Semi-continuous function versions of Urysohn Lemma

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Abstract. Some characterizations of semi-stratifiable and k-semi-stratifiable spaces in terms of semi-continuous functions, analogous to Urysohn Lemma, were established in this paper. Some applications of them in the insertion of functions are given as well.

1. Introduction, basic definitions and notations

To study general topology, the concept of continuous functions is fundamental and much work has been done concerning the existence and extension of continuous functions and the collection of continuous (bounded) real-valued functions on a space (see for instance [4]). Amongst this work one finds the famous results of Tietze and Urysohn which are central to the theory of normal spaces.

Urysohn Lemma. A space X is normal if and only if for each ordered pair (A, U) of subsets of X, with A closed, U open and $A \subseteq U$, there is a continuous function $f_{A,U} : X \to [0,1]$ such that $A \subseteq f_{A,U}^{-1}(0)$ and $X - U \subseteq f_{A,U}^{-1}(1)$.

For perfectly normal spaces, there is a strengthening form of Urysohn Lemma as follows.

Theorem 1.1. ([3]) *A space X is perfectly normal if and only if for each ordered pair (A, U) of subsets of X, with A closed, U open and $A \subseteq U$, there is a continuous function $f_{A,U} : X \to [0,1]$ such that $A = f_{A,U}^{-1}(0)$ and $X - U = f_{A,U}^{-1}(1)$.*

In 1966, Borges [1] first gave a characterization of stratifiable spaces in terms of continuous functions separating pairs of disjoint, closed sets, but, in 2001, Lane, Nyikos, and Pan [6] provided a strengthening form as follows.

Theorem 1.2. ([6]) *A space X is stratifiable if and only if for each ordered pair (A, U) of subsets of X, with A closed, U open and $A \subseteq U$, there is a continuous function $f_{A,U} : X \to [0,1]$ such that $A = f_{A,U}^{-1}(0)$, $X - U = f_{A,U}^{-1}(1)$ and $f_{A,U} \geqslant f_{B,V}$ whenever $A \subseteq B$ and $U \subseteq V$.*

The semi-stratifiable and k-semi-stratifiable spaces are very important generalized metrizable spaces. Many interesting properties of them refer to [7]. In 2007, Yan and Yang [12] established some characterizations, analogous to Urysohn Lemma, of semi-stratifiable and perfect spaces in terms of semi-continuous functions as follows.

Theorem 1.3. ([12]) *A space X is semi-stratifiable if and only if for each ordered pair (A, U) of subsets of X, with A closed, U open and $A \subseteq U$, there is a lower semi-continuous function $f_{A,U} : X \to [0,1]$ such that $A = f_{A,U}^{-1}(0)$, $X - U = f_{A,U}^{-1}(1)$ and $f_{A,U} \geqslant f_{B,V}$ whenever $A \subseteq B$ and $U \subseteq V$.*

Theorem 1.4. ([12]) *A space X is perfect if and only if for each ordered pair (A, U) of subsets of X, with A closed, U open and $A \subseteq U$, there is a lower semi-continuous function $f_{A,U} : X \to [0,1]$ such that $A = f_{A,U}^{-1}(0)$ and $X - U = f_{A,U}^{-1}(1)$.*

Now, it is natural to ask whether there exist other semi-continuous function versions of the Urysohn Lemma for *k*-semi-stratifiable spaces. Can the continuous functions be replaced by semi-continuous functions in Theorem 1.1? We give affirmative answers to the questions above. In the second section of this paper, some semi-continuous function versions of the Urysohn Lemma are obtained for *k*-semi-stratifiable spaces and so on. In the third section, we give some applications of them in the insertion of functions.

Throughout this paper, a space means a topological space and all spaces are assumed to be T_1.

A real-valued function f defined on a space X is lower (upper) semi-continuous if for any real number r, the set $\{x \in X : f(x) > r\}$ ($\{x \in X : f(x) < r\}$) is open.

Definition 1.5. ([12]) A real-valued function f defined on a space X is *K*-lower (*K*-upper) semi-continuous if for any compact set K of X, f has a minimum (maximum) value on K.

Before stating the main results of this paper, we shall introduce some notations. $k(X)$, $o(X)$ are the sets of all closed, open subsets of X, respectively. $f \leqslant g$ means $f(x) \leqslant g(x)$ for each $x \in X$, where f and g are real-valued functions defined on the space X. The $\mathbb{N}$ represents the set of all non-negative integers. We also write $(A, B) \prec (C, D)$ whenever $A \subseteq C$ and $B \subseteq D$. $C(X)$, $USC(X)$, $LSC(X)$, $UKL(X)$ are the sets of all continuous, upper semi-continuous, lower semi-continuous, upper and *K*-lower semi-continuous functions from X into $[0,1]$, respectively. Also,

$$ko(X) = \{(F, U) \in k(X) \times o(X) : F \subseteq U\},$$

$$LU(X) = \{(g, f) \in LSC(X) \times USC(X) : g \leqslant f\},$$

$$UL(X) = \{(g, f) \in USC(X) \times LSC(X) : g \leqslant f\},$$

$$UC(X) = \{(g, f) \in USC(X) \times C(X) : g \leqslant f\},$$

$$CL(X) = \{(g, f) \in C(X) \times LSC(X) : g \leqslant f\}.$$

Let X be a space. If $A \subseteq X$, we write χ_A for the characteristic function on A, that is, a function $\chi_A : X \to [0,1]$ defined by

$$\chi_A(x) = \begin{cases} 1 & x \in A, \\ 0 & x \notin A. \end{cases}$$

One easily verifies that if $A \in k(X)$, then $\chi_A \in USC(X)$; and $\chi_A \in LSC(X)$, if $A \in o(X)$.

Definition 1.6. ([8]) A space X is said to be *k*-semi-stratifiable, if there is an operator U assigning to each closed set F, a sequence $U(F) = (U(j, F))_{j \in \mathbb{N}}$ of open sets such that,

(1) $F \subseteq U(j, F)$ for each $j \in \mathbb{N}$;
(2) if $D \subseteq F$, then $U(j, D) \subseteq U(j, F)$ for each $j \in \mathbb{N}$;
(3) $\bigcap_{j \in \mathbb{N}} U(j, F) = F$, and for every compact subset K of X, if $K \cap F = \emptyset$, then there exists some $j_0 \in \mathbb{N}$ such that $K \cap U(j_0, F) = \emptyset$.

Definition 1.7. ([2]) A space X is said to be semi-stratifiable, if there is an operator U assigning to each closed set F, a sequence $U(F) = (U(n, F))_{n \in \mathbb{N}}$ of open sets such that,

(1) $F \subseteq U(n, F)$ for each $n \in \mathbb{N}$;
(2) if $D \subseteq F$, then $U(n, D) \subseteq U(n, F)$ for each $n \in \mathbb{N}$;
(3) $\bigcap_{n \in \mathbb{N}} U(n, F) = F$.

2. Various versions of Urysohn Lemma

In this section some semi-continuous function forms of Urysohn Lemma are stated for various classical spaces.

Let X be a space and $(D_n)_{n\in\mathbb{N}}$ a decreasing sequence of sets of X. Define a real-valued function $f_{(D_n)} : X \to [0,1]$ as follows:

$$(*) \qquad f_{(D_n)}(x) = \begin{cases} 1 & x \in X - D_0, \\ \frac{1}{2^{j+1}} & x \in D_j - D_{j+1}, \\ 0 & x \in \bigcap_{j\in\mathbb{N}} D_j. \end{cases}$$

Lemma 2.1. *Let X be a space and $(D_j)_{j\in\mathbb{N}}$ a decreasing sequence of sets of X. Then the function $f_{(D_n)}$ defined by $(*)$ has the following properties:*

(1) if the $(D_j)_{j\in\mathbb{N}}$ is a sequence of open sets, then the $f_{(D_n)}$ is an upper semi-continuous function;

(2) if the $(D_j)_{j\in\mathbb{N}}$ is a sequence of closed sets, then the $f_{(D_n)}$ is a lower semi-continuous function;

(3) $\bigcap_{j\in\mathbb{N}} D_j = f_{(D_n)}^{-1}(0)$ and $X - D_0 = f_{(D_n)}^{-1}(1)$;

(4) if $(D_n) \prec (E_n)$, that is, $D_n \subseteq E_n$ for each $n \in \mathbb{N}$, then $f_{(E_n)} \leqslant f_{(D_n)}$;

(5) if for any compact set K of X such that $\bigcap_{j\in\mathbb{N}} D_j \cap K = \emptyset$ there exists $j_0 \in \mathbb{N}$ such that $D_{j_0} \cap K = \emptyset$, then $f_{(D_n)}$ is a K-lower semi-continuous function.

Proof. The property (3) directly follows from the definition of $f_{(D_n)}$. The proofs of (1) and (2) are similar, so it suffices to prove (2), (4) and (5).

To prove (2), it is enough to show that the set $\{x \in X : f_{(D_n)}(x) \leqslant r\}$ is closed for any real number r. Without loss of generality, we can actually assume $r \in [0,1]$. Thus, (*i*) if $r = 1$, then $\{x \in X : f_{(D_n)}(x) \leqslant r\} = X$; (*ii*) if $r = 0$, then $\{x \in X : f_{(D_n)}(x) \leqslant r\} = \bigcap_{j\in\mathbb{N}} D_j$; (*iii*) if $0 < r < 1$, then there exists $j_0 \in \mathbb{N}$ such that $\frac{1}{2^{j_0+1}} \leq r < \frac{1}{2^{j_0}}$ and $\{x \in X : f_{(D_n)}(x) \leqslant r\} = D_{j_0}$. Since the $(D_j)_{j\in\mathbb{N}}$ is a sequence of closed sets, for each case the set $\{x \in X : f_{(D_n)}(x) \leqslant r\}$ is closed, which implies that $f_{(D_n)}$ is a lower semi-continuous function.

Now we prove (4). First, observe that if $x \notin D_{j_0}$, then $f_{(D_n)}(x) \geqslant \frac{1}{2^{j_0}}$. Take any $x \in X$. If $x \in \bigcap_{j\in\mathbb{N}} E_j$, then $f_{(E_n)}(x) = 0 \leqslant f_{(D_n)}(x)$. On the other hand, if $x \notin \bigcap_{j\in\mathbb{N}} E_j$, then there exists $j_0 \in \mathbb{N}$ such that $x \in E_{j_0-1} - E_{j_0}$ (set $E_{-1} = X$) which implies $f_{(E_n)}(x) = \frac{1}{2^{j_0}} \leqslant f_{(D_n)}(x)$, since $x \notin E_{j_0} \supseteq D_{j_0}$.

According to Definition 1.5, to prove (5) it suffices to show that $f_{(D_n)}$ has a minimum value on K for any compact set K in X. If $\bigcap_{j\in\mathbb{N}} D_j \cap K \neq \emptyset$, then take a point $x_0 \in \bigcap_{j\in\mathbb{N}} D_j \cap K$, thus $f_{(D_n)}(x_0) = 0$ which is a minimum value on K. On the other hand, if $\bigcap_{j\in\mathbb{N}} D_j \cap K = \emptyset$, then there exists $j_0 \in \mathbb{N}$ such that $D_{j_0-1} \cap K \neq \emptyset$ and $D_{j_0} \cap K = \emptyset$ by the hypothesis. Thus there is an $x_0 \in D_{j_0-1} \cap K$ (set $D_{-1} = X$) such that $f_{(D_n)}(x_0) = \frac{1}{2^{j_0}}$ which is a minimum value on K. the proofs are finished. $\square$

Proposition 2.2. ([12]) *If $f : X \to \mathbb{R}^+$ is a lower (an upper) semi-continuous function and $g : X \to \mathbb{R}^+$ is an upper (a lower) semi-continuous function, then $\frac{f}{g}$ is a lower (an upper) semi-continuous function on X into $\mathbb{R}^+$.*

Theorem 2.3. *A space X is perfectly normal if and only if for each $(A, U) \in ko(X)$, there exists $(g_{A,U}, f_{A,U}) \in LU(X)$ such that $A = f_{A,U}^{-1}(0) = g_{A,U}^{-1}(0)$ and $X - U = f_{A,U}^{-1}(1) = g_{A,U}^{-1}(1)$.*

Note. In fact, from the fact that every continuous function is both upper and lower semi continuous it follows that the necessity of conditions, but we shall directly construct an upper semi continuous function and a lower semi continuous function, which satisfy the necessity of conditions in Theorem 2.3.

Proof. Necessity. Assume that the space X is perfectly normal. Take any $(A, U) \in ko(X)$. Then $(U', A') \in ko(X)$, where $A' = X - A$ and $U' = X - U$. Since the space X is perfectly normal, one can easily find decreasing sequences $(V_i)_{i\in\mathbb{N}}$ and $(W_i)_{i\in\mathbb{N}}$ of open sets such that $A = \bigcap_{i\in\mathbb{N}} V_i = \bigcap_{i\in\mathbb{N}} \overline{V_i}, \overline{V_0} \subseteq U, U' = \bigcap_{i\in\mathbb{N}} W_i = \bigcap_{i\in\mathbb{N}} \overline{W_i}$ and $\overline{W_0} \subseteq A'$. According to (1), (2), (3) and (4) in Lemma 2.1, we have $(h_{\overline{(V_i)}}, h_{(V_i)}) \in LU(X)$ and $(h_{\overline{(W_i)}}, h_{(W_i)}) \in LU(X)$ such that $A = h_{\overline{(V_i)}}^{-1}(0) = h_{(V_i)}^{-1}(0), 1 = h_{\overline{(V_i)}}(x) = h_{(V_i)}(x)$ for each $x \in U', U' = h_{\overline{(W_i)}}^{-1}(0) = h_{(W_i)}^{-1}(0)$ and $1 = h_{\overline{(W_i)}}(x) = h_{(W_i)}(x)$ for each $x \in A$, where each $h_{(.)}$ is defined by $(*)$ above. Now define two functions

as follows: $g_{A,U}(x) = \frac{h_{\overline{(V_i)}}(x)}{1+h_{(W_i)}(x)}$ and $f_{A,U}(x) = \frac{h_{(V_i)}(x)}{1+h_{\overline{(W_i)}}(x)}$ for all $x \in X$. Clearly, $g_{A,U} \in L(X)$ and $f_{A,U} \in U(X)$ by Proposition 2.2. Since $h_{\overline{(V_i)}} \leqslant h_{(V_i)}$ and $h_{\overline{(W_i)}} \leqslant h_{(W_i)}$, one has $g_{A,U} \leqslant f_{A,U}$, that is, $(g_{A,U}, f_{A,U}) \in LU(X)$. $A = f_{A,U}^{-1}(0) = g_{A,U}^{-1}(0)$ is obvious. We shall prove $X - U = f_{A,U}^{-1}(1)$. Take any point $x \in U' = X - U$; from $h_{(V_i)}(x) = 1$ and $h_{\overline{(W_i)}}(x) = 0$ it follows that $f_{A,U}(x) = \frac{h_{(V_i)}(x)}{1+h_{\overline{(W_i)}}(x)} = \frac{1}{1+0} = 1$, which implies $X - U \subseteq f_{A,U}^{-1}(1)$. On the other hand, take any point $x \notin U' = X - U$; according to $h_{(V_i)}(x) \leqslant 1$ and $h_{\overline{(W_i)}}(x) > 0$, we have $f_{A,U}(x) = \frac{h_{(V_i)}(x)}{1+h_{\overline{(W_i)}}(x)} < 1$, which implies $X - U \supseteq f_{A,U}^{-1}(1)$. Thus the proof of $X - U = f_{A,U}^{-1}(1)$ is completed. Similarly, one can prove $X - U = g_{A,U}^{-1}(1)$.

Conversely, take any $F \in k(X)$. To show that the space X is perfectly normal, it suffices to prove that there is a sequence $(V_j)_{j\in\mathbb{N}}$ of open sets in X such that $F = \bigcap_{j\in\mathbb{N}} V_j = \bigcap_{j\in\mathbb{N}} \overline{V_j}$. Clearly, $(F, X) \in ko(X)$. According to the hypothesis there exists $(g_{F,X}, f_{F,X}) \in LU(X)$ such that $F = f_{F,X}^{-1}(0) = g_{F,X}^{-1}(0)$. Set $V_j = \{x \in X : f_{F,X}(x) < \frac{1}{2^{j+1}}\}$ and $V'_j = \{x \in X : g_{F,X}(x) \leqslant \frac{1}{2^{j+1}}\}$ for each $j \in \mathbb{N}$. Clearly, $(V_j)_{j\in\mathbb{N}}$ is a sequence of open sets. We claim that $F = \bigcap_{j\in\mathbb{N}} V_j = \bigcap_{j\in\mathbb{N}} \overline{V_j}$. In fact, it is very easy to verify that $F = \bigcap_{j\in\mathbb{N}} V_j \subseteq \bigcap_{j\in\mathbb{N}} \overline{V_j} \subseteq \bigcap_{j\in\mathbb{N}} V_j' = F$, since $(g_{F,X}, f_{F,X}) \in LU(X)$. Thus the space X is perfectly normal. $\square$

Corollary 2.4. *A space X is perfectly normal if and only if for each $A \in k(X)$, there exists $(g_A, f_A) \in LU(X)$ such that $A = f_A^{-1}(0) = g_A^{-1}(0)$.*

Proof. Assume that the space X is perfectly normal. Take any $A \in k(X)$. Clearly, $(A, X) \in ko(X)$. According to Theorem 2.3 there exists $(g_{A,X}, f_{A,X}) \in LU(X)$ such that $A = f_{A,X}^{-1}(0) = g_{A,X}^{-1}(0)$.

Conversely, to show that the space X is perfectly normal, it is enough to find a decreasing sequence $(V_n)_{n\in\mathbb{N}}$ of open sets such that $A = \bigcap_{n\in\mathbb{N}} V_n = \bigcap_{n\in\mathbb{N}} \overline{V_n}$ for each $A \in k(X)$. By the hypothesis, there exists $(g_A, f_A) \in LU(X)$ such that $A = f_A^{-1}(0) = g_A^{-1}(0)$ for each $A \in k(X)$. Set $V_n = f_A^{-1}[0, \frac{1}{2^{n+1}})$ and $V'_n = g_A^{-1}[0, \frac{1}{2^{n+1}}]$ for each $n \in \mathbb{N}$. Because of the ordered pair $(g_A, f_A) \in LU(X)$ such that $A = f_A^{-1}(0) = g_A^{-1}(0)$, one can easily verify that $A = \bigcap_{n\in\mathbb{N}} V_n \subseteq \bigcap_{n\in\mathbb{N}} \overline{V_n} \subseteq \bigcap_{n\in\mathbb{N}} V'_n = A, ,$ which implies $A = \bigcap_{n\in\mathbb{N}} V_n = \bigcap_{n\in\mathbb{N}} \overline{V_n}$. The proof is completed. $\square$

Theorem 2.5, which is similar to Urysohn Lemma, is a characterization of k-semi-stratifiable in terms of upper and K-lower semi-continuous functions.

Theorem 2.5. *A space X is k-semi-stratifiable if and only if for each $(A, W) \in ko(X)$, there is a function $f_{A,W} \in UKL(X)$ such that $A = f_{A,W}^{-1}(0)$, $X - W = f_{A,W}^{-1}(1)$ and $f_{A,W} \geqslant f_{B,V}$ whenever $(A, W) \prec (B, V)$.*

Proof. Assume that the space X is k-semi-stratifiable. Then there is an operator U satisfying (1), (2) and (3) in Definition 1.6. Take any $(A, W) \in ko(X)$. By leting $\sigma(A, W)_0 = W$ and $\sigma(A, W)_{i+1} = U(i, A) \cap W$ for each $i \in \mathbb{N}$. Clearly, we have (i) $A = \bigcap_{i\in\mathbb{N}} \sigma(A, W)_i$; (ii) $\sigma(A, W)_i \subseteq \sigma(B, V)_i$ whenever $(A, W) \prec (B, V)$; (iii) for any compact set K, if $A = \bigcap_{i\in\mathbb{N}} \sigma(A, W)_i \cap K = \emptyset$, then there is $i_0 \in \mathbb{N}$ such that $\sigma(A, W)_{i_0} \cap K = \emptyset$. Now define $f_{A,W} = f_{(\sigma(A,W)_i)}$, where $f_{(\sigma(A,W)_i)}$ is defined by (*) above. From Lemma 2.1 it follows that $f_{A,W} \in UKL(X)$ such that $A = f_{A,W}^{-1}(0)$, $X - W = f_{A,W}^{-1}(1)$ and $f_{A,W} \geqslant f_{B,V}$ whenever $(A, W) \prec (B, V)$.

Conversely, to show that X is k-semi-stratifiable, it is enough to prove that there is an operator U assigning to each $F \in k(X)$, a sequence of open sets $U(F) = (U(n, F))_{n\in\mathbb{N}}$ which satisfies (1), (2) and (3) in Definition 1.6. Clearly, $(F, X) \in ko(X)$ for each $F \in k(X)$. By the hypothesis there exists $f_{F,X} \in UKL(X)$ such that $F = f_{F,X}^{-1}(0)$ and $f_{E,X} \leqslant f_{F,X}$ whenever $F \subseteq E$, where $E \in k(X)$. Set $U(n, F) = \{x \in X : f_{F,X}(x) < \frac{1}{2^{n+1}}\}$ for each $n \in \mathbb{N}$. It is very easy to verify that $F = \bigcap_{n\in\mathbb{N}} U(n, F)$, because of $F = f_{F,X}^{-1}(0)$. By the hypothesis, we have $f_{F,X} \geqslant f_{E,X}$ whenever $F \subseteq E$. Thus $U(n, F) \subseteq U(n, E)$ for each $n \in \mathbb{N}$. Take any compact set K such that $K \cap F = \emptyset$. Since the function $f_{F,X}$ is K-lower and $F = f_{F,X}^{-1}(0)$, there is $x_0 \in K$ such that $0 < f_{F,X}(x_0) \leqslant f_{F,X}(x)$ for each $x \in K$. Therefore, there is an $i \in \mathbb{N}$ such that $\frac{1}{2^{i+1}} \leqslant f_{F,X}(x_0)$. Thus $U(i, F) = \{x \in X : f_{F,X}(x) < \frac{1}{2^{i+1}}\} \cap K = \emptyset$. The proof is completed. $\square$

Corollary 2.6. *A space X is k-semi-stratifiable if and only if for each $A \in k(X)$, there exists $f_A \in UKL(X)$ such that $A = f_A^{-1}(0)$ and $f_A \geqslant f_B$ whenever $A \subseteq B$.*

Proof. Suppose that the space X is k-semi-stratifiable. Clearly, $(A, X) \in ko(X)$ for each $A \in k(X)$. Thus according to Theorem 2.5 there is an $f_{A,X} \in UKL(X)$, which satisfies the conditions.

Conversely, to show that X is k-semi-stratifiable, it is enough to prove that there is an operator U assigning to each closed set A, a sequence of open sets $U(A) = (U(n, A))_{n \in \mathbb{N}}$ which satisfies (1), (2) and (3) in Definition 1.6. For each $A \in k(X)$, there is a function $f_A \in UKL(X)$ such that $A = f_A^{-1}(0)$ and $f_A \geqslant f_B$ whenever $A \subseteq B$ by the hypothesis. Set $U(n, A) = \{x \in X : f_A(x) < \frac{1}{2^{n+1}}\}$ for each $n \in \mathbb{N}$. One can easily verify that the operator U, where $U(A) = (U(n, A))_{n \in \mathbb{N}}$ for each closed set A of X, satisfies (1), (2) and (3) in Definition 1.6. $\square$

According to the proofs of Theorem 2.5 and Corollary 2.6, it is not difficult to find that the lower semi-continuous functions of the Theorems 1.3 and 1.4 can be replaced by upper semi-continuous functions, which is surprised. Thus we have the following theorems and corollaries.

Theorem 2.7. *A space X is semi-stratifiable if and only if for each $(A, U) \in ko(X)$, there exists $f_{A,U} \in USC(X)$ such that $A = f_{A,U}^{-1}(0)$, $X - U = f_{A,U}^{-1}(1)$ and $f_{A,U} \geqslant f_{B,V}$ whenever $(A, U) \prec (B, V)$.*

Corollary 2.8. *A space X is semi-stratifiable if and only if for each $A \in k(X)$, there exists $f_A \in USC(X)$ such that $A = f_A^{-1}(0)$ and $f_A \geqslant f_B$ whenever $A \subseteq B$.*

Theorem 2.9. *A space X is perfect if and only if for each $(A, U) \in ko(X)$, there exists $f_{A,U} \in USC(X)$ such that $A = f_{A,U}^{-1}(0)$ and $X - U = f_{A,U}^{-1}(1)$.*

Corollary 2.10. *A space X is perfect if and only if for each $A \in k(X)$, there exists $f_A \in USC(X)$ such that $A = f_A^{-1}(0)$.*

3. Applications

In this section, we give some applications on the insertion of functions. Corollary 3.1 was proved in [12]. Using Corollary 2.8, we give another simple proof.

Corollary 3.1. ([12]) *A space X is semi-stratifiable if and only if there exists a map $\Phi : LSC(X) \to USC(X)$ such that for any $h \in LSC(X)$, $0 \leqslant \Phi(h) \leqslant h$, $\Phi(h) \leqslant \Phi(h')$ if $h \leqslant h'$ and $0 < \Phi(h)(x) < h(x)$ whenever $h(x) > 0$.*

Proof. Assume that the space X is semi-stratifiable. Take any $h \in LSC(X)$. By letting $E_{n,h} = \{x \in X : h(x) \leqslant \frac{1}{2^{n+1}}\}$ for each $n \in \mathbb{N}$. Clearly, $E_{n,h} \supseteq E_{n,h'}$ whenever $h \leqslant h'$ for each $n \in \mathbb{N}$. According to Corollary 2.8 there is a sequence $\{f_{E_{n,h}}\}_{n \in \mathbb{N}}$ of upper semi-continuous functions such that $E_{n,h} = f_{E_{n,h}}^{-1}(0)$ and $f_{E_{n,h}} \leqslant f_{E_{n,h'}}$ whenever $h \leqslant h'$ for each $n \in \mathbb{N}$. Define a map as follows: $\Phi(h)(x) = \sum_{n=0}^{\infty} \frac{1}{2^{n+2}} \times \frac{1}{2^{n+2}} f_{E_{n,h}}(x)$ for each $x \in X$. Now we prove that the map Φ is required. $\Phi(h) \leqslant \Phi(h')$ is clear whenever $h \leqslant h'$. Firstly, to show $\Phi(h) \in USC(X)$, it is enough to show that for each $x_0 \in X$ and $r \in [0, 1]$ such that $\Phi(h)(x_0) < r$, there is an open set V of X such that $x_0 \in V$ and $\Phi(h)(x) < r$ for each $x \in V$. Clearly, there exist $\varepsilon > 0$ and $n_0 \in \mathbb{N}$ such that $\sum_{n=n_0}^{\infty} \frac{1}{2^{n+2}} \times \frac{1}{2^{n+2}} f_{E_{n,h}}(x) < \varepsilon$ for each $x \in X$ and $\Phi(h)(x_0) + 2\varepsilon < r$. Since $g = \sum_{n=0}^{n_0-1} \frac{1}{2^{n+2}} \times \frac{1}{2^{n+2}} f_{E_{n,h}}$ is an upper semi-continuous function and $g(x_0) \leqslant \Phi(h)(x_0) < r - 2\varepsilon$, there exists an open neighborhood V of x_0 such that $g(x) < r - 2\epsilon$ for each $x \in V$. Thus, for each $x \in V$ we have $\Phi(h)(x) = \sum_{n=0}^{\infty} \frac{1}{2^{n+2}} \times \frac{1}{2^{n+2}} f_{E_{n,h}}(x) = \sum_{n=0}^{n_0-1} \frac{1}{2^{n+2}} \times \frac{1}{2^{n+2}} f_{E_{n,h}}(x) + \sum_{n=n_0}^{\infty} \frac{1}{2^{n+2}} \times \frac{1}{2^{n+2}} f_{E_{n,h}}(x) < r - 2\varepsilon + \varepsilon < r$ which implies $\Phi(h) \in USC(X)$. Now show $0 \leqslant \Phi(h) \leqslant h$ and $0 < \Phi(h)(x) < h(x)$ whenever $h(x) > 0$ for any $h \in LSC(X)$. Take any $x \in X$. If $h(x) = 0$, then $x \in \bigcap_{n \in \mathbb{N}} E_{n,h}$ and so $f_{n,h}(x) = 0$ for each $n \in \mathbb{N}$. Thus $\Phi(h)(x) = 0$. If $h(x) > 0$, then $h(x) > \frac{1}{2^{i+1}}$ and $x \notin E_{i,h}$ for some $i \in \mathbb{N}$. Let $m = min\{i : x \notin E_{i,h}\}$, then $0 < \Phi(h)(x) = \sum_{n=0}^{m-1} \frac{1}{2^{n+2}} \times \frac{1}{2^{n+2}} f_{E_{n,h}}(x) + \sum_{n=m}^{\infty} \frac{1}{2^{n+2}} \times \frac{1}{2^{n+2}} f_{E_{n,h}}(x) = \sum_{n=m}^{\infty} \frac{1}{2^{n+2}} \times \frac{1}{2^{n+2}} f_{E_{n,h}}(x) \leqslant \sum_{n=m}^{\infty} \frac{1}{2^{n+2}} = \frac{1}{2^{m+1}} < h(x)$.

Conversely, choose any $A \in k(X)$. By the hypothesis, one can easily verify that $\Phi(1 - \chi_A) \in USC(X)$ such that $A = \Phi(1 - \chi_A)^{-1}(0)$ and $\Phi(1 - \chi_A) \leqslant \Phi(1 - \chi_B)$ whenever $A \supseteq B$. From Corollary 2.8 it follows that X is semi-stratifiable. $\square$

Remark. In fact, we have also proved that a space X is perfect if and only if there exists a map $\Phi : LSC(X) \to USC(X)$ such that for any $h \in LSC(X), 0 \leqslant \Phi(h) \leqslant h$ and $0 < \Phi(h)(x) < h(x)$ whenever $h(x) > 0$, which was proved by Yan and Yang in [12].

The following π_i means the ith (i=1,2) projection. Using the same way in the proof of Corollary 3.1, one can easily prove Corollary 3.2 according to Corollary 2.4, which was proved in [11], so we omit the proof.

Corollary 3.2. ([11]) *A space X is perfectly normal if and only if there is an operator $\Phi : LSC(X) \to LU(X)$ such that for any $h \in LSC(X), 0 \leqslant \pi_1(\Phi(h)) \leqslant \pi_2(\Phi(h)) \leqslant h$ and $0 < \pi_1(\Phi(h))(x) \leqslant \pi_2(\Phi(h))(x) < h(x)$ whenever $h(x) > 0$.*

Lemma 3.3. ([5]) *A space X is normal if and only if for each upper semi-continuous function $g : X \to [0,1]$ and lower semi-continuous function $h : X \to [0,1]$ such that $g \leqslant h$, there is a continuous function $f : X \to [0,1]$ such that $g \leqslant f \leqslant h$.*

Theorem 3.4. *For any space X, the following statements are equivalent:*

(1) the space X is perfectly normal;

(2) there is an operator $\Phi : LSC(X) \to LU(X)$ such that for any $h \in LSC(X), 0 \leqslant \pi_1(\Phi(h)) \leqslant \pi_2(\Phi(h)) \leqslant h$ and $0 < \pi_1(\Phi(h))(x) \leqslant \pi_2(\Phi(h))(x) < h(x)$ whenever $h(x) > 0$;

(3) there is an operator $\Phi : CL(X) \to LU(X)$ such that for any $(f,g) \in CL(X), f \leqslant \pi_1(\Phi(f,g)) \leqslant \pi_2(\Phi(f,g)) \leqslant g$ and $f(x) < \pi_1(\Phi(f,g))(x) \leqslant \pi_2(\Phi(f,g))(x) < g(x)$ whenever $f(x) < g(x)$;

(4) there is an operator $\Phi : UC(X) \to LU(X)$ such that for any $(f,g) \in UC(X), f \leqslant \pi_1(\Phi(f,g)) \leqslant \pi_2(\Phi(f,g)) \leqslant g$ and $f(x) < \pi_1(\Phi(f,g))(x) \leqslant \pi_2(\Phi(f,g))(x) < g(x)$ whenever $f(x) < g(x)$;

(5) there is an operator $\Phi : UL(X) \to LU(X)$ such that for any $(f,g) \in UL(X), f \leqslant \pi_1(\Phi(f,g)) \leqslant \pi_2(\Phi(f,g)) \leqslant g$ and $f(x) < \pi_1(\Phi(f,g))(x) \leqslant \pi_2(\Phi(f,g))(x) < g(x)$ whenever $f(x) < g(x)$.

Proof. (1)⇔(2). It follows from Corollary 3.2.

(2)⇒(3). Assume that the Φ_0 is an operator in (2). Take any $(f,g) \in CL(X)$, then $g - f \in LSC(X)$. Thus we can define an operator $\Phi : CL(X) \to LU(X)$ as follows: $\Phi((f,g)) = (\pi_1(\Phi_0(g-f))+f, \pi_2(\Phi_0(g-f))+f)$. We assert that the operator Φ has the required properties. According to $0 \leqslant \pi_1(\Phi_0(g - f)) \leqslant \pi_2(\Phi_0(g - f)) \leqslant g - f$, $f \leqslant \pi_1(\Phi(f,g)) \leqslant \pi_2(\Phi(f,g)) \leqslant g$ is obvious. If $f(x) < g(x)$, then $(g - f)(x) = g(x) - f(x) > 0$. Thus $0 < \pi_1(\Phi_0(g - f))(x) \leqslant \pi_2(\Phi_0(g - f))(x) < (g - f)(x)$, which is equivalent to $f(x) < \pi_1(\Phi_0(g - f))(x) + f(x) \leqslant \pi_2(\Phi_0(g - f))(x) + f(x) < (g - f)(x) + f(x) = g(x)$, that is, $f(x) < \pi_1(\Phi(f,g))(x) \leqslant \pi_2(\Phi(f,g))(x) < g(x)$.

(3)⇒(4). Assume that the Φ_0 is an operator in (3). Take any $(f,g) \in UC(X)$, then $(1 - g, 1 - f) \in CL(X)$. Thus, one can define an operator $\Phi : UC(X) \to LU(X)$ as follows: $\Phi(f,g) = (1 - \pi_2(\Phi_0(1 - g, 1 - f)), 1 - \pi_1(\Phi_0(1 - g, 1 - f)))$. One can easily verify that the operator Φ has the required properties.

(4)⇒(5). Assume that the Φ_0 is an operator in (4). First, we prove that (4) implies (2), which implies that (1), (2), (3) and (4) are equivalent. Take any $h \in LSC(X)$, then $(1 - h, 1) \in UC(X)$. Thus one can define an operator $\Phi' : LSC(X) \to LU(X)$ as follows: $\Phi'(h) = (1 - \pi_2(\Phi_0(1 - h, 1)), 1 - \pi_1(\Phi_0(1 - h, 1)))$. One can easily verify that the operator Φ' satisfies the conditions of (2), which implies the space X is normal. Thus we can also assume that there is an operator Φ_1 satisfying the conditions in (3). Take any $(f,g) \in UL(X)$. There exists $h_{f,g} \in C(X)$ such that $f \leqslant h_{f,g} \leqslant g$ by Lemma 3.3. Thus $(f, h_{f,g}) \in UC(X)$ and $(h_{f,g}, g) \in CL(X)$. Now define an operator $\Phi : UL(X) \to LU(X)$ as follows: $\Phi(f,g) = (\frac{\pi_1(\Phi_0(f,h_{f,g}))+\pi_1(\Phi_1(h_{f,g},g))}{2}, \frac{\pi_2(\Phi_0(f,h_{f,g}))+\pi_2(\Phi_1(h_{f,g},g))}{2})$. One can easily verify that the operator Φ has the required properties in (5).

(5)⇒(1). Let Φ be an operator in (5). Clearly, $(0, 1 - \chi_A) \in UL(X)$ for any $A \in k(X)$. Thus one can easily verify that $(\pi_1(\Phi(0, 1 - \chi_A)), \pi_2(\Phi(0, 1 - \chi_A))) \in LU(X)$ such that $A = \pi_1(\Phi(0, 1 - \chi_A))^{-1}(0) = \pi_2(\Phi(0, 1 - \chi_A))^{-1}(0)$. From Corollary 2.4 it follows that X is perfectly normal. □

Corollary 3.5 was proved by Michael in [9]. We give another proof.

Corollary 3.5. ([9]) *A space X is perfectly normal if and only if for each upper semi-continuous function $g : X \to [0,1]$ and lower semi-continuous function $f : X \to [0,1]$ such that $g \leqslant f$, there is a continuous function $h : X \to R$ such that $g \leqslant h \leqslant f$ and $g(x) < h(x) < f(x)$ whenever $g(x) < f(x)$.*

Proof. Assume that the space X is perfectly normal. Take any $(g, f) \in UL(X)$. According to (5) in Theorem 3.4, there exists $(k_1, k_2) \in LU(X)$ such that $g \leqslant k_1 \leqslant k_2 \leqslant f$ and $g(x) < k_1(x) \leqslant k_2(x) < f(x)$ whenever $g(x) < f(x)$. For $(g, k_1) \in UL(X)$, there is also a $(h_1, h_2) \in LU(X)$ such that $g \leqslant h_1 \leqslant h_2 \leqslant k_1$ and $g(x) < h_1(x) \leqslant h_2(x) < k_1(x)$ whenever $g(x) < k_1(x)$. Since X is normal, there is a function $j \in C(X)$, which is required, such that $h_2 \leqslant j \leqslant k_1$ by Lemma 3.3. In fact, $g \leqslant j \leqslant f$ is obvious. We also have $g(x) < h_2(x) \leqslant j(x) \leqslant k_1(x) < f(x)$ whenever $g(x) < f(x)$.

Conversely, it is obvious, since for any $(g, f) \in UL(X)$, there is continuous function h such that $g \leqslant h \leqslant f$ and $g(x) < h(x) < f(x)$ whenever $g(x) < f(x)$ by the hypothesis. $(h, h) \in LU(X)$ is obvious. From the (5) in Theorem 3.4 it follows that X is perfectly normal. $\square$

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