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# On the Power Graph of a Finite Group

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**Abstract.** The power graph $P(G)$ of a group $G$ is the graph whose vertex set is the group elements and two elements are adjacent if one is a power of the other. In this paper, we consider some graph theoretical properties of a power graph $P(G)$ that can be related to its group theoretical properties. As consequences of our results, simple proofs for some earlier results are presented.

## 1. Introduction

All groups and graphs in this paper are finite. Throughout the paper, we follow the terminology and notation of [11, 12] for groups and [18] for graphs.

Groups are the main mathematical tools for studying symmetries of an object and symmetries are usually related to graph automorphisms, when a graph is related to our object. Groups linked with graphs have been arguably the most famous and productive area of algebraic graph theory, see [1, 11] for details. The power graphs is a new representation of groups by graphs. These graphs were first used by Chakrabarty et al. [4] by using semigroups. It must be mentioned that the authors of [4] were motivated by some papers of Kelarev and Quinn [8–10] regarding digraphs constructed from semigroups. We also encourage interested readers to consult papers by Cameron and his co-workers on power graphs constructed from finite groups [2, 3].

Suppose $G$ is a finite group. The *power graph* $P(G)$ is a graph in which $V(P(G)) = G$ and two distinct elements $x$ and $y$ are adjacent if and only if one of them is a power of the other. If $G$ is a finite group then it can be easily seen that the power graph $P(G)$ is a connected graph of diameter 2. In [4], it is proved that for a finite group $G$, $P(G)$ is complete if and only if $G$ is a cyclic group of order 1 or $p^m$, for some prime number $p$ and positive integer $m$.

Following [12, 13], two finite groups $G$ and $H$ are said to be conformal if and only if they have the same number of elements of each order. In [13], the following question was investigated:

**Question:** *For which natural numbers n are any two conformal groups of order n isomorphic?*

Let $G$ be a group and $x \in G$. We denote by $o(x)$ the order of $x$ and $G$ is said to be EPO−group, if all non-trivial element orders of $G$ are prime. An EPPO−group is that its element orders are prime power.

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The set of all elements order of $G$ is called its *spectrum*, denoted by $\pi_e(G)$, A maximal subgroup $H$ of $G$ is denoted by $H < \cdot\, G$ and the set of all elements of $G$ of order $k$ is denoted by $\Omega_k(G)$.

Suppose $\Gamma$ is a graph. A subset $X$ of the vertices of $\Gamma$ is called a *clique* if the induced subgraph on $X$ is a complete graph. The maximum size of a clique in $\Gamma$ is called the *clique number* of $\Gamma$ and denoted by $\omega(\Gamma)$. A subset $Y$ of $V(\Gamma)$ is an *independent set* if the induced subgraph on $X$ has no edges. The maximum size of an independent set is called the *independence number* of $G$ and denoted by $\alpha(G)$. The *chromatic number* of $\Gamma$ is the smallest number of colors needed to color the vertices of $\Gamma$ so that no two adjacent vertices share the same color. This number is denoted by $\chi(\Gamma)$.

Throughout this paper our notation is standard and they are taken from the standard books on graph theory and group theory such as [12, 18].

## 2. Main Results

Suppose $G$ is a finite group of order $n$. Chakrabarty, Ghosh and Sen [4] proved that the number of edges of $P(G)$ can be computed by the following formula:

$$e = \frac{1}{2}\sum_{a\in G}\{2o(a) - \phi(o(a)) - 1\},$$

where $\phi$ is the Euler's totient function. In the case that $G$ is cyclic, we have:

$$e = \frac{1}{2}\sum_{d|n}\{2d - \phi(d) - 1\}\phi(d).$$

Moreover, $P(Z_n)$ is nonplanar when $\phi(n) > 7$ or $n = 2^m$, $m \geq 3$. Finally, if $n \geq 3$ then $P(Z_n)$ is Hamiltonian.

Suppose $D(n)$ denotes the set of all positive divisors of $n$. It is well-known that $(D(n), |)$ is a distributive lattice. $D(n)$ is a Boolean algebra if and only if $n$ is square-free. In the following theorem we apply the structure of this lattice to compute the clique and chromatic number of $P(Z_n)$.

**Lemma 1** Suppose $G$ is a group and $A \subseteq G$. The vertices of $A$ constitute a complete subgraph in $P(G)$ if and only if $\{\langle x\rangle \mid x \in A\}$ is a chain.

*Proof* Suppose $C$ is a clique in $P(G)$. To prove that $\{\langle x\rangle \mid x \in C\}$ is a chain, we proceed by induction on $|V(C)|$. If $|C| = 2$ the result is obvious. If $V(C) = \{x_1, x_2, \cdots, x_n\}$ then by induction hypothesis, $\{\langle x_i\rangle \mid 1 \leq i \leq n-1\}$ is a chain in $P(G)$. Without loss of generality we can assume that $1 \subseteq \langle x_1\rangle \subseteq \langle x_2\rangle \subseteq \cdots \subseteq \langle x_{n-1}\rangle$. Consider $t = max\{i \mid \langle x_i\rangle \subseteq \langle x_n\rangle\}$. If $t = n-1$ then the result is proved. Otherwise, $\langle x_t\rangle \subseteq \langle x_n\rangle \subseteq \langle x_{t+1}\rangle$, as desired. Conversely, by definition of power graph, every chain of cyclic subgroups is a clique. □

**Theorem 2** Suppose $n = p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_r^{\alpha_r}$, where $p_1 < p_2 < ... < p_r$ are prime numbers. Then

$$\omega(P(Z_n)) = \chi(P(Z_n)) = p_r^{\alpha_r} + \sum_{j=0}^{r-2}(p_{r-j-1}^{\alpha_{r-j-1}} - 1)\left(\prod_{i=0}^{j}\phi(p_{r-i}^{\alpha_{r-i}})\right).$$

*Proof* Define the relation $\sim$ on $Z_n$ by $a \sim b$ if and only if they have the same order. Then it can easily seen that $\sim$ is an equivalence relation on $Z_n$ and $\frac{Z_n}{\sim}$ can be equipped with an order such that $\frac{Z_n}{\sim} \cong D(n)$. Here $\frac{a}{\sim} \leq \frac{b}{\sim}$ if and only if $o(a)|o(b)$. Choose an element $a \in Z_n$. By our definition, the elements of $\frac{a}{\sim}$ are adjacent in $P(Z_n)$. Moreover, for each chain $\frac{v_1}{\sim}, \frac{v_2}{\sim}, \cdots, \frac{v_t}{\sim}$ of elements in $\frac{Z_n}{\sim}$, $\bigcup_{i=1}^{t}\frac{v_i}{\sim}$ is a complete subgraph of $P(Z_n)$. For an arbitrary element $\frac{u}{\sim} \in \frac{Z_n}{\sim}$, define $d(\frac{o}{\sim}, \frac{u}{\sim})$ to be the same as distance between corresponding elements of $D(n)$.

To find a maximal complete subgraph of $P(Z_n)$, by Lemma 1 it is enough to obtain a maximal chain

$$Q : \frac{a_0}{\sim} = \frac{o}{\sim}, \frac{a_1}{\sim}, \frac{a_2}{\sim}, \cdots, \frac{a_l}{\sim}, \frac{n}{\sim} = \frac{a_{l+1}}{\sim} \tag{1}$$

such that $Q$ has the maximum length, $\frac{a_1}{\sim} \cup \frac{a_2}{\sim} \cup \cdots \cup \frac{a_l}{\sim}$ has the maximum possible size and $l+1 = \alpha_1 + \cdots + \alpha_r$. To do this, it is enough to choose $a_1$ to be an element of order $p_r$, $a_2$ to be an element of order $p_r^2$, ...., $a_{\alpha_r}$ to be an element of order $p_r^{\alpha_r}$, $a_{\alpha_{r+1}}$ to be an element of order $p_r^{\alpha_r} p_{r-1}$ and so on. Therefore,

$$\begin{aligned}
\omega(P(Z_n)) &= |\frac{a_0}{\sim}| + |\frac{a_1}{\sim}| + \cdots + |\frac{a_{l+1}}{\sim}| \\
&= (\phi(p_r) + \phi(p_r^2) + \cdots + \phi(p_r^{\alpha_r})) \\
&+ \phi(p_r^{\alpha_r})(\phi(p_{r-1}) + \cdots + \phi(p_{r-1}^{\alpha_{r-1}})) \\
&+ \cdots \\
&+ \phi(p_r^{\alpha_r}) \cdots \phi(p_2^{\alpha_2})(\phi(p_1) + \cdots + \phi(p_1^{\alpha_1})) + 1 \\
&= p_r^{\alpha_r} + \sum_{j=0}^{r-2} (p_{r-j-1}^{\alpha_{r-j-1}} - 1) \left( \prod_{i=0}^{j} \phi(p_{r-i}^{\alpha_{r-i}}) \right).
\end{aligned}$$

To complete the proof we have to prove that $\omega(P(Z_n)) = \chi(P(Z_n))$ and this is a direct consequence of the strong perfect graph theorem [5]. □

The *exponent* of a finite group $G$ is defined as the least common multiple of all elements of $G$, denoted by $Exp(G)$. It is easy to see that if $G$ is nilpotent then there exists an element $a \in G$ such that $o(a) = Exp(G)$. Such groups are said to be *full exponent*.

**Theorem 3** Suppose that $G$ is a full exponent group and $n = Exp(G) = p_1^{\beta_1} p_2^{\beta_2} \cdots p_r^{\beta_r}$, where $p_1 < p_2 < ... < p_r$ are prime numbers. If $x$ is an element of order $n$ then

$$\omega(P(G)) = \chi(P(G)) = p_r^{\beta_r} + \sum_{j=0}^{r-2} (p_{r-j-1}^{\beta_{r-j-1}} - 1) \left( \prod_{i=0}^{j} \phi(p_{r-i}^{\beta_{r-i}}) \right).$$

*Proof* By Lemma 1, a subset $A$ of $G$ constitutes a clique in $P(G)$ if and only if $\{\langle x \rangle \mid x \in A\}$ is a chain. To obtain a maximal clique in $P(G)$, we have to choose a chain $1 \subseteq \langle x_1 \rangle \subseteq \langle x_2 \rangle \subseteq \cdots \subseteq \langle x_t \rangle$ such that $o(x_t) = o(x)$ and $1 + \sum_{i=1}^{t} \varphi(o(x_i))$ has maximum value among all possible chains of subgroups of $\langle x \rangle$. Now a similar argument as given in the proof of Theorem 2, completes the proof. □

Our calculations on the small group library of GAP [15] suggest the following conjecture:

**Conjecture 1:** The Theorem 3 is correct in general.

**Corollary 4** Let $G$ be a finite group. Then the power graph $P(G)$ is planar if and only if $\pi_e(G) \subseteq \{1, 2, 3, 4\}$.

*Proof* Suppose $P(G)$ is planar. Then $P(G)$ does not have the complete graph $K_5$ as its induced subgraph and the Theorem 3 concludes the result. Conversely, if $\pi_e(G) \subseteq \{1, 2, 3, 4\}$ then it can easily seen that $P(G)$ can be embedded into sphere, as desired. □

In [4, Lemma 4.7], the authors proved that if $G$ is a cyclic group of order $n$, $n \geq 3$ and $\phi(n) > n$ then $P(G)$ is not planar. Also, in [4, Lemma 4.8] it is proved that a cyclic group of order $2^n$, $n \geq 3$, is not planar. In the following corollary we apply Corollary 4 to find a simple classification for planarity of the power graph of cyclic groups.

**Corollary 5** The power graph of a cyclic group of order $n$ is planar if and only if $n = 2, 3, 4$.

In what follows, $U_n$ denotes the groups of units in the ring $Z_n$. In the following corollary a new simple proof for [4, Lemma 4.10] is presented.

**Corollary 6** The power graph of $U_n$ is planar if and only if $n|240$.

*Proof* Suppose $n = p_1^{e_1} p_2^{e_2} \cdots p_k^{e_k}$, where $p_1, p_2, \cdots, p_k$ are distinct primes. Then by [7, Theorems 6.11, 6.13 and Corollary 6.14], $U_{p^e}$ is cyclic for odd $p$, $U_2 \cong 1$, $U_4 \cong Z_2$, $U_{2^n} \cong Z_2 \times Z_{2^{n-2}}$ and $U_n \cong U_{p_1^{e_1}} \times \cdots \times U_{p_k^{e_k}}$. Therefore, by Corollary 4, $n|240$. □

Consider the dihedral group $D_{2n}$ presented by

$$D_{2n} = \langle x, y \mid x^n = y^2 = e \ \& \ y^{-1}xy = x^{-1} \rangle.$$

From [4, Corollary 4.3], we can deduce that the number of edges of $P(D_{2n})$ is given by $e = \frac{1}{2}\sum_{d|n}\{2d\phi(d) - \phi(d)^2\} + n$. This graph is neither Eulerian nor hamitonian, since the group has elements of order 2.

By corollary 5, it is easy to prove the power graph of a dihedral group of order $2n$ is planar if and only if $n = 2, 3, 4$.

**Corollary 7** $\chi(P(D_{2n})) = \omega(P(D_{2n})) = \chi(P(Z_n))$.

*Proof* Notice that the power graph $P(D_{2n})$ is a union of $P(Z_n)$ and $n$ copy of $K_2$ that share in the identity element of $D_{2n}$. □

The semi−dihedral group $SD_{2^n}$ is presented by

$$SD_{2^n} = \langle x, y \mid x^{2^{n-1}} = y^2 = 1, yxy = r^{2^{n-2}-1} \rangle.$$

**Corollary 8** The power graph $P(SD_{2^n})$ is a union of a complete graph of order $2^n$ and $2^n$ copies of $K_2$ that share in the identity vertex. This graph is non-Eulerian, non-hamiltonian and nonplanar, for $n \geq 3$. Moreover, $\chi(P(SD_{2^n})) = \omega(P(SD_{2^n})) = \alpha(P(SD_{2^n})) = 2^n$.

Following [6] we assume that $P$ is a finite partially ordered set (poset for short) which possesses a *rank function* $r : P \longrightarrow \mathbb{N}$ with the property that $r(p) = 0$, for some minimal element $p$ of $P$ and $r(q) = r(p) + 1$ whenever $q$ covers $p$. Let $N_k := \{p \in P \ : \ r(p) = k\}$ be its $k^{th}$ level and let $r(P) := max\{r(p) \ : \ p \in P\}$ be the rank of $P$. An *antichain* or *Sperner family* in $P$ is a subset of pairwise incomparable elements of $P$. It is clear that each level is an antichain. The *width* (*Dilworth* or *Sperner number*) of $P$ is the maximum size $d(P)$ of an antichain of $P$. The poset $P$ is said to have the *Sperner property* if $d(P) = max_k|N_k|$. A *k−family* in $P$, $1 \leq k \leq r(P)$, is a subset of $P$ containing no $(k+1)$−chain in $P$, and $P$ has the *strong Sperner property* if for each $k$ the largest size of a $k$−family in $P$ equals the largest size of a union of $k$ levels.

**Theorem 9** Suppose that $n = p_1^{\beta_1} \cdots p_r^{\beta_r}$ is the prime decomposition of $n$ and $m = \beta_1 + \cdots + \beta_r$. Then $\alpha(P(Z_n))$ is the coefficient of the middle or the two middle term of $\Pi_{j=1}^{m}(1 + x + \cdots + x^{\beta_j})$.

*Proof* It is well-known that the lattice of divisors of a natural number, ordered by divisibility, has strong Sperner property and so its largest antichain is its largest rank level. □

Let $\Gamma$ be a graph. The minimum number of vertices of $\Gamma$ which need to be removed to disconnect the remaining vertices of $\Gamma$ from each other is called the *connectivity* of $\Gamma$, denoted by $\kappa(\Gamma)$. If $G$ is finite group then we define:

$$M(G) = \{x \in G \ ; \ \langle x \rangle < \cdot \ G\}.$$

**Theorem 10** Suppose $G$ is a non-cyclic group and $x \in G$ such that $\langle x \rangle < \cdot \ G$. Define $r(x) = \cup_{y \in M(G) - \langle x \rangle} (\langle x \rangle \cap \langle y \rangle)$. Then,

$$\kappa(P(G)) \leq Min\{|r(x)| \ ; \ \langle x \rangle < \cdot \ G\}.$$

*Proof* Suppose $\langle x\rangle$ is a maximal cyclic subgroup of $G$. We claim that $r(x)$ is a cut set of $P(G)$. Since $G$ is noncyclic, there exists another maximal cyclic subgroup $\langle y\rangle$ different from $\langle x\rangle$. If $r(x)$ is not a cut set of $P(G)$ then there exists a shortest path $Q : x = x_0, x_1, x_2, ..., x_{n-1}, x_n = y$ in $P(G)$ connecting $x$ and $y$. Without loss of generality we can assume that $x_{2k}, 0 \le k \le \lceil \frac{n}{2} \rceil$, are generators of maximal cyclic subgroups of $G$. Thus, $x_1 \in \langle x\rangle \cap \langle x_2\rangle \subseteq r(x)$ contradict by our assumption. This completes the proof. □

For a finite group $G$, the set of all maximal cyclic subgroups of $G$ is denoted by $MaxCyc(G)$.

**Lemma 11** Suppose $G$ is a non-cyclic finite group, $S \subseteq G - M(G)$, $MaxCyc(G) = \{\langle x_1\rangle, ..., \langle x_r\rangle\}$ and $A = \{x_1, ..., x_r\}$. $S$ is a minimal cut set with this property that each component of $P(G) - S$ has exactly one element of $A$ if and only if $S = \cup_{x \in M(G)} r(x)$.

*Proof* If $S = \cup_{x\in M(G)} r(x)$ then by an argument similar to the proof of Theorem 10, one can see that if $x, y \in M(G)$ and $\langle x\rangle \neq \langle y\rangle$ then $\{x_1, x_3, \cdots\} \subseteq S$, where $x = x_0, x_1, x_2, ..., x_{n-1}, x_n = y$ is a shortest path in $P(G)$ connecting $x$ and $y$. Therefore, if $x, y \in M(G)$, $\langle x\rangle \neq \langle y\rangle$ then $x$ and $y$ are not in the same component of $P(G) - S$.

Conversely, we assume that $S$ is a cut set with this property that each component of $P(G) - S$ has exactly one element of $A$ and $x, y \in A$. Suppose $ł \in \langle x\rangle \cap \langle y\rangle$ and $ł \notin S$. Then $ł$ is adjacent to $x$ and $y$ and so there exists a component of $P(G) - S$ containing both of $x$ and $y$, a contradiction. Therefore, $\cup_{x\in M(G)} r(x) \subseteq S$. On the other hand, we assume that $z \in S$ and $\langle t\rangle$ is a maximal cyclic subgroup of $G$ containing $z$. By minimality of $S$, there are at least two components $X_1$ and $X_2$ of $P(G) - S$ such that $z$ is adjacent to a vertex $v_1 \in X_1$ and a vertex $v_2 \in X_2$. Without loss of generality, we can assume that $X_1$ is the component containing $t$ and $v_1 = t$. Obviously, $\langle v_2\rangle \nsubseteq \langle t\rangle$ and so there exists a vertex $t' \in A \cap X_2$ such that $\langle v_2\rangle \subseteq \langle t'\rangle$. Since $z$ is adjacent to $v_2$, $\langle z\rangle \subseteq \langle v_2\rangle$ or $\langle v_2\rangle \subseteq \langle z\rangle$. If $\langle z\rangle \subseteq \langle v_2\rangle$ then $z \in \langle t\rangle \cap \langle t'\rangle$, as desired. If $\langle v_2\rangle \subseteq \langle z\rangle$ then $v_2$ is adjacent to $t$ which is impossible. This completes our argument. □

It is easily seen that the power graph of a $p-$group $Q$ is a union of some complete graphs of order $p$ which share in identity vertex if and only if $Q$ has exponent $p$. In the following theorem we investigate the same problem for an arbitrary group.

**Theorem 12** $P(G)$ is a union of complete graphs which share the identity element of $G$ if and only if $G$ is an EPPO-group and for every maximal cyclic subgroup $A$ and $B$ with $A \neq B$, $A \cap B = \{e\}$.

*Proof* Suppose there exist $x \in G$ and prime numbers $p_1$ and $p_2$ such that $p_1, p_2 | o(x)$. Then the cyclic subgroup $\langle x\rangle$ is containing non-adjacent elements $x_1$ of order $p_1$ and $x_2$ of order $p_2$. Since $x_1$ and $x_2$ are adjacent to $x$, they are in the same block of $P(G)$, a contradiction. If $A = \langle a\rangle$ and $B = \langle b\rangle$ are maximal cyclic subgroup of $G$ such that $e \neq x \in A \cap B$ then $x$, $a$ and $b$ are mutually adjacent and so $A \subseteq B$ or $B \subseteq A$, which is impossible. Conversely, we assume that maximal cyclic subgroups of $G$ have prime power order and for every maximal cyclic subgroup $A$ and $B$ with $A \neq B$, $A \cap B = \{e\}$. By Lemma 11, $S = \cup_{x\in M(G)} r(x) = \{e\}$. On the other hand, if $MaxCyc(G) = \{\langle x_1\rangle, ..., \langle x_r\rangle\}$ and $A = \{x_1, ..., x_r\}$ then by Lemma 11, each component of $P(G) - \{e\}$ is of form $\langle x_i\rangle - \{e\}$, for some $i$, $1 \le i \le r$, which is a complete subgraph of $P(G)$. This completes the proof. □

**Corollary 13** If $G$ is an EPO$-$group then $P(G)$ is a union of some complete graphs which share in the identity element of $G$.

**Lemma 14** A finite group $G$ is *EPPO* if and only if the vertices of every maximal clique of $P(G)$ is a maximal cyclic subgroup of $G$.

*Proof* ($\Longleftarrow$) Suppose $H$ is a maximal clique in $P(G)$ and $x \in H$. If $o(x)$ has at least two prime divisors $p$ and $q$ then there are elements of these orders in $H$ which is impossible.

($\Longrightarrow$) By Lemma 1, we map the maximal clique $H$ in $P(G)$ to the chain $1 \subseteq \langle x_1\rangle \subseteq \langle x_2\rangle \subseteq \cdots \subseteq \langle x_t\rangle$. Then $x_t$ has prime power order $p^\alpha$ and since $G$ is *EPPO* group, $p^\alpha = 1 + \varphi(p) + \cdots + \varphi(p^\alpha)$. This implies that $H = \langle x_t\rangle$. □

A Chinese group theorist Wujie Shi [14] conjectured that a finite group and a finite simple group are isomorphic if they have the same orders and sets of element orders, see also [16, Question 12.39]. Vasiliev, Grechkoseeva and Mazurov gave an affirmative answer to this question in [17]. In the following theorem this result is applied to obtain a new characterization of finite simple groups by their power graphs.

**Theorem 15** If $G_1$ is one of the following finite groups

a) A simple group,
b) A cyclic group,
c) A symmetric group,
d) A diheral group,
e) A generalised quaternion group,

and $G_2$ is a finite group such that $P(G_1) \cong P(G_2)$ then $G_1 \cong G_2$.

*Proof* Since $P(G_1) \cong P(G_2)$, by [3, Corollary 3] $G_1$ and $G_2$ have the same numbers of elements of each order. To prove (a) it is enough to use this corollary and the main result of [17] mentioned in Introduction.

b) If $P(G_2) \cong P(Z_n)$ then by the mentioned result of Cameron, $G_2$ have to exists an element of order $n$.

c) By [14], $G_2 \cong S_n$ if and only if $\pi_e(G_2) = \pi_e(S_n)$ and $|G_2| = |S_n|$, proving the part (c).

d) Suppose $P(G_2) \cong P(D_{2n})$ then $|G_2| = 2n$ and $G_2$ has an element $a$ of order $n$. Since $G$ has the same number of elements of order 2 as the dihedral group $D_{2n}$, we can choose an element $b$ of order 2 in $G_2$ such that $\langle a\rangle \cap \langle b\rangle = 1$. This implies that $G_2$ is a semi-direct product of the cyclic group $Z_n$ by $Z_2$. Therefore, $G_2 \cong D_{2n}$.

e) Suppose $Q_{4n}$ denotes the generalized quaternion group of order $4n$ and $P(G_2) \cong P(Q_{4n})$. Then $|S| > 1$, where $S$ is the set of vertices of the power graph $P(G_2)$ which are joined to all other vertices. We now apply [3, Proposition 4] to deduce that $G_2$ is isomorphic to $Q_{4n}$. □

Let $p$ be an odd prime number. Two groups of order $2p^2$ have isomorphic power graph if and only if they are isomorphic. This is a direct consequence of [13, Lemma 1]. In [2, Theorem 1], Peter Cameron characterized abelian groups by their power graphs. In the following theorem a simple proof for this result is presented.

**Theorem 16** If $G_1$ and $G_2$ are finite abelian groups such that $P(G_1) \cong P(G_2)$ then $G_1 \cong G_2$.

*Proof* Suppose $G_1$ and $G_2$ are finite abelian groups such that $P(G_1) \cong P(G_2)$. Then by [3, Corollary 3], $G_1$ and $G_2$ are conformal. On the other hand, by [12, pp 107-109], finite abelain conformal groups are isomorphic. Therefore, $G_1 \cong G_2$. □

Suppose $p$ is prime. Then there are five groups of order $p^3$ up to isomorphism. From the cyclic decomposition of finite abelian groups, there are three abelian groups isomorphic to $G_1 \cong Z_p \times Z_p \times Z_p$, $G_2 \cong Z_p \times Z_{p^2}$, $G_3 \cong Z_{p^3}$. There are also two non-abelian groups, $G_4$ and $G_5$, of order $p^3$. If $p = 2$ then these groups are isomorphic to $D_8$ and $Q_8$, respectively. If $p$ is odd then

$$G_4 \cong \langle a, b | a^{p^2} = b^p = bab^{p-1}a^{p^2-p-1} = e\rangle,$$

is a non-abelian group of order $p^3$. It has $p^2 - 1$ elements of order $p$, which fall into two conjugacy classes, of sizes $p - 1$ and $p^2 - p$; and $p^3 - p^2$ elements of order $p^2$, forming a single conjugacy class. There is also another group isomorphic to semi-direct product $Z_{p^2} \propto Z_p$. It has $p^3 - 1$ elements of order $p$ falling into three conjugacy classes of sizes $p - 1$, $p^2 - p$ and $p^3 - p^2$. Suppose $G = G_1$ and $H = G_4$. An easy calculation shows that $P(G) \cong P(H)$. Therefore, non-cyclic abelian groups cannot be characterized by their power graphs.

**Theorem 17** Let $G$ be a finite group. The power graph $P(G)$ is bipartite if and only if $G$ is an elementary abelian group of even order.

*Proof* Suppose $P(G)$ is bipartite. If an odd prime $p$ divides $|G|$ then the complete graph $K_p$ can be embedded into $P(G)$, a contradiction. On the other hand, if $G$ has an element of order 4 then $P(G)$ is containing a copy of $K_4$ which is impossible. Therefore, $G$ is an elementary abelian group of even order. The converse is trivial. □

A matching on a graph $G$ is a set of edges of $G$ such that no two of them share a vertex in common. A maximum matching of $G$ is a matching with the largest size among all matchings in $G$. A vertex cover of $G$ is a subset $Q \subseteq V(G)$ that contains at least one end point of each edge. The König-Egerváry theorem [18, Theorem 3.1.16], states that in any bipartite graph, the number of edges in a maximum matching equals the number of vertices in a minimum vertex cover.

**Theorem 18** The power graph $P(Z_{p^n})$ has the maximum number of edges among all power graphs of $p-$groups of order $p^n$.

*Proof* Suppose $G$ is a non-cyclic $p-$groups of order $p^n$. We construct a bipartite graph $\Gamma = (X, Y)$ as follows:

$$X = G, Y = Z_{p^n} \text{ and } E(\Gamma) = \{ab \mid a \in X,\ b \in Y \textit{ and } o(a) \leq o(b)\}.$$

We first assume that $\Gamma$ has a perfect matching $M$ and $f : G \longrightarrow Z_{p^n}$ is a bijective mapping such that for each $a \in G$, $a$ and $f(a)$ are saturated by $M$. Thus, $o(a) - \varphi(o(a)) \leq o(f(a)) - \varphi(o(f(a))$ and since $G$ is not cyclic,

$$\sum_{a\in G}[2o(a) - \varphi(o(a))] < \sum_{a\in G}[2o(f(a)) - \varphi(o(f(a))].$$

But $\frac{1}{2}\left[\sum_{a\in G}[2o(a) - \varphi(o(a))] - 1\right]$ and $\frac{1}{2}\left[\sum_{a\in G}[2o(f(a)) - \varphi(o(f(a))] - 1\right]$ are the number of edges in $P(G)$ and $P(Z_{p^n})$, respectively. So, it is enough to prove that $\Gamma$ has a perfect matching. By König-Egerváry theorem we have to show that a minimum vertex cover of $\Gamma$ has exactly $p^n$ elements. Suppose that $A$ is a minimum vertex cover of $\Gamma$ and $p^\gamma = max\{o(x) \mid x \in G\}$. If $A = X$ then there is nothing to prove that $|A| = p^n$. Otherwise, elements of orders $p^\gamma + 1, p^{\gamma+2}, \cdots, p^n$ of $Y$ are adjacent to all elements of $G$ and so these elements are in $A$. We claim that $A$ contains all elements of $Y$ of order $p^k, k \leq \gamma$. Define,

$$L_k = \{(x, y) \in X \times Y \mid o(x) = o(y) = p^k\},$$

where $k \leq \gamma$. By our definition, if $(x, y) \in L_k$ then $x$ is adjacent to $y$ and so if $(x, y) \in L_k$ then $x \in A$ or $y \in A$. One can easily seen that $L_k$ induces a complete bipartite induced subgraph of $\Gamma$ and hence $\Omega_{p^k}(G) \subseteq A$ or $\Omega_{p^k}(Z_{p^n}) \subseteq A$. Since $|\Omega_{p^k}(G)| \leq |\Omega_{p^k}(Z_{p^n})|$, by minimality we can assume that $\Omega_{p^k}(Z_{p^n}) \subseteq A$, where $1 \leq k \leq \gamma$. Therefore, $A = Y$ and $\Gamma$ has a perfect matching. This completes the proof. □

**Corollary 19** If $G$ is a non-cyclic $p-$group of order $p^n$ then $\sum_{x\in G} o(x) < \sum_{x\in Z_{p^n}} o(x)$.

Our calculations with groups of small order suggest the following conjecture:

**Conjecture 2:** The power graph $P(Z_n)$ has the maximum number of edges among all power graphs of groups of order $n$.

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