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Statistical convergence of double sequences in fuzzy normed spaces

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Abstract. In this paper, we study the concepts of statistically convergent and statistically Cauchy double sequences in the framework of fuzzy normed spaces which provide better tool to study a more general class of sequences. We also introduce here statistical limit point and statistical cluster point for double sequences in this framework and discuss the relationship between them.

1. Introduction and preliminaries

By modifying own studies on fuzzy topological vector spaces, Katsaras [13] first introduced the notion of fuzzy seminorm and norm on a vector space and later on Felbin [7] gave the concept of a fuzzy normed space (for short, FNS) by applying the notion fuzzy distance of Kaleva and Seikala [11] on vector spaces. Further, Xiao and Zhu [29] improved a bit the Felbin's definition of fuzzy norm of a linear operator between FNSs. Recently, Bag and Samanta [2] has given another notion of boundedness in FNS and introduced another type of boundedness of operators. With the novelty of their approach they can introduce the fuzzy dual spaces and some important analogues of fundamental theorems in classical functional analysis [3].

In many branches of science and engineering we often come across double sequences, i.e. sequences of matrices and certainly there are situations where either the idea of ordinary convergence does not work or the underlying space does not serve our purpose. Therefore to deal with such situations we have to introduce some new type of measures which can provide a better tool and a suitable framework. In particular, we are interested to put forward our studies to deal with the sequences of chaotic behaviour.

The idea of statistical convergence was introduced by Fast [6] and Steinhaus [28] independently in the same year 1951 and since then several generalizations and application of this concept have been investigated by various authors, e.g. [9], [12], [20], [21], [22], [24] and [25]. Recently, fuzzy version of this concept were discussed in [15], [16], [18], [19] and [27].

In this paper we shall study the concept of convergence, statistical convergence and statistically Cauchy for double sequences in the framework of fuzzy normed spaces. Finally, Section 3 is devoted to introduce limit point, thin subsequence, non-thin subsequence, statistical limit point and statistical cluster point of double sequences in fuzzy normed spaces and find relations among these concepts.

Firstly, we recall some notations and basic definitions used in this paper.

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According to Mizumoto and Tanaka [14], a fuzzy number is a mapping $x : \mathbb{R} \to [0,1]$ over the set $\mathbb{R}$ of all real numbers. A fuzzy number x is convex if $x(t) \geq \min\{x(s), x(r)\}$ where $s \leq t \leq r$. If there exists a $t_0 \in \mathbb{R}$ such that $x(t_0) = 1$, then x is called normal. For $0 < \alpha \leq 1$, α-level set of an upper semi continuous convex normal fuzzy number η (denoted by $[\eta]_\alpha$) is a closed interval $[a_\alpha, b_\alpha]$, where $a_\alpha = -\infty$ and $b_\alpha = +\infty$ admissible. When $a_\alpha = -\infty$, for instance, then $[a_\alpha, b_\alpha]$ means the interval $(-\infty, b_\alpha]$. Similar is the case when $b_\alpha = +\infty$. A fuzzy number x is called non-negative if $x(t) = 0$, for all $t < 0$. We denoted the set of all convex, normal, upper semicontinuous fuzzy real numbers by $L(\mathbb{R})$ and the set of all non-negative, convex, normal, upper semicontinuous fuzzy real numbers by $L(\mathbb{R}^*)$. Given a number $r \in \mathbb{R}$, we define a corresponding fuzzy number $\tilde{r}$ by

$$\tilde{r}(t) = \begin{cases} 1 & \text{if } t = r, \\ 0 & \text{otherwise.} \end{cases}$$

As α-level sets of a convex fuzzy number is an interval, there is a debate in the nomenclature of fuzzy numbers/fuzzy real numbers. In [5], Dubois and Prade suggested to call this as fuzzy interval.

A partial ordering $\leq$ on $L(\mathbb{R})$ is defined by $u \leq v$ if and only if $u_\alpha^- \leq v_\alpha^-$ and $u_\alpha^+ \leq v_\alpha^+$ for all $\alpha \in [0,1]$, where $[u]_\alpha = [u_\alpha^-, u_\alpha^+]$ and $[v]_\alpha = [v_\alpha^-, v_\alpha^-]$. The strict inequality in $L(\mathbb{R})$ is defined by $u \prec v$ if and only if $u_\alpha^- < v_\alpha^-$ and $u_\alpha^+ < v_\alpha^+$ for all $\alpha \in [0,1]$. For $k > 0$, ku is defined as $ku(t) = u(t/k)$ and $(0u)(t)$ is defined to be $\tilde{0}(t)$.

According to Mizumoto and Tanaka [14], the arithmetic operations $\oplus, \ominus, \otimes$ on $L(\mathbb{R}) \times L(\mathbb{R})$ are defined by

$$\begin{aligned} (x \oplus y)(t) &= \sup_{s \in \mathbb{R}} \min\{x(s), y(t-s)\}, (x \ominus y)(t) = \sup_{s \in \mathbb{R}} \min\{x(s), y(s-t)\}, \text{ and} \\ (x \otimes y)(t) &= \sup_{s \in \mathbb{R}, s \neq 0} \min\{x(s), y(t/s)\}, \end{aligned}$$

for all $t \in \mathbb{R}$.

Let $u, v \in L(\mathbb{R})$. Define

$$D(u,v) = \sup_{\alpha \in [0,1]} \max\{|u_\alpha^- - v_\alpha^-|, |u_\alpha^+ - v_\alpha^+|\},$$

then D is called the supremum metric on $L(\mathbb{R})$. Let $(u_n) \subset L(\mathbb{R})$ and $u \in L(\mathbb{R})$. We say that a sequence (u_n) converges to u in the metric D (for short, D-converges to u), written as $u_n \xrightarrow{D} u$ or (D)-$\lim_{n \to \infty} u_n = u$ if $\lim_{n \to \infty} D(u_n, u) = 0$.

In [7] Felbin introduced the concept of fuzzy normed linear space by applying the notion fuzzy distance of Kaleva and Seikkala [11] on vector spaces. Recently, Şençimen and Pehlivan [27] gave a slightly simplified version of this FNS as follows.

Let X is a vector space over $\mathbb{R}$, $\|.\| : X \to L^*(\mathbb{R})$, mapping $L, R : [0,1] \times [0,1] \to [0,1]$ be symmetric, non-decreasing in both arguments and satisfy $L(0,0) = 0$ and $R(1,1) = 1$.

Write $[\|x\|]_\alpha = [\|x\|_\alpha^-, \|x\|_\alpha^+]$ for $x \in X$ and $0 \leq \alpha \leq 1$. Suppose that for all $x \in X$, $x \neq \theta$, $\inf_{\alpha \in [0,1]} \|x\|_\alpha^- > 0$, where θ is the zero vector of X.

The quadruple $(X, \|.\|)$ is said to be fuzzy normed space (for short FNS) if the following conditions are satisfied for every $x, y \in X$ and $s, t \in \mathbb{R}$:

(i) $\|x\| = \tilde{0}$ if and only if $x = \theta$,
(ii) $\|\alpha x\| = |\alpha| \|x\|$, $\alpha \in \mathbb{R}$,
(iii) $\|x + y\|(s + t) \geq L(\|x\|(s), \|y\|(t))$ whenever $s \leq \|x\|_1^-$, $t \leq \|y\|_1^-$ and $s + t \leq \|x + y\|_1^-$,
(iv) $\|x + y\|(s + t) \leq R(\|x\|(s), \|y\|(t))$ whenever $s \geq \|x\|_1^-$, $t \geq \|y\|_1^-$ and $s + t \geq \|x + y\|_1^-$,

In this case $\|.\|$ is called a fuzzy norm.

In the sequel we take $L(x, y) = \min(x, y)$ and $R(x, y) = \max(x, y)$ for all $x, y \in [0,1]$.

Example 1.1. Let $(X, ||.||_C)$ be a ordinary normed linear space. Then a fuzzy norm $||.||$ on Xcan be obtained as

$$||x||(t) = \begin{cases} 0 & \text{if } 0 \le t \le a||x||_C \text{ or } t \ge b||x||_C, \\ \frac{t}{(1-a)||x||_C} - \frac{a}{1-a} & \text{if } a||x||_C \le t \le ||x||_C, \\ \frac{-t}{(b-1)||x||_C} + \frac{b}{b-1} & \text{if } ||x||_C \le t \le b||x||_C, \end{cases} \tag{1.1.1}$$

where $||x||_C$ is the ordinary norm of $x(\neq \theta)$, $0 < a < 1$ and $1 < b < \infty$. For $x = \theta$, define $||x|| = \tilde{0}$. Hence $(X, ||.||)$ is a FNS. This fuzzy norm is called triangular fuzzy norm.
Let us consider the topological structure of a FNS $(X, ||.||)$. For any $\epsilon > 0$,$\alpha \in [0, 1]$ and $x \in X$, the (ϵ, α)-neighborhood of x is the set

$$\mathcal{N}_x(\epsilon, \alpha) := \{y \in X : ||x - y||^+_\alpha < \epsilon\}.$$

2. Statistically convergent and statistically Cauchy double sequences

Before proceeding further, we should recall some of the basic concepts on statistical convergence.

Let K be a subset of $\mathbb{N}$, the set of natural numbers. Then the *asymptotic density* of K, denoted by $\delta(K)$ (see [8],[28]), is defined as

$$\delta(K) = \lim_n \frac{1}{n}|\{k \le n : k \in K\}|,$$

where the vertical bars denote the cardinality of the enclosed set.

A number sequence $x = (x_k)$ is said to be *statistically convergent* to the number ℓ if for each $\epsilon > 0$, the set $K(\epsilon) = \{k \le n : |x_k - \ell| > \epsilon\}$ has asymptotic density zero, i.e.

$$\lim_n \frac{1}{n}|\{k \le n : |x_k - \ell| > \epsilon\}| = 0.$$

In this case we write $st\text{-}\lim x = \ell$.

Notice that every convergent sequence is statistically convergent to the same limit, but its converse need not be true.

A double sequence $x = (x_{jk})$ is said to be *Pringsheim's convergent* (or *P-convergent*) if for given $\epsilon > 0$ there exists an integer N such that $|x_{jk} - \ell| < \epsilon$ whenever $j, k > N$. We shall write this as

$$\lim_{j,k\to\infty} x_{jk} = \ell,$$

where j and k tending to infinity independent of each other (cf.[23]).

Let $K \subseteq \mathbb{N} \times \mathbb{N}$ be a two-dimensional set of positive integers and let $K(m, n)$ be the numbers of (j, k) in K such that $j \le m$ and $k \le n$. Then the two-dimensional analogue of natural density can be defined as follows [17].

The *lower asymptotic density* of the set $K \subseteq \mathbb{N} \times \mathbb{N}$ is defined as

$$\underline{\delta_2}(K) = \liminf_{m,n} \frac{K(m, n)}{mn}.$$

In case the sequence $(K(m, n)/mn)$ has a limit in Pringsheim's sense then we say that K has a *double natural density* and is defined as

$$\lim_{m,n} \frac{K(m, n)}{mn} = \delta_2(K).$$

For example, let $K = \{(i^2, j^2) : i, j \in \mathbb{N}\}$. Then

$$\delta_2(K) = \lim_{m,n} \frac{K(m, n)}{mn} \le \lim_{m,n} \frac{\sqrt{m}\sqrt{n}}{mn} = 0,$$

i.e. the set K has double natural density zero, while the set $\{(i, 2j) : i, j \in \mathbb{N}\}$ has double natural density 1/2.

Note that, if we set $m = n$, we have a two dimensional natural density due two Christopher [4].

A real double sequence $x = (x_{jk})$ is said to be *statistically convergent* [17] to the number ℓ if for each $\epsilon > 0$, the set

$$\{(j,k), j \leq m \text{ and } k \leq n :| x_{jk} - \ell |\geq \epsilon\}$$

has double natural density zero. In this case we write $st_2\text{-}\lim_{j,k} x_{jk} = \ell$.

Now we study the concept of convergence, statistical convergence and statistically Cauchy for double sequences in fuzzy normed spaces. We define the following:

Definition 2.1. Let $(X, \|.\|)$ be a FNS. Then a double sequence (x_{jk}) is said to be *convergent* to $x \in X$ with respect to the fuzzy norm on X if for every $\epsilon > 0$ there exists a number $N = N(\epsilon)$ such that

$$D(\|x_{jk} - x\|, \tilde{0}) < \epsilon \ \text{ for all } \ j,k \geq N.$$

In this case we write $x_{jk} \xrightarrow{FN} x$. This means that for every $\epsilon > 0$ there exists a number $N = N(\epsilon)$ such that

$$\sup_{\alpha\in[0,1]} \|x_{jk} - x\|_\alpha^+ = \|x_{jk} - x\|_0^+ < \epsilon$$

for all $j,k \geq N$. In terms of neighborhoods, we have $x_{jk} \xrightarrow{FN} x$ provided that for any $\epsilon > 0$ there exists a number $N = N(\epsilon)$ such that $x_{jk} \in \mathcal{N}_x(\epsilon, 0)$ whenever $j,k \geq N$.

Definition 2.2. Let $(X, \|.\|)$ be a FNS. We say that a double sequence (x_{jk}) is said to be *statistically convergent* to $x \in X$ with respect to the fuzzy norm on X if for every $\epsilon > 0$,

$$\delta_2(\{(j,k) \in \mathbb{N}\times\mathbb{N} : D(\|x_{jk} - x\|, \tilde{0}) \geq \epsilon\}) = 0.$$

This implies that for each $\epsilon > 0$, the set

$$K(\epsilon) := \{(j,k) \in \mathbb{N}\times\mathbb{N} : \|x_{jk} - x\|_0^+ \geq \epsilon\}$$

has natural density zero; namely, for each $\epsilon > 0$, $\|x_{jk} - x\|_0^+ < \epsilon$ for almost all j,k. In this case we write $st_2(FN)\text{-}\lim \|x_{jk} - x\| = \tilde{0}$ or $x_{jk} \xrightarrow{st_2(FN)} x$.

In terms of neighborhoods, we have $x_{jk} \xrightarrow{st_2(FN)} x$ if for every $\epsilon > 0$,

$$\delta_2(\{(j,k) \in \mathbb{N}\times\mathbb{N} : x_{jk} \notin \mathcal{N}_x(\epsilon, 0)\}) = 0,$$

i.e., for each $\epsilon > 0$, $(x_{jk}) \in \mathcal{N}_x(\epsilon, 0)$ for almost all j,k.

A useful interpretation of the above definition is the following:

$$x_{jk} \xrightarrow{st_2(FN)} x \ \text{ iff } \ st_2(FN)\text{-}\lim \|x_{jk} - x\|_0^+ = 0.$$

Note that $st_2(FN)\text{-}\lim \|x_{jk} - x\|_0^+ = 0$ implies that

$$st_2(FN)\text{-}\lim \|x_{jk} - x\|_\alpha^- = st_2(FN)\text{-}\lim \|x_{jk} - x\|_\alpha^+ = 0$$

for each $\alpha \in [0,1]$, since

$$0 \leq \|x_{jk} - x\|_\alpha^- \leq \|x_{jk} - x\|_\alpha^+ \leq \|x_{jk} - x\|_0^+$$

holds for every $j,k \in \mathbb{N}$ and for each $\alpha \in [0,1]$. Hence the result.

Remark 2.1. If a double sequence (x_{jk}) in a fuzzy normed space $(X, \|.\|)$ is convergent then it is also statistically convergent but converse need not be true, which can be seen by the following example.

Example 2.1. Let $(\mathbb{R}^m, \|.\|)$ be a FNS and $x = (x_{jk})_{j,k=1}^m \in \mathbb{R}^m$ be a fixed nonzero vector, where the fuzzy norm on $\mathbb{R}^m$ is defined as in (1.1.1) such that $\|x\|_C = \left(\sum_{n=1}^{m}\sum_{j=1}^{m}|x_{nj}|^2\right)^{1/2}$. Now we define a double sequence (x_{nj}) in $\mathbb{R}^m$ as

$$x_{nj} = \begin{cases} x\,; \text{ if } n = j = k^2, k \in \mathbb{N} \\ \theta\,; \text{ otherwise.} \end{cases}$$

Then we see that for any ϵ satisfying $0 < \epsilon \le b\|x\|_C$ where $1 < b < \infty$, we have

$$K(\epsilon) = \{(j,k) \in \mathbb{N} \times \mathbb{N} : \|x_{nj} - \theta\|_0^+ \ge \epsilon\} = \{(1,1),(4,4),(9,9),\cdots\}.$$

Hence $\delta_2(K(\epsilon)) = 0$. If we choose $\epsilon > b\|x\|_C$ then $K(\epsilon) = \emptyset$ and hence $\delta_2(\emptyset) = 0$, that is $(x_{nj}) \stackrel{st_2(FN)}{\longrightarrow} \theta$. However (x_{nj}) is not convergent in $(\mathbb{R}^m, \|.\|)$.

Definition 2.3. Let $(X, \|.\|)$ be a FNS. Then a double sequence (x_{jk}) is said to be *statistically Cauchy* with respect to the fuzzy norm on X if for every $\epsilon > 0$ there exist $N = N(\epsilon)$ and $M = M(\epsilon)$ such that for all $j, p \ge N$; $k, q \ge M$

$$\delta_2(\{(j,k) \in \mathbb{N} \times \mathbb{N}, j \le n \text{ and } k \le m : \|x_{jk} - x_{pq}\|_0^+ \ge \epsilon\}) = 0.$$

Theorem 2.1. *Let (x_{jk}) and (y_{jk}) be a double sequences in a FNS $(X, \|.\|)$ such that $x_{jk} \stackrel{st_2(FN)}{\longrightarrow} x$ and $y_{jk} \stackrel{st_2(FN)}{\longrightarrow} y$ for all $x, y \in X$. Then we have the following:*

(i) $(x_{jk} + y_{jk}) \stackrel{st_2(FN)}{\longrightarrow} x + y$,

(ii) $\alpha x_{jk} \stackrel{st_2(FN)}{\longrightarrow} \alpha x$, $\alpha \in \mathbb{R}$,

(iii) $st_2(FN)\text{-}\lim \|x_{jk}\| = \|x\|$.

Proof. (i) Suppose that $x_{jk} \stackrel{st_2(FN)}{\longrightarrow} x$ and $y_{jk} \stackrel{st_2(FN)}{\longrightarrow} x$. Since $\|.\|_0^+$ is a norm in the usual sense, we get

$$\|(x_{jk} + y_{jk}) - (x + y)\|_0^+ \le \|x_{jk} - x\|_0^+ + \|y_{jk} - y\|_0^+ \tag{2.1.1}$$

for all $j, k \in \mathbb{N}$. Write

$$K(\epsilon) := \{(j,k) \in \mathbb{N} \times \mathbb{N} : \|(x_{jk} + y_{jk}) - (x + y)\|_0^+ \ge \epsilon\},$$
$$K_1(\epsilon) := \{(j,k) \in \mathbb{N} \times \mathbb{N} : \|x_{jk} - x\|_0^+ \ge \epsilon/2\},$$
$$K_2(\epsilon) = \{(j,k) \in \mathbb{N} \times \mathbb{N} : \|y_{jk} - y\|_0^+ \ge \epsilon/2\}.$$

From (2.1.1) that $K(\epsilon) \subseteq K_1(\epsilon) \cup K_2(\epsilon)$. Now by assumption we have $\delta_2(K_1(\epsilon)) = \delta_2(K_2(\epsilon)) = 0$. This yields $\delta_2(K(\epsilon)) = 0$, i.e., (i) holds.

(ii) is obvious.

(iii) Since $\|.\|_\alpha^-$ and $\|.\|_\alpha^+$ are norms in the usual sense, we have

$$0 \le |\|x_{jk}\|_\alpha^- - \|x\|_\alpha^-| \le \|x_{jk} - x\|_\alpha^-$$

and

$$0 \le |\|x_{jk}\|_\alpha^+ - \|x\|_\alpha^+| \le \|x_{jk} - x\|_\alpha^+$$

for all $\alpha \in [0,1]$. Therefore

$$0 \le \max\{|\|x_{jk}\|_\alpha^- - \|x\|_\alpha^-|, |\|x_{jk}\|_\alpha^+ - \|x\|_\alpha^+|\} \le \|x_{jk} - x\|_\alpha^+$$

for all $\alpha \in [0,1]$. Taking supremum over $\alpha \in [0,1]$, we get

$$0 \le D(\|x_{jk}\|, \|x\|) \le \|x_{jk} - x\|_0^+.$$

Hence, we have $st_2(FN)$-$\|x_{jk}\| = \|x\|$ by Definition 5 in [26]. □

Theorem 2.2. *Let* $(X, \|.\|)$ *be a FNS. If a double sequence* (x_{jk}) *for which there is a double sequence* (y_{jk}) *that is convergent such that* $x_{jk} = y_{jk}$ *for almost all* j, k *then* (x_{jk}) *is a statistically convergent to* x *with respect to the fuzzy norm on* X.

Proof. Suppose that $x_{jk} = y_{jk}$ for almost all j, k and $y_{jk} \xrightarrow{FN} x$. Let $\epsilon > 0$. Then

$$\begin{aligned} &\{(j,k), j \leq m \text{ and } k \leq n : \|x_{jk} - x\|_0^+ \geq \epsilon\} \\ &\subseteq \{(j,k), j \leq m \text{ and } k \leq n : x_{jk} \neq y_{jk}\} \cup \{(j,k), j \leq m \text{ and } k \leq n : \|y_{jk} - y\|_0^+ > \epsilon\}, \end{aligned} \tag{2.2.1}$$

for each m, n. Since $y_{jk} \xrightarrow{FN} x$, the second set on the right hand side of (2.2.1) contains a finite number of elements, say $p = p(\epsilon)$. Therefore

$$\lim_{m,n\to\infty} \frac{1}{mn} |\{(j,k), j \leq m \text{ and } k \leq n : \|x_{jk} - x\|_0^+ \geq \epsilon\}| \leq \lim_{m,n\to\infty} \frac{1}{mn} \left|\{(j,k), j \leq m \text{ and } k \leq n : x_{jk} \neq y_{jk}\}\right| + \lim_{j,k\to\infty} \frac{p}{mn} = 0,$$

since $x_{jk} = y_{jk}$ for almost all j, k. Hence $\|x_{jk} - x\|_0^+ < \epsilon$ for almost all j, k. Hence (x_{jk}) is statistically convergent with respect to the fuzzy norm on X. □

Theorem 2.3. *Let* $(X, \|.\|)$ *be a FNS. Then every statistically convergent double sequence* (x_{jk}) *is statistically Cauchy with respect to the fuzzy norm on* X.

Proof is easy and hence omitted.

Theorem 2.4. *Let* (x_{jk}) *be a double sequence in FNS* $(X, \|.\|)$ *and denote* $E_N(\epsilon) := \{(j,k) \in \mathbb{N}\times\mathbb{N} : \|x_{jk} - x_{NM}\|_0^+ \geq \epsilon\}$. *If* (x_{jk}) *is statistically Cauchy, then for every* $\epsilon > 0$ *there exists* $A \subset \mathbb{N}\times\mathbb{N}$ *with* $\delta_2(A) = 0$ *such that* $\|x_{nm} - x_{jk}\|_0^+ < \epsilon$ *for all* $(n,m), (j,k) \notin A$.

Proof. For a given $\epsilon > 0$, write $A = E_N(\epsilon/2)$. Since (x_{jk}) is statistically Cauchy, we can write $\delta_2(A) = 0$. Then, for any $(n,m), (j,k) \notin A$, we have $\|x_{jk} - x_{NM}\|_0^+ < \epsilon/2$ and $\|x_{nm} - x_{NM}\|_0^+ < \epsilon/2$. Hence $\|x_{nm} - x_{jk}\|_0^+ < \epsilon$ for all $(n,m), (j,k) \notin A$. □

Definition 2.4. A fuzzy norm $|||.|||$ on a vector space X is called fuzzy equivalent to a fuzzy norm $\|.\|$, written as $|||.||| \sim \|.\|$, on X if there exist $\mu, \nu \in L(\mathbb{R})$ and $\mu, \nu \succ \tilde{0}$ such that for all $x \in X$,

$$\mu \otimes \|x\| \preceq |||x||| \preceq \nu \otimes \|x\|,$$

for all $x \in X$.

Theorem 2.5. *Let* X *be a vector space over* $\mathbb{R}$ *and let* $\|.\|$ *and* $|||.|||$ *be fuzzy equivalent fuzzy norms on* X. *Let* (x_{jk}) *be a double sequence in* X. *Then*

(i) (x_{jk}) *is statistically convergent to* x *in* $(X, \|.\|)$ *iff* (x_{jk}) *is statistically convergent to* x *in* $(X, |||.|||)$.
(ii) (x_{jk}) *is statistically Cauchy in* $(X, \|.\|)$ *iff* (x_{jk}) *is statistically Cauchy in* $(X, |||.|||)$.

Proof. (i) Let (x_{jk}) be statistically convergent to x in $(X, \|.\|)$. Since $\|.\|$ and $|||.|||$ are fuzzy equivalent, there exist $\mu, \nu \in L(\mathbb{R})$ and $\mu, \nu \succ \tilde{0}$ such that

$$\mu \otimes \|x_{jk} - x\| \preceq |||x_{jk} - x||| \preceq \nu \otimes \|x_{jk} - x\|$$

for all $(x_{jk}), x \in X$. Thus

$$\mu_0^+ \|x_{jk} - x\|_0^+ \leq |||x_{jk} - x|||_0^+ \leq \nu_0^+ \|x_{jk} - x\|_0^+.$$

By assumption, we have $st_2(FN)$-$\lim \|x_{jk} - x\|_0^+ = 0$. Hence $st_2(FN)$-$\lim |||x_{jk} - x|||_0^+ = 0$, i.e., $x_{jk} \xrightarrow{st_2(FN)} x$ in $(X, |||.|||)$. Similarly, if $x_{jk} \xrightarrow{st_2(FN)} x$ then $x_{jk} \xrightarrow{st_2(FN)} x$ in $(X, \|.\|)$.

(ii) Let (x_{jk}) be statistically Cauchy in $(X, \|.\|)$. Since $\|.\|$ and $|||.|||$ are fuzzy equivalent, there exist $\mu, \nu \in L(\mathbb{R})$ and $\mu, \nu > \tilde{0}$ such that

$$\mu_0^+ \|x\|_0^+ \leq |||x|||_0^+ \leq \nu_0^+ \|x\|_0^+$$

for all $x \in X$. For any $\epsilon > 0$, there exist $N = N(\epsilon)$ and $M = M(\epsilon)$ such that for all $j, p > N; k, q > M$

$$\|x_{jk} - x_{pq}\|_0^+ < \epsilon/\nu_0^+$$

for almost all j, k. Hence

$$|||x_{jk} - x_{pq}|||_0^+ \leq \nu_0^+ \|x_{jk} - x_{pq}\|_0^+ < \epsilon$$

for almost all j, k. Hence (x_{jk}) is statistically Cauchy in $(X, |||.|||)$. Similarly, if (x_{jk}) is statistically Cauchy in $(X, |||.|||)$ then it is statistically Cauchy in $(X, \|.\|)$. □

3. Statistical limit point and statistical cluster point for double sequences

Statistical limit point for single sequence (x_k) has been define and studied by Fridy [10]; and for fuzzy number by Aytar [1]. In this section, we define and study the notions of thin subsequence, non-thin subsequence, statistical limit point and statistical cluster point for double sequences with respect to the fuzzy normed spaces.

Definition 3.1. Let (x_{jk}) be a double sequence in FNS $(X, \|.\|)$. An element $x \in X$ is said to be *limit point* of the double sequence (x_{jk}) with respect to the fuzzy norm on X if there is subsequence of (x_{jk}) that converges to x with respect to the fuzzy norm on X. We denote by $L_{FN}(x_{jk})$, the set of all limit points of the double sequence (x_{jk}).

Definition 3.2. Let (x_{jk}) be a double sequence in FNS $(X, \|.\|)$ and $(x_{j_m k_m})$ be a subsequence of (x_{jk}). Write $K = \{(j_m, k_m) : j_1 < j_2 < \cdots ; k_1 < k_2 < \cdots\}$ subset of $\mathbb{N} \times \mathbb{N}$. If $\delta_2(K) = 0$ then we say that $(x_{j_m k_m})$ is *thin subsequence* of (x_{jk}). A subsequence $(x_{j_m k_m})$ is said to be *non-thin subsequence* provided that $\delta_2(k) > 0$ or $\delta_2(k)$ does not exist, namely, $\bar{\delta}_2(k) > 0$.

Definition 3.3. Let (x_{jk}) be a double sequence in FNS $(X, \|.\|)$. An element $x \in X$ is said to be *statistical limit point* of the double sequence (x_{jk}) provided that there exists a non-thin subsequence of (x_{jk}) that converges to x with respect to the fuzzy norm on X. By $\Lambda_{FN}(x_{jk})$, we denote the set of all statistical limit points of the double sequence (x_{jk}).

Definition 3.4. Let (x_{jk}) be a double sequence in FNS $(X, \|.\|)$. We say that an element $x \in X$ is said to be *statistical cluster point* of the double sequence (x_{jk}) with respect to the fuzzy norm on X provided that for every $\epsilon > 0$,

$$\bar{\delta}_2(\{(j,k) \in \mathbb{N} \times \mathbb{N} : \|x_{jk} - x\|_0^+ < \epsilon\}) > 0.$$

By $\Gamma_{FN}(x_{jk})$, we denote the set of all statistical limit points of the double sequence (x_{jk}).

Remark 3.1. An element $x \in \Gamma_{FN}(x_{jk})$ implies

$$\bar{\delta}_2(\{(j,k) \in \mathbb{N} \times \mathbb{N} : \|x_{jk} - x\|_\alpha^+ < \epsilon\}) > 0.$$

and

$$\bar{\delta}_2(\{(j,k) \in \mathbb{N} \times \mathbb{N} : \|x_{jk} - x\|_\alpha^- < \epsilon\}) > 0.$$

for all $\epsilon > 0$ and $\alpha \in [0, 1]$.

Theorem 3.1. *Let $(X, \|.\|)$ be a FNS. Then for every double sequence (x_{jk}) in X, we have*

$$\Lambda_{FN}(x_{jk}) \subseteq \Gamma_{FN}(x_{jk}) \subseteq L_{FN}(x_{jk}).$$

Proof. Let $x \in \Lambda_{FN}(x_{jk})$. Then there exists a non-thin subsequence $(x_{j_m k_m})$ of the double sequence (x_{jk}) that converges to x, namely, $\bar{\delta}_2(K) = d > 0$. Since

$$\{(j,k) \in \mathbb{N}\times\mathbb{N} : \|x_{jk} - x\|_0^+ < \epsilon\} \supseteq \{(j_m, k_m) \in \mathbb{N}\times\mathbb{N} : \|x_{j_m k_m} - x\|_0^+ < \epsilon\}$$

for every $\epsilon > 0$ and so

$$\{(j,k) \in \mathbb{N}\times\mathbb{N} : \|x_{jk} - x\|_0^+ < \epsilon\} \supseteq K \setminus \{(j_m, k_m) \in \mathbb{N}\times\mathbb{N} : \|x_{j_m k_m} - x\|_0^+ \geq \epsilon\}.$$

Since $(x_{j_m k_m}) \xrightarrow{FN} x$, the set $\{(j_m, k_m) \in \mathbb{N}\times\mathbb{N} : \|x_{j_m k_m} - x\|_0^+ \geq \epsilon\}$ is finite for any $\epsilon > 0$. Hence we have

$$\bar{\delta}_2(\{(j,k) \in \mathbb{N}\times\mathbb{N} : \|x_{jk} - x\|_0^+ < \epsilon\}) \geq \bar{\delta}_2(K) - \bar{\delta}_2(\{(j_m, k_m) \in \mathbb{N}\times\mathbb{N} : \|x_{j_m k_m} - x\|_0^+ \geq \epsilon\}) = d > 0.$$

Thus, for every $\epsilon > 0$

$$\bar{\delta}_2(\{(j,k) \in \mathbb{N}\times\mathbb{N} : \|x_{jk} - x\|_0^+ < \epsilon\}) > 0,$$

i.e., $x \in \Gamma_{FN}(x_{jk})$.

Let $x \in \Gamma_{FN}(x_{jk})$. For every $\epsilon > 0$, write

$$\bar{\delta}_2(\{(j,k) \in \mathbb{N}\times\mathbb{N} : \|x_{jk} - x\|_0^+ < \epsilon\}) > 0.$$

This means that there are infinitely many terms of the double sequence (x_{jk}) in every $(\epsilon, 0)$-neighborhood of x, i.e., $x \in L_{FN}(x_{jk})$. Hence the result. □

Theorem 3.2. *Let (x_{jk}) be a double sequence in a FNS $(X, \|.\|)$. Then $\Lambda_{FN}(x_{jk}) = \Gamma_{FN}(x_{jk}) = \{x\}$, provided $x_{jk} \xrightarrow{st_2(FN)} x$.*

Proof. Let $x_{jk} \xrightarrow{st_2(FN)} x$. Then $x \in \Gamma_{FN}(x_{jk})$. Now suppose that there exists atleast one $y \in \Gamma_{FN}(x_{jk})$ such that $y \neq x$. Thus there exists $\epsilon > 0$ such that

$$\{(j,k) \in \mathbb{N}\times\mathbb{N} : \|x_{jk} - x\|_0^+ \geq \epsilon\} \supseteq \{(j,k) \in \mathbb{N}\times\mathbb{N} : \|x_{jk} - y\|_0^+ < \epsilon\}$$

holds. Hence

$$\bar{\delta}_2(\{(j,k) \in \mathbb{N}\times\mathbb{N} : \|x_{jk} - x\|_0^+ \geq \epsilon\}) \geq \bar{\delta}_2(\{(j,k) \in \mathbb{N}\times\mathbb{N} : \|x_{jk} - y\|_0^+ < \epsilon\}).$$

Since $x_{jk} \xrightarrow{st_2(FN)} x$, we have $\delta_2(\{(j,k) \in \mathbb{N}\times\mathbb{N} : \|x_{jk} - x\|_0^+ \geq \epsilon\}) = 0$, which implies

$$\bar{\delta}_2(\{(j,k) \in \mathbb{N}\times\mathbb{N} : \|x_{jk} - x\|_0^+ \geq \epsilon\}) = 0.$$

Thus

$$\bar{\delta}_2(\{(j,k) \in \mathbb{N}\times\mathbb{N} : \|x_{jk} - y\|_0^+ < \epsilon\}) = 0,$$

which is a contradiction to $y \in \Gamma_{FN}(x_{jk})$. Therefore, $\Gamma_{FN}(x_{jk}) = \{x\}$.

On the other hand, since $x_{jk} \xrightarrow{st_2(FN)} x$. By Theorem 2.2 and Definition 3.3, we get $x \in \Lambda_{FN}(x_{jk})$. Hence by using Theorem 3.1, we get $\Lambda_{FN}(x_{jk}) = \Gamma_{FN}(x_{jk}) = \{x\}$. □

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