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Parametric quintic spline solution for sixth order two point boundary value problems

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Abstract. In this paper, parametric quintic spline method is presented to solve a linear special case sixth order two point boundary value problems with two different cases of boundary conditions. The method presented in this paper has been shown to be second and fourth order accurate. Boundary equations are derived for both the cases of boundary conditions. Convergence analysis of these methods are discussed. The presented method is tested on four numerical examples of linear sixth order boundary value problems. Comparison is made to show the practical usefulness of the presented method.

1. Introduction

We consider a numerical solution of the following linear sixth order two point boundary value problem:

$$u^{(6)}(x) + g(x)u(x) = q(x), \quad a \le x \le b, \tag{1.1}$$

subject to the boundary conditions
Case I:

$$\begin{aligned} u(a) - \gamma_0 = u'(a) - \delta_0 = u''(a) - \eta_0 = 0, \\ u(b) - \gamma_1 = u'(b) - \delta_1 = u''(b) - \eta_1 = 0, \end{aligned} \tag{1.2a}$$

Case II: or

$$\begin{aligned} u(a) - \gamma_0 = u''(a) - \eta_0 = u^{(4)}(a) - \zeta_0 = 0, \\ u(b) - \gamma_1 = u''(b) - \eta_1 = u^{(4)}(b) - \zeta_1 = 0, \end{aligned} \tag{1.2b}$$

where $\gamma_j, \delta_j, \eta_j$ and ζ_j, $j = 0, 1$ are finite real constants. The functions $g(x)$ and $q(x)$ are continuous on the interval $[a, b]$. A boundary value problem of this type arises in astrophysics; the narrow convecting layers bounded by stable layers which are believed to surround A-type stars can be modelled by sixth order boundary value problems [2]. In [9], Glatzmaier also notes that dynamo action in some stars may be modelled by such equations. Theorems which list the conditions for the existence and uniqueness of solution of such problems are thoroughly discussed in a book by Agarwal [1], though no numerical methods are contained there in.

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When an infinite horizontal layer of fluid is heated from below and is subjected to the action of rotation, instability sets in. When this instability is as ordinary convection, the ordinary differential equation is sixth order; when instability sets in as overstability, it is modelled by eighth order ordinary differential equation. Suppose, now, that a uniform magnetic field is also applied across the fluid in the same direction as gravity. When instability sets in now as ordinary convection, it is modeled by tenth order boundary value problem. When instability sets in as overstability, it is modelled by twelfth order boundary value problem [6]. Further discussions of sixth order boundary value problems are given in [3, 4]. Perturbation method for nonlinear analysis of engineering problems have been given in [10].

Boutayeb and Twizell [5] and Twizell [24] developed finite difference methods for solving such boundary value problems. Wazawz [27] used decomposition and modified domain decomposition methods to investigate the solution of sixth order boundary value problems. Numerical solutions were introduced implicitly by Chawla and Katti [7]. Ritz's method based on variational theory [10] and variational iteration methods [16] have been applied for the solution of sixth order boundary value problems. M. El-Gamel et al. [8] used Sinc-Galerkin method, Loghmani and Ahmadinia [13] used sixth degree B-spline functions, Ramadan et al. [17] used septic nonpolynomial spline functions and Siddiqi and Akram [21] solved linear problem using polynomial septic spline solution for sixth order boundary value problems. Twizell and Boutayeb [25] developed a family of numerical methods for the solutions of special and general sixth order boundary value problems with application to Benard layer eigenvalue problems. Noor and Mohyud-Din [15] have applied homotopy perturbation method for solving linear and nonlinear sixth order boundary value problems. In [20], Siddiqi et al. developed quintic spline method for the numerical solution of linear special sixth order boundary value problems. Siddiqi and Twizell [19] presented the solution of sixth order boundary value problems using sextic spline and Siraj-Ul-Islam et al. [22] solved such problems using nonpolynomial septic spline at mid points.

In this paper, we have developed a new spline function method for sixth order boundary value problems by using parametric quintic spline and have solved special case linear boundary value problems. Analysis of the method shows second and fourth order accuracy for arbitrary choices of parameters p, q and s developed in next section.

In Section 2, we have described a parametric quintic spline method and its consistency relations and have developed a new method for problem (1.1) subject to the boundary conditions (1.2a-1.2b). Section 3 is devoted to the development of the boundary equations for both the cases of boundary conditions. The class of methods are discussed in Section 4. The parametric quintic spline solution approximating the analytical solution of the sixth order boundary value problems is determined by using the consistency relation involving the sixth order derivatives and the values of the spline along with the end conditions in Section 5. Convergence analysis is discussed in Section 6. In Section 7, four examples are considered for the usefulness of the method developed in the paper and numerical results are compared with existing methods.

2. Description of the method

In order to develop the parametric quintic spline approximate solution for sixth order boundary value problem (1.1) with (1.2a-1.2b), we discretize the interval $[a, b]$ into N equal subintervals using the grid points $x_j = a + jh,\ j = 0(1)N,\ x_0 = a,\ x_N = b$ and

$$h = \frac{(b-a)}{N} \tag{2.1}$$

where N is a positive integer.

In this section, we present the formulation of the parametric quintic spline interpolant, $S_\Delta(x, \tau) \in C^4[a, b]$ (see [11, 12, 18]) and derive the spline relation for our method. For this, we consider the mesh

$$\Delta = \{x_j,\ j = 0(1)N\}.$$

Let $S_\Delta(x,\tau) = S_\Delta(x)$ be a parametric quintic spline satisfying the following differential equation in the subinterval $[x_{j-1}, x_j]$;

$$\begin{aligned} S_\Delta{}^{(4)}(x) - \tau^2 S_\Delta{}''(x) &= (F_j - \tau^2 M_j)\frac{x - x_{j-1}}{h} + (F_{j-1} - \tau^2 M_{j-1})\frac{x_j - x}{h} \\ &= Q_j z + Q_{j-1}\bar{z}, \end{aligned} \tag{2.2}$$

where

$$Q_j = F_j - \tau^2 M_j, \;\; S_\Delta{}''(x_j) = M_j, \;\; S_\Delta{}^{(4)}(x_j) = F_j, \;\; \tau > 0.$$

Solving differential equation (2.2), we get

$$S_\Delta(x) = A_1 + A_2 x + A_3\cosh\tau x + A_4\sinh\tau x - \frac{1}{\tau^2}\Big[Q_j\frac{(x - x_{j-1})^3}{6h} + Q_{j-1}\frac{(x_j - x)^3}{6h}\Big]. \tag{2.3}$$

To develop the consistency relations between the value of spline and its derivatives at knots, let

$$\left.\begin{aligned} S_\Delta(x_{j-1}) = u_{j-1}, \;\; S_\Delta(x_j) = u_j, \\ S''_\Delta(x_{j-1}) = M_{j-1}, \;\; S''_\Delta(x_j) = M_j. \end{aligned}\right\} \tag{2.4}$$

To define spline in terms of u_j's, M_j's and F_j's, the coefficients introduced in Eq.(2.3) are calculated as

$$\begin{aligned} A_1 &= \frac{1}{h}(u_{j-1}x_j - u_j x_{j-1}) - \frac{A_3}{h}(x_j\cosh\tau x_{j-1} - x_{j-1}\cosh\tau x_j) - \frac{A_4}{h}(x_j\sinh\tau x_{j-1} - x_{j-1}\sinh\tau x_j) \\ &\quad + \frac{h}{6\tau^2}(x_j Q_{j-1} - x_{j-1}Q_j), \\ A_2 &= \frac{1}{h}(u_j - u_{j-1}) - \frac{A_3}{h}(\cosh\tau x_j - \cosh\tau x_{j-1}) - \frac{A_4}{h}(\sinh\tau x_j - \sinh\tau x_{j-1}) + \frac{h}{6\tau^2}(Q_j - Q_{j-1}), \\ A_3 &= -\frac{1}{\tau^2\sinh\tau h}(M_j\sinh\tau x_{j-1} - M_{j-1}\sinh\tau x_j) - \frac{1}{\tau^4\sinh\tau h}(Q_j\sinh\tau x_{j-1} - Q_{j-1}\sinh\tau x_j), \\ A_4 &= \frac{1}{\tau^2\sinh\tau h}(M_j\cosh\tau x_{j-1} - M_{j-1}\cosh\tau x_j) + \frac{1}{\tau^4\sinh\tau h}(Q_j\cosh\tau x_{j-1} - Q_{j-1}\cosh\tau x_j). \end{aligned} \tag{2.5}$$

Substituting these values in Eq. (2.3), we obtain

$$S_\Delta(x) = zu_j + \bar{z}u_{j-1} + \frac{h^2}{3!}[q_3(z)M_j + q_3(\bar{z})M_{j-1}] + \Big(\frac{h}{\omega}\Big)^4\Big[q_2(z) - \Big(\frac{\omega^2}{3!}\Big)q_3(z)\Big]F_j + \Big(\frac{h}{\omega}\Big)^4\Big[q_2(\bar{z}) - \Big(\frac{\omega^2}{3!}\Big)q_3(\bar{z})\Big]F_{j-1}, \tag{2.6}$$

where

$$z = \frac{x - x_{j-1}}{h}, \;\; \bar{z} = 1 - z, \;\; q_3(z) = z^3 - z, \;\; S_\Delta(x_j) = u_j,$$

$$q_2(z) = \frac{\sinh(\omega z)}{\sinh(\omega)} - z, \;\; \omega = \tau h, \;\; \tau \geq 0,$$

$$q_3(0) = q_2(0) = 0, \;\; q_3(\pm 1) = q_2(\pm 1) = 0.$$

Replacing j by $j+1$ in Eq. (2.6), we get the spline valid in the interval $[x_j, x_{j+1}]$.
Continuity of first and third derivative implies that

$$\begin{aligned} \text{(i)}\ M_{j+1}+4M_j+M_{j-1} &= \frac{6}{h^2}(u_{j+1}-2u_j+u_{j-1})-6h^2(\alpha_1F_{j+1}+2\beta_1F_j+\alpha_1F_{j-1}), \\ \text{(ii)}\ M_{j+1}-2M_j+M_{j-1} &= h^2(\alpha F_{j+1}+2\beta F_j+\alpha F_{j-1}). \end{aligned} \tag{2.7}$$

From Eq. (2.7) we obtain

$$M_j = \frac{1}{h^2}(u_{j+1}-2u_j+u_{j-1})-h^2(pF_{j+1}+p_0F_j+pF_{j-1}). \tag{2.8}$$

From Eqs. (2.7) and (2.8), we obtain the following useful relation

$$pF_{j-2}+qF_{j-1}+sF_j+qF_{j+1}+pF_{j+2} = \frac{1}{h^4}(u_{j-2}-4u_{j-1}+6u_j-4u_{j+1}+u_{j+2}). \tag{2.9}$$

To make the system (2.9) consistent with the boundary value problem (1.1) subject to the boundary conditions (1.2a-1.2b), the finite difference formula of order h^4 is used, which leads to the following relation

$$-F_{j-2}+16F_{j-1}-30F_j+16F_{j+1}-F_{j+2} = 12h^2u_j^{(6)}+O(h^6),\ j=2(1)N-2 \tag{2.10}$$

Eqs. (2.9) and (2.10) leads to the following relation :

$$\begin{aligned} (16p+q)F_{j-1}+(-30p+s)F_j+(16p+q)F_{j+1} &= \frac{1}{h^4}(u_{j-2}-4u_{j-1}+6u_j-4u_{j+1}+u_{j+2})+12ph^2u_j^{(6)}+pO(h^6), \\ & j=2(1)N-2. \end{aligned} \tag{2.11}$$

Moreover, following Lucas [14] for quintic spline, it can be written as

$$F_{j-1}-2F_j+F_{j+1} = h^2u_j^{(6)}+O(h^8),\ j=1(1)N-1. \tag{2.12}$$

From Eqs. (2.11) and (2.12), the following relation is obtained:

$$(4p+q)u_{j-1}^{(6)}+(-6p+s)u_j^{(6)}+(4p+q)u_{j+1}^{(6)} = \frac{1}{h^6}(u_{j-3}-6u_{j-2}+15u_{j-1}-20u_j+15u_{j+1}-6u_{j+2}+u_{j+3}),\ j=3(1)N-3. \tag{2.13}$$

Remark. Our method (2.13) reduces to Siddiqi et al. [20] based on quintic spline when $(p,q,s) \to \frac{1}{120}(1,26,66)$.

3. Development of boundary equations

The relation (2.13) gives $(N-5)$ equations in $(N-1)$ unknowns u_j, $j=1(1)N-1$. We require four more equations, two at each end of the range of integration in order to have closed form solution for u_j. For the discretization of the boundary conditions, we define

Case I:

$$\text{(i)}\ \sum_{k=0}^{4} a_ku_k + c_1hu_0' + c_2h^2u_0'' + h^6\sum_{k=0}^{5} d_ku_k^{(6)} + t_1 = 0,\ j=1,$$

$$\text{(ii)}\ \sum_{k=1}^{5} a_k^*u_k + c_3hu_0' + c_4h^2u_0'' + h^6\sum_{k=1}^{6} d_k^*u_k^{(6)} + t_2 = 0,\ j=2,$$

$$\text{(iii)} \sum_{k=N-5}^{N-1} a_k^* u_k - c_3 h u_N' + c_4 h^2 u_N'' + h^6 \sum_{k=N-6}^{N-1} d_k^* u_k^{(6)} + t_{N-2} = 0, \ j = N-2,$$

$$\text{(iv)} \sum_{k=N-4}^{N} a_k u_k - c_1 h u_N' + c_2 h^2 u_N'' + h^6 \sum_{k=N-5}^{N} d_k u_k^{(6)} + t_{N-1} = 0, \ \ j = N-1. \tag{3.1}$$

Case II:

$$\text{(i)} \sum_{k=0}^{4} a_k u_k + b_1 h^2 u_0'' + m_1 h^4 u_0^{(4)} + h^6 \sum_{k=0}^{5} d_k u_k^{(6)} + t_1 = 0, \ j = 1,$$

$$\text{(ii)} \sum_{k=0}^{5} a_k^* u_k + b_1^* h^2 u_0'' + m_1^* h^4 u_0^{(4)} + h^6 \sum_{k=1}^{6} d_k^* u_k^{(6)} + t_2 = 0, \ j = 2,$$

$$\text{(iii)} \sum_{k=N-5}^{N} a_k^* u_k + b_2^* h^2 u_N'' + m_2^* h^4 u_N^{(4)} + h^6 \sum_{k=N-6}^{N-1} d_k^* u_k^{(6)} + t_{N-2} = 0, \ j = N-2,$$

$$\text{(iv)} \sum_{k=N-4}^{N} a_k u_k + b_2 h^2 u_N'' + m_2 h^4 u_N^{(4)} + h^6 \sum_{k=N-5}^{N} d_k u_k^{(6)} + t_{N-1} = 0, \ j = N-1. \tag{3.2}$$

where a_k, c_1, c_2, c_3, c_4, b_1, m_1, d_k, a_k^*, b_1^*, m_1^*, d_k^*, b_2^*, m_2^*, b_2 and m_2 are arbitrary parameters to be determined at $j = 1, 2, N-2, N-1$ for second and fourth order methods.

4. Class of methods

By expanding (2.13) in Taylor series about x_j, we obtain the following local truncation error t_j as

$$\begin{aligned} t_j &= h^6(1-2p-2q-s)u_j^{(6)} + h^8\Big(\frac{1}{4} - 4p - q\Big)u_j^{(8)} + h^{10}\Big(\frac{105840}{10!} - \frac{1}{12}(4p+q)\Big)u_j^{(10)} \\ &\quad + h^{12}\Big(\frac{1013760}{12!} - \frac{1}{360}(4p+q)\Big)u_j^{(12)} + O(h^{14}). \end{aligned} \tag{4.1}$$

By using the above equation and by eliminating the coefficients of the various powers of h for different choices of parameters p, q and s, we obtain the following class of methods:

4.1. Second order methods

Case I:

The value of unknown coefficients of boundary equations for second order at each end are given by

$$\text{(i)}(a_0, a_1, a_2, a_3, a_4) = \Big(-\tfrac{415}{9}, 64, -24, \tfrac{64}{9}, -1 \Big),$$

$$(c_1, c_2, d_0, d_1, d_2, d_3, d_4, d_5) = \Big(-\tfrac{100}{3}, -8, -\tfrac{8}{35}, \tfrac{16}{21}, 0, 0, 0, 0 \Big).$$

$$\text{(ii)}(a_1^*, a_2^*, a_3^*, a_4^*, a_5^*) = \Big(-\tfrac{3799}{415}, \tfrac{7909}{415}, -\tfrac{6186}{415}, \tfrac{2491}{415}, -1 \Big),$$

$$(c_3, c_4, d_1^*, d_2^*, d_3^*, d_4^*, d_5^*, d_6^*) = \Big(-\tfrac{270}{83}, -\tfrac{822}{415}, -\tfrac{318}{7295}, \tfrac{1012}{1003}, 0, 0, 0, 0 \Big).$$

$$\text{(iii)}(a_{N-5}^*, a_{N-4}^*, a_{N-3}^*, a_{N-2}^*, a_{N-1}^*) = \Big(-1, \tfrac{2491}{415}, -\tfrac{6186}{415}, \tfrac{7909}{415}, -\tfrac{3799}{415} \Big),$$

$$(d_{N-6}^*, d_{N-5}^*, d_{N-4}^*, d_{N-3}^*, d_{N-2}^*, d_{N-1}^*) = \Big(0, 0, 0, 0, \tfrac{1012}{1003}, -\tfrac{318}{7295} \Big).$$

$$\text{(iv)}(a_{N-4}, a_{N-3}, a_{N-2}, a_{N-1}, a_N) = \Big(-1, \tfrac{64}{9}, -24, 64, -\tfrac{415}{9} \Big),$$

$$(d_{N-5}, d_{N-4}, d_{N-3}, d_{N-2}, d_{N-1}, d_N) = \Big(0, 0, 0, 0, \tfrac{16}{21}, -\tfrac{8}{35} \Big) \tag{4.2}$$

and the local truncation error is

$$t_j = \begin{cases} (-\frac{28800}{3})h^8 u_j^{(8)} + O(h^9), \, j = 1, N-1, \\ \\ (\frac{135731}{14})h^8 u_j^{(8)} + O(h^9), \, j = 2, N-2. \end{cases} \tag{4.3}$$

Case II:
The value of unknown coefficients of boundary equations for second order at each end are given by

$$\begin{aligned}
&(\mathrm{i})(a_0, a_1, a_2, a_3, a_4) = (5, -14, 14, -6, 1),\\
&(b_1, m_1, d_0, d_1, d_2, d_3, d_4, d_5) = \Big(-2, \tfrac{5}{6}, \tfrac{29}{180}, -1, 0, 0, 0, 0\Big).\\
&(\mathrm{ii})(a_0^*, a_1^*, a_2^*, a_3^*, a_4^*, a_5^*) = (-4, 14, -20, 15, -6, 1),\\
&(b_1^*, m_1^*, d_1^*, d_2^*, d_3^*, d_4^*, d_5^*, d_6^*) = \Big(1, \tfrac{1}{12}, \tfrac{1}{180}, \tfrac{-361}{360}, 0, 0, 0, 0\Big).\\
&(\mathrm{iii})(a_{N-5}^*, a_{N-4}^*, a_{N-3}^*, a_{N-2}^*, a_{N-1}^*, a_N^*) = (1, -6, 15, -20, 14, -4),\\
&(b_2^*, m_2^*, d_{N-6}^*, d_{N-5}^*, d_{N-4}^*, d_{N-3}^*, d_{N-2}^*, d_{N-1}^*) = \Big(1, \tfrac{1}{12}, 0, 0, 0, 0, \tfrac{-361}{360}, \tfrac{1}{180}\Big).\\
&(\mathrm{iv})(a_{N-4}, a_{N-3}, a_{N-2}, a_{N-1}, a_N) = (1, -6, 14, -14, 5),\\
&(b_2, m_2, d_{N-5}, d_{N-4}, d_{N-3}, d_{N-2}, d_{N-1}, d_N) = \Big(-2, \tfrac{5}{6}, 0, 0, 0, 0, -1, \tfrac{29}{180}\Big)
\end{aligned} \tag{4.4}$$

and the local truncation error is

$$t_j = \begin{cases} (\frac{9580}{8!})h^8 u_j^{(8)} + O(h^9), \, j = 1, N-1, \\ \\ (\frac{9966}{8!})h^8 u_j^{(8)} + O(h^9), \, j = 2, N-2. \end{cases} \tag{4.5}$$

Case 1: For $(p, q, s) = \left(\frac{1}{120}, \frac{25}{120}, \frac{17}{30}\right)$ the truncation error is given by

$$t_j = \left(\frac{1}{120}\right)h^8 u_j^{(8)} + O(h^{10}), \, j = 3(1)N-3. \tag{4.6}$$

Case 2: For $(p, q, s) = (0, 0, 1)$ the truncation error is given by

$$t_j = \left(\frac{1}{4}\right)h^8 u_j^{(8)} + O(h^{10}), \, j = 3(1)N-3. \tag{4.7}$$

4.2. Fourth order methods

Case I:
The value of unknown coefficients of boundary equations for fourth order at each end are given by

$(\mathrm{i})(a_0, a_1, a_2, a_3, a_4) = \Big(-\frac{415}{9}, 64, -24, \frac{64}{9}, -1\Big),$

$(c_1, c_2, d_0, d_1, d_2, d_3, d_4, d_5) = \Big(-\frac{100}{3}, -8, \frac{4}{495}, \frac{19}{63}, \frac{2}{9}, \frac{1}{189}, 0, 0\Big).$

$(\mathrm{ii})(a_1^*, a_2^*, a_3^*, a_4^*, a_5^*) = \Big(-\frac{3799}{415}, \frac{7909}{415}, -\frac{6186}{415}, \frac{2491}{415}, -1\Big),$

$(c_3, c_4, d_1^*, d_2^*, d_3^*, d_4^*, d_5^*, d_6^*) = \Big(-\frac{270}{83}, -\frac{822}{415}, \frac{1105}{5878}, \frac{1415}{2551}, \frac{1187}{5551}, \frac{262}{29529}, 0, 0\Big).$

$(\mathrm{iii})(a_{N-5}^*, a_{N-4}^*, a_{N-3}^*, a_{N-2}^*, a_{N-1}^*) = \Big(-1, \frac{2491}{415}, -\frac{6186}{415}, \frac{7909}{415}, -\frac{3799}{415}\Big),$

$(d_{N-6}^*, d_{N-5}^*, d_{N-4}^*, d_{N-3}^*, d_{N-2}^*, d_{N-1}^*) = \Big(0, 0, \frac{262}{29529}, \frac{1187}{5551}, \frac{1415}{2551}, \frac{1105}{5878}\Big).$

(iv)$(a_{N-4}, a_{N-3}, a_{N-2}, a_{N-1}, a_N) = \left(-1, \frac{64}{9}, -24, 64, -\frac{415}{9}\right)$,

$$(d_{N-5}, d_{N-4}, d_{N-3}, d_{N-2}, d_{N-1}, d_N) = \left(0, 0, \frac{1}{189}, \frac{2}{9}, \frac{19}{63}, \frac{4}{495}\right) \tag{4.8}$$

and the local truncation error is

$$t_j = \begin{cases} (-\frac{15552}{3})h^{10}u_j^{(10)} + O(h^{11}), j = 1, N-1, \\ \\ (\frac{30412}{97})h^{10}u_j^{(10)} + O(h^{11}), j = 2, N-2. \end{cases} \tag{4.9}$$

Case II:
The value of unknown coefficients of boundary equations for fourth order at each end are given by

(i)$(a_0, a_1, a_2, a_3, a_4) = (5, -14, 14, -6, 1)$,
$(b_1, m_1, d_0, d_1, d_2, d_3, d_4, d_5) = \left(-2, \frac{5}{6}, -\frac{323}{5040}, -\frac{1133}{2016}, -\frac{101}{504}, -\frac{25}{2016}, 0, 0\right)$.
(ii)$(a_0^*, a_1^*, a_2^*, a_3^*, a_4^*, a_5^*) = (-4, 14, -20, 15, -6, 1)$,
$(b_1^*, m_1^*, d_1^*, d_2^*, d_3^*, d_4^*, d_5^*, d_6^*) = \left(1, \frac{1}{12}, -\frac{729}{3013}, -\frac{444}{875}, -\frac{1735}{6991}, \frac{29}{86688}, 0, 0\right)$.
(iii)$(a_{N-5}^*, a_{N-4}^*, a_{N-3}^*, a_{N-2}^*, a_{N-1}^*, a_N^*) = (1, -6, 15, -20, 14, -4)$,
$(b_2^*, m_2^*, d_{N-6}^*, d_{N-5}^*, d_{N-4}^*, d_{N-3}^*, d_{N-2}^*, d_{N-1}^*) = \left(1, \frac{1}{12}, 0, 0, \frac{29}{86688}, -\frac{1735}{6991}, -\frac{444}{875}, -\frac{729}{3013}\right)$.
(iv)$(a_{N-4}, a_{N-3}, a_{N-2}, a_{N-1}, a_N) = (1, -6, 14, -14, 5)$,

$$(b_2, m_2, d_{N-5}, d_{N-4}, d_{N-3}, d_{N-2}, d_{N-1}, d_N) = \left(-2, \frac{5}{6}, 0, 0, -\frac{25}{2016}, -\frac{101}{504}, -\frac{1133}{2016}, -\frac{323}{5040}\right) \tag{4.10}$$

and the local truncation error is

$$t_j = \begin{cases} (-\frac{13046}{10!})h^{10}u_j^{(10)} + O(h^{11}), j = 1, N-1, \\ \\ (\frac{(146306/3)}{10!})h^{10}u_j^{(10)} + O(h^{11}), j = 2, N-2. \end{cases} \tag{4.11}$$

Case 1: For $(p, q, s) = \left(0, \frac{1}{4}, \frac{1}{2}\right)$ the truncation error is given by

$$t_j = \left(\frac{1}{120}\right)h^{10}u_j^{(10)} + O(h^{12}), j = 3(1)N-3. \tag{4.12}$$

Case 2: For $(p, q, s) = \left(-\frac{1}{12}, \frac{7}{12}, 0\right)$ the truncation error is given by

$$t_j = \left(\frac{1}{120}\right)h^{10}u_j^{(10)} + O(h^{12}), j = 3(1)N-3. \tag{4.13}$$

5. Spline solution

The parametric quintic spline solution of the problem (1.1) subject to the boundary conditions (1.2a-1.2b) is based on the system of linear equations given by equations (2.13) and (3.1,3.2). If $U = [u_1, u_2, ..., u_{N-1}]^T$, $\overline{U} = [\overline{u}_1, \overline{u}_2, ..., \overline{u}_{N-1},]^T$, $V = [v_1, v_2, ..., v_{N-1}]^T$ and $T = [t_1, t_2, ..., t_{N-1}]^T$ be the column vectors.Here, U denotes the exact solution, $\overline{U}$ the approximate solution and T denotes truncation error. Also $E = (e_j) = u_j - \overline{u_j}$, $j = 1, 2, ..., N-1$; is the discretization error. Then we can describe the method in following matrix form

(i) $LU = V + T$,
(ii) $L\overline{U} = V$, (5.1)
(iii) $LE = T$.
Also, we have

$$L = A + h^6 BG \ \ and \ \ G = diag(g_j). \tag{5.2}$$

The septadiagonal matrix A is given by

$$A = \begin{bmatrix} a_1 & a_2 & a_3 & a_4 & & & & \\ a_1^* & a_2^* & a_3^* & a_4^* & a_5^* & & & \\ -6 & 15 & -20 & 15 & -6 & 1 & & \\ 1 & -6 & 15 & -20 & 15 & -6 & 1 & \\ & & \ddots & \ddots & \ddots & & & \\ & & & \ddots & \ddots & \ddots & & \\ & 1 & -6 & 15 & -20 & 15 & -6 & 1 \\ & & 1 & -6 & 15 & -20 & 15 & -6 \\ & & & a_{N-5}^* & a_{N-4}^* & a_{N-3}^* & a_{N-2}^* & a_{N-1}^* \\ & & & & a_{N-4} & a_{N-3} & a_{N-2} & a_{N-1} \end{bmatrix} \tag{5.3}$$

and the matrix B has the form:

$$B = \begin{bmatrix} -d_1 & -d_2 & -d_3 & -d_4 & -d_5 & & & & \\ -d_1^* & -d_2^* & -d_3^* & -d_4^* & -d_5^* & -d_6^* & & & \\ 0 & 4p+q & -6p+s & 4p+q & 0 & 0 & & & \\ 0 & 0 & 4p+q & -6p+s & 4p+q & 0 & 0 & & \\ & & \ddots & \ddots & \ddots & & & & \\ & & & \ddots & \ddots & \ddots & & & \\ & 0 & 0 & 4p+q & -6p+s & 4p+q & 0 & 0 \\ & & 0 & 0 & 4p+q & -6p+s & 4p+q & 0 \\ & & -d_{N-6}^* & -d_{N-5}^* & -d_{N-4}^* & -d_{N-3}^* & -d_{N-2}^* & -d_{N-1}^* \\ & & & -d_{N-5} & -d_{N-4} & -d_{N-3} & -d_{N-2} & -d_{N-1} \end{bmatrix}. \tag{5.4}$$

Moreover,
Case I:

$$\begin{aligned} v_1 &= -a_0\gamma_0 - c_1 h\delta_0 - c_2 h^2\eta_0 - h^6 d_0(q_0 - g_0\gamma_0) - h^6(d_1q_1 + d_2q_2 + d_3q_3 + d_4q_4 + d_5q_5), \quad j = 1, \\ v_2 &= -c_3h\delta_0 - c_4h^2\eta_0 - h^6(d_1^*q_1 + d_2^*q_2 + d_3^*q_3 + d_4^*q_4 + d_5^*q_5 + d_6^*q_6), \quad j = 2, \end{aligned}$$

Case II:

$$\begin{aligned} v_1 &= -a_0\gamma_0 - b_1h^2\eta_0 - m_1h^4\zeta_0 - h^6 d_0(q_0 - g_0\gamma_0) - h^6(d_1q_1 + d_2q_2 + d_3q_3 + d_4q_4 + d_5q_5), \quad j = 1, \\ v_2 &= -a_0^*\gamma_0 - b_1^*h^2\eta_0 - m_1^*h^4\zeta_0 - h^6(d_1^*q_1 + d_2^*q_2 + d_3^*q_3 + d_4^*q_4 + d_5^*q_5), \quad j = 2, \end{aligned}$$

Case I and II:

$$\begin{aligned} v_3 &= -\gamma_0 + h^6[(4p+q)q_2 + (-6p+s)q_3 + (4p+q)q_4], \quad j = 3, \\ v_j &= h^6[(4p+q)q_{j-1} + (-6p+s)q_j + (4p+q)q_{j+1}], \quad j = 4(1)N-4, \\ v_{N-3} &= -\gamma_1 + h^6[(4p+q)q_{N-4} + (-6p+s)q_{N-3} + (4p+q)q_{N-2}], \quad j = N-3, \end{aligned}$$

Case I:

$$\begin{aligned} v_{N-2} &= c_3h\delta_1 - c_4h^2\eta_1 - h^6(d_{N-6}^*q_{N-6} + d_{N-5}^*q_{N-5} + d_{N-4}^*q_{N-4} + d_{N-3}^*q_{N-3} + d_{N-2}^*q_{N-2} + d_{N-1}^*q_{N-1}), \quad j = N-2, \\ v_{N-1} &= -a_N\gamma_1 + c_1h\delta_1 - c_2h^2\eta_1 - h^6 d_N(q_N - g_N\gamma_1) - h^6(d_{N-5}q_{N-5} + d_{N-4}q_{N-4} + d_{N-3}q_{N-3} \\ &\quad + d_{N-2}q_{N-2} + d_{N-1}q_{N-1}), \quad j = N-1, \end{aligned}$$

Case II:

$$\begin{aligned} v_{N-2} &= -a_N^*\gamma_1 - b_2^*h^2\eta_1 - m_2^*h^4\zeta_1 - h^6(d_{N-6}^*q_{N-6} + d_{N-5}^*q_{N-5} + d_{N-4}^*q_{N-4} + d_{N-3}^*q_{N-3} \\ &\quad + d_{N-2}^*q_{N-2} + d_{N-1}^*q_{N-1}), \quad j = N-2, \\ v_{N-1} &= -a_N\gamma_1 - b_2h^2\eta_1 - m_2h^4\zeta_1 - h^6 d_N(q_N - g_N\gamma_1) - h^6(d_{N-5}q_{N-5} + d_{N-4}q_{N-4} + d_{N-3}q_{N-3} \\ &\quad + d_{N-2}q_{N-2} + d_{N-1}q_{N-1}), \quad j = N-1. \end{aligned} \tag{5.5}$$

6. Convergence analysis

In this section, we investigate the convergence analysis of the presented method along with (3.1,3.2) based on parametric quintic spline. To derive the error bound $||E||_\infty$, the error Eq.(5.1)(iii) can be rewritten in the following form

$$E = L^{-1}T = (A + h^6BG)^{-1}T = (I + A^{-1}h^6BG)^{-1}A^{-1}T.$$

We get

$$||E||_\infty \leq \frac{||A^{-1}||_\infty ||T||_\infty}{1 - h^6||A^{-1}||_\infty ||B||_\infty ||G||_\infty} \tag{6.1}$$

provided that $||A^{-1}||_\infty ||B||_\infty ||G||_\infty < 1$.
Case I:
It is shown in [2] that

$$||A^{-1}||_\infty = \frac{83(b-a)^6 - 1088h^4(b-a)^2 + 19344h^5(b-a)}{3824640h^6} = O(h^{-6}). \tag{6.2}$$

Case II:
Twizell [24] have shown that

$$||A^{-1}||_\infty \leq \frac{(b-a)^6}{512h^6} = O(h^{-6}). \tag{6.3}$$

Lemma 6.1: The matrix L given by Eq.(5.2) is nonsingular provided that

$$||B||_\infty |g(x)| \varpi < 1. \tag{6.4}$$

where $\varpi = \frac{83(b-a)^6 - 1088h^4(b-a)^2 + 19344h^5(b-a)}{3824640h^6}$ for case I, $\varpi = \frac{(b-a)^6}{512}$ for case II and $||B||_\infty$ is a finite number.

The proof of this lemma follows from the following statement [26]:
If T is a square matrix of order N and $||T|| < 1$, then $(I + T)$ is nonsingular.

As a consequence of lemma 6.1, the discrete boundary value problem (5.1)(ii) has a unique solution if $||B||_\infty |g(x)| \varpi < 1$.
Now from Eqs.(4.2-4.13), we investigate the following two cases:

Case 1: Second order methods

We have from the Eqs. (4.2-4.7)

$$||T||_\infty = K_2h^8M_8, \tag{6.5}$$

where

$$M_8 = \max_{a \leq x \leq b} |u^{(8)}(x)|$$

and K_2 is a finite number. It follows that

$$||E||_\infty \leq \frac{K_2\varpi M_8h^2}{[1 - \varpi||B||_\infty |g(x)|]} = G_2(h^2) \tag{6.6}$$

where $G_2 = \frac{K_2 \varpi M_8}{[1-\varpi \|B\|_\infty |g(x)|]}$.

Case 2: Fourth order methods

We have from the Eqs. (4.8-4.13)

$$\|T\|_\infty = K_4 h^{10} M_{10}, \tag{6.7}$$

where

$$M_{10} = \max_{a \le x \le b} |u^{(10)}(x)|$$

and K_4 is a finite number. It follows that

$$\|E\|_\infty \le \frac{K_4 \varpi M_{10} h^4}{[1-\varpi \|B\|_\infty |g(x)|]} = G_4(h^4) \tag{6.8}$$

where $G_4 = \frac{K_4 \varpi M_{10}}{[1-\varpi \|B\|_\infty |g(x)|]}$.

We summarize the above results in the following theorem:

Theorem 6.1. *If $U = [u_1, u_2, ..., u_{N-1}]^T$, $\overline{U} = [\overline{u}_1, \overline{u}_2, ..., \overline{u}_{N-1},]^T$, $V = [v_1, v_2, ..., v_{N-1}]^T$ and $T = [t_1, t_2, ..., t_{N-1}]^T$ be the column vectors. Here, U denotes the exact solution, $\overline{U}$ the approximate solution and T denotes truncation error. Further, if $E = (e_j) = u_j - \overline{u_j}$, $j = 1(1)N-1$; is the discretization error, then*
(i) $\| E \| = O(h^2)$ is a second order method which is given by (6.6),
(ii) $\| E \| = O(h^4)$ is a fourth order method which is given by (6.8).

7. Numerical examples and results

In this section, we consider four numerical examples for linear sixth order boundary value problems illustrating the comparative performance of our scheme (2.13) over other existing methods. Series of numerical experiments are carried out using various values of h. All calculations are performed in MATLAB 7.

Example 1. Consider the following boundary value problem [20]:

$$u^{(6)} - u = -6\exp(x), \quad 0 \le x \le 1,$$

$$\left.\begin{array}{c} u(0) = 1, \quad u(1) = 0, \\ u'(0) = 0, \quad u'(1) = -e, \\ u''(0) = -1, \quad u''(1) = -2e \end{array}\right\} \tag{7.1}$$

with the analytical solution

$$u(x) = (1-x)\exp(x). \tag{7.2}$$

Example 2. Consider the following boundary value problem [20]:

$$u^{(6)} + u = -6\cos(x), \quad 0 \le x \le 1,$$

$$\left.\begin{array}{c} u(0) = 0, \quad u(1) = 0, \\ u'(0) = -1, \quad u'(1) = \sin 1, \\ u''(0) = 2, \quad u''(1) = 2\cos 1, \end{array}\right\} \tag{7.3}$$

with the analytical solution

$$u(x) = (x-1)\sin(x). \tag{7.4}$$

These examples have been solved by using our scheme (2.13) with different values of $h = 2^{-m}, m = 3, 4, 5, 6$ for second and fourth order methods. The maximum absolute errors in solution and comparison with [20] for Example 1 and Example 2 are tabulated in Table 1 and 2 respectively.

Table 1. Maximum absolute errors for example 1

Methods (p, q, s)	$h = \frac{1}{8}$	$h = \frac{1}{16}$	$h = \frac{1}{32}$	$h = \frac{1}{64}$
Fourth order methods				
$(0, \frac{1}{4}, \frac{1}{2})$	6.64×10^{-9}	1.04×10^{-9}	7.66×10^{-11}	9.39×10^{-11}
$(-\frac{1}{12}, \frac{7}{12}, 0)$	6.64×10^{-9}	1.04×10^{-9}	7.66×10^{-11}	9.39×10^{-11}
Second order methods				
$(\frac{1}{120}, \frac{25}{120}, \frac{17}{30})$	2.37×10^{-7}	1.27×10^{-8}	2.29×10^{-9}	6.44×10^{-10}
$(0, 0, 1)$	1.06×10^{-6}	2.66×10^{-7}	6.66×10^{-8}	1.66×10^{-8}
[20]	3.65×10^{-6}	3.02×10^{-7}	2.14×10^{-8}	1.23×10^{-9}

Table 2. Maximum absolute errors for example 2

Methods (p, q, s)	$h = \frac{1}{8}$	$h = \frac{1}{16}$	$h = \frac{1}{32}$	$h = \frac{1}{64}$
Fourth order methods				
$(0, \frac{1}{4}, \frac{1}{2})$	2.95×10^{-9}	4.5×10^{-10}	3.65×10^{-11}	5.92×10^{-11}
$(-\frac{1}{12}, \frac{7}{12}, 0)$	2.95×10^{-9}	4.5×10^{-10}	3.65×10^{-11}	5.92×10^{-11}
Second order methods				
$(\frac{1}{120}, \frac{25}{120}, \frac{17}{30})$	1.29×10^{-7}	6.92×10^{-9}	1.29×10^{-9}	2.70×10^{-10}
$(0, 0, 1)$	6.05×10^{-7}	1.51×10^{-7}	3.78×10^{-8}	9.51×10^{-9}
[20]	1.84×10^{-6}	1.40×10^{-7}	9.48×10^{-9}	5.63×10^{-10}

Example 3. Consider the following boundary value problem [22]:

$$u^{(6)} + xu = -(24 + 11x + x^3)\exp(x), \quad 0 \le x \le 1,$$

$$\left.\begin{array}{c} u(0) = 0, \quad u(1) = 0, \\ u''(0) = 0, \quad u''(1) = -4e, \\ u^{(4)}(0) = -8, \quad u^{(4)}(1) = -16e, \end{array}\right\} \tag{7.5}$$

with the analytical solution

$$u(x) = x(1 - x)\exp(x). \tag{7.6}$$

Example 4. We consider the following boundary value problem [22]:

$$u^{(6)} + u = 6[2x\cos(x) + 5\sin(x)], \quad -1 \le x \le 1,$$

$$\left.\begin{array}{c} u(-1) = 0, \quad u(1) = 0, \\ u''(-1) = -4\cos(-1) + 2\sin(-1), \quad u''(1) = 4\cos(1) + 2\sin(1), \\ u^{(4)}(-1) = 8\cos(-1) - 12\sin(-1), \quad u^{(4)}(1) = -8\cos(1) - 12\sin(1) \end{array}\right\} \tag{7.7}$$

with the analytical solution

$$u(x) = (x^2 - 1)\sin(x). \tag{7.8}$$

Examples 3 and 4 have been solved by using our scheme (2.13) with different values of $h = 2^{-m}, m = 3, 4, 5, 6$ for second and fourth order methods. The maximum absolute errors in solution and comparison with [22] for examples 3 and 4 are tabulated in Table 3 and 4 respectively.

Table 3. Maximum absolute errors for example 3

Methods (p, q, s)	$h = \frac{1}{8}$	$h = \frac{1}{16}$	$h = \frac{1}{32}$	$h = \frac{1}{64}$
Fourth order methods				
$(0, \frac{1}{4}, \frac{1}{2})$	2.25×10^{-7}	2.19×10^{-8}	1.94×10^{-9}	1.35×10^{-9}
$(-\frac{1}{12}, \frac{7}{12}, 0)$	2.25×10^{-7}	2.19×10^{-8}	1.94×10^{-9}	1.35×10^{-9}
[22]	2.39×10^{-4}	3.43×10^{-6}	7.34×10^{-8}	-
Second order methods				
$(\frac{1}{120}, \frac{25}{120}, \frac{17}{30})$	2.31×10^{-4}	1.86×10^{-5}	1.97×10^{-6}	3.26×10^{-7}
$(0, 0, 1)$	5.01×10^{-4}	1.24×10^{-4}	3.09×10^{-5}	7.71×10^{-6}
[22]	2.99×10^{-2}	7.00×10^{-3}	1.80×10^{-3}	-

Table 4. Maximum absolute errors for example 4

Methods (p, q, s)	$h = \frac{1}{4}$	$h = \frac{1}{8}$	$h = \frac{1}{16}$	$h = \frac{1}{32}$
Fourth order methods				
$(0, \frac{1}{4}, \frac{1}{2})$	2.60×10^{-6}	1.65×10^{-7}	1.02×10^{-8}	3.47×10^{-9}
$(-\frac{1}{12}, \frac{7}{12}, 0)$	2.60×10^{-6}	1.65×10^{-7}	1.02×10^{-8}	3.47×10^{-9}
[22]	6.97×10^{-4}	3.60×10^{-5}	7.44×10^{-7}	-
Second order methods				
$(\frac{1}{120}, \frac{25}{120}, \frac{17}{30})$	6.15×10^{-4}	5.84×10^{-5}	5.09×10^{-6}	5.97×10^{-7}
$(0, 0, 1)$	7.31×10^{-4}	1.74×10^{-4}	4.29×10^{-5}	1.07×10^{-5}
[22]	1.23×10^{-2}	2.80×10^{-3}	1.60×10^{-3}	-

8. Conclusion

Parametric quintic spline method is developed for the approximate solution of sixth order two point boundary value problems. A class of methods are presented for solving such problems. This shows that our methods are better in the sense of accuracy and applicability. These have been verified by the maximum absolute errors given in tables 1-4 for four examples.

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