus, $\phi(\alpha) - \alpha^2 = O(\alpha)$, when $\alpha \to \pm\infty$. □

Lemma 3.16. *Let $(\mathbf{X}, \|\cdot\|)$ be a 2-dimensional normed space and denote its unit circle by $S_{\mathbf{X}}$. Then, there is a dense subset D of $S_{\mathbf{X}}$ such that if $u \in D$ and $u \perp v$ (B), the function*

$$\phi(\alpha) = 3 \int_0^1 \|(1-t)u + t\alpha v\|^2 \, dt$$

satisfies

$$\phi(\alpha) = 1 + O(\alpha^2) \quad \text{when } \alpha \to 0. \tag{3.12}$$

Proof. Since $\dim(\mathbf{X}) = 2$, then the norm $\|\cdot\|$ is twice differentiable for almost every $u \in S_{\mathbf{X}}$ by Lemma 3.12 [3, p. 22]. Let D be the subset of $S_{\mathbf{X}}$ consists of all points where the norm $\|\cdot\|$ is twice differentiable. We conclude that D is a dense subset of $S_{\mathbf{X}}$. Denote $\varphi(x) = \|x\|$, then for any $u \in D$, the derivative φ'_u is a linear functional and the second derivative φ''_u is a bilinear functional. Furthermore, we have the following

$$\lim_{\|z\|\to 0} \left| \frac{\|u+z\| - \|u\| - \varphi'_u(z) - \varphi''_u(z,z)}{\|z\|^2} \right| = 0. \tag{3.13}$$

Let $u \in D$ and $u \perp v$ (B), where $\|v\| = 1$. Set $z = \frac{t}{1-t}\alpha v$ $(t \in (0,1))$, therefore, when $\alpha \to 0$, $\|z\| \to 0$. Since $u \perp v$ (B), $\varphi'_u(v) = 0$, and (3.13) gives us

$$\lim_{\alpha\to 0} \left| \frac{\left\| u + \left(\frac{t}{1-t}\alpha\right) v \right\| - \|u\| - \left(\frac{t}{1-t}\alpha\right)^2 \varphi''_u(v,v)}{\left(\frac{t}{1-t}\alpha\right)^2 \|v\|^2} \right| = 0,$$

i.e., for any $\epsilon > 0$, there exists $\delta_0 > 0$, such that for any $|\alpha| < \delta_0$

$$\left| \frac{\left\| u + \left(\frac{t}{1-t}\alpha\right) v \right\| - 1}{\left(\frac{t}{1-t}\alpha\right)^2} - \varphi''_u(v,v) \right| < \epsilon.$$

Furthermore,

$$\left| \frac{\left\| u + \frac{t}{(1-t)}\alpha v \right\| - 1}{\frac{t^2}{(1-t)^2}\alpha^2} \right| < \epsilon + |\varphi''_u(v,v)| = M.$$

Equivalently, we have,

$$\left| \left\| u + \frac{t}{1-t}\alpha v \right\| - 1 \right| < M \frac{t^2}{(1-t)^2}\alpha^2.$$

Note that for any $t \in (0,1)$, $\left\| u + \frac{1-t}{t}\alpha v \right\| + 1 \to 2$ when $\alpha \to 0$. Set $\epsilon = 1$, then there exists δ_1 such that for any $|\alpha| < \delta_1$, we have

$$\left| \left\| u + \frac{t}{1-t}\alpha v \right\| + 1 - 2 \right| < 1,$$

i.e.,

$$\left\| u + \frac{t}{1-t}\alpha v \right\| + 1 < 1 + 2 = 3.$$

Now, for any $|\alpha| < \min\{\delta_0, \delta_1\}$, we have

$$\begin{aligned}
|\phi(\alpha) - 1| &= \left| 3\left(\int_0^1 \|(1-t)u + t\alpha v\|^2 \, dt - \frac{1}{3} \right) \right| \\
&\leq 3 \int_0^1 \left| \|(1-t)u + t\alpha v\|^2 - \|(1-t)u\|^2 \right| dt \\
&= 3 \int_0^1 (1-t)^2 \left| \left\| u + \frac{t}{1-t}\alpha v \right\|^2 - \|u\|^2 \right| dt \\
&= 3 \int_0^1 (1-t)^2 \left(\left\| u + \frac{t}{1-t}\alpha v \right\| + 1 \right) \left| \left\| u + \frac{t}{1-t}\alpha v \right\| - 1 \right| dt \\
&< 9 \int_0^1 (1-t)^2 M \frac{t^2}{(1-t)^2}\alpha^2 \, dt = 3M\alpha^2,
\end{aligned}$$

i.e., $\phi(\alpha) - 1 = O(\alpha^2)$, when $\alpha \to 0$. □

The last two results conclude that the homogeneity (also, the right-additivity) of HH-C-orthogonality is a necessary and sufficient condition for the normed space to be an inner product space.

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