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Local Approximation Properties for certain King type Operators

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Abstract. In this paper, we consider a certain King type operators which includes general families of Szász-Mirakjan, Baskakov, Post-Widder and Stancu operators. By introducing two parameter family of Lipschitz type space, which provides global approximation for the above mentioned operators, we obtain the rate of convergence of this class. Furthermore, we give local approximation results by using the first and the second modulus of continuity.

1. Introduction

King was the first, who constructed a non-trivial Bernstein operators preserving the test functions 1 and x^2 and provide a better error estimation for $0 \leq x < \dfrac{1}{3}$ [8]. Research in this direction followed by several authors in [4],[5],[6],[7],[9],[11] and [13]. The generalization of King-type operators was given in [1].

Now, let $p \in \mathbb{N}_0 := \{0\} \cup \mathbb{N} = \{0, 1, 2, ...\}$, $I \subset [0, \infty)$, $\omega_p(x) := (1 + x^p)^{-1}$, $e_k(x) := x^k$ $(k = 0, 1, 2)$ for $x \in I$. Throughout the paper, we consider the following function spaces:
$C(I)$: The space of all real valued continuous functions on I.
$C_B(I)$: The space of all real valued continuous bounded functions on I endowed with the norm

$$\left\|f\right\| = sup_{x\in I}\left|f(x)\right|.$$

$B_p(I)$: The space of all functions $f : I \to \mathbb{R}$, for which $f\omega_p$ is bounded on I, endowed with the norm

$$\left\|f\right\|_p = sup_{x\in I}\omega_p(x)\left|f(x)\right|.$$

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$C_p(I) : f \in B_p(I)$ for which $f\omega_p$ is uniformly continuous on I.

$E[0,\infty) := \left\{ f \in C[0,\infty) : \lim\limits_{x\to\infty} \dfrac{f(x)}{1+x^2} \text{ is convergent} \right\}$.

In this paper we consider a certain class of positive linear operators $\{L_n\}_{n\in\mathbb{N}}$ satisfying the following conditions:

(i) $L_n : C_p(I) \to B_p(I)$ for every $p \in \mathbb{N}$.

(ii) Let $L_n(e_0, x) = 1$, $L_n(e_1, x) = x$ for $x \in I$.

(iii) There exist numbers $a, b \geq 0$, $a^2 + b^2 > 0$ and a positive increasing unbounded sequence (λ_n) such that

$$L_n(e_2; x) = x^2 + \frac{ax^2 + bx}{\lambda_n} \text{ for every } x \in I.$$

Note that the requirements (ii) and (iii) are the same as in [13, see Eqs. (2.4)-(2.5)]. Furthermore, in [1], the family of linear positive operators L_n satisfying

$$L_n(e_0, x) = 1, L_n(e_1, x) = x, L_n(e_2, x) = a_n x^2 + b_n x + c_n$$

was considered by O. Agratini. Hence choosing

$$a_n := 1 + \frac{a}{\lambda_n}, b_n := \frac{b}{\lambda_n}, c_n := 0$$

one can obtain the operators considered in this paper.

Taking into account the above family, for all $f \in C_p$, we consider the operators

$$T_n(f; x) = L_n(f, u_n(x)), \quad x \in I,\ n \geq n_0 \tag{1}$$

where (u_n) is a sequence of continuous functions on $[0, \infty)$ such that

$$u_n(0) = 0,\ \ 0 \leq u_n(x) \leq x \text{ for every } x \in I \subset [0,\infty) \text{ and } n \in \mathbb{N}. \tag{2}$$

Note that by taking

$$u_n(x) = u_n^*(x) := \frac{-b + \sqrt{b^2 + 4\lambda_n(a+\lambda_n)x^2}}{2(a+\lambda_n)},$$

one can obtain the King-type operators (a class of operators preseving $e_2(x)$) considered earlier by Rempulska and Tomczak [13] , where they obtained some global results for the sub-family of these operators. The family $T_n(f, x)$, satisfying the above conditions, includes many well known operators such as Szász- Mirakjan ($a = 0, b = 1, \lambda_n = n$), Baskakov ($a = 1, b = 1, \lambda_n = n$), Post-Widder ($a = 1, b = 0, \lambda_n = n$) and Stancu operators ($a = 1, b = 1, \lambda_n = n - 1$).

It is obvious by (ii) and (iii) that

$$T_n(e_0; x) = 1,$$
$$T_n(e_1, x) = u_n(x),$$
$$T_n(e_2; x) = u_n^2(x) + \frac{au_n^2(x) + bu_n(x)}{\lambda_n}$$

for every $x \in I$. Furthermore,

$$T_n(\varphi_x(t), x) = u_n(x) - x,$$
$$T_n(\varphi_x^2(t), x) = (x - u_n(x))^2 + \frac{au_n^2(x) + bu_n(x)}{\lambda_n}, \tag{3}$$

for every $x, t \in I$, where $\varphi_x(t) := (t - x)$.

Now we start with giving the following Korovkin-type approximation theorem for the operators T_n given by (1).

Theorem 1.1. *Let (u_n) be a sequence of functions on I satisfying* (2). *If*

$$\lim_{n\to\infty} u_n(x) = x \tag{4}$$

uniformly with respect to $x \in [0, b]$ with $b > 0$, then, for all $f \in E[0, \infty)$, we have

$$\lim_{n\to\infty} T_n(f; x) = f(x)$$

uniformly with respect to $x \in [0, b]$.

Proof. For a fixed $b > 0$, consider the lattice homomorphism $H_b : C[0, +\infty) \to C[0, b]$ defined by $H_b(f) := f|_{[0,b]}$ for every $f \in C[0, +\infty)$. In this case, from (4), we see that, for each $i = 0, 1, 2$,

$$\lim_{n\to\infty} H_b\left(T_n(e_i)\right) = H_b(e_i) \text{ uniformly on } [0, b].$$

Hence, with the universal Korovkin-type property with respect to monotone operators (see Theorem 4.1.4 (*vi*) of [2, p. 199]), we obtain that, for all $f \in E[0, \infty)$,

$$\lim_{n\to\infty} T_n(f; x) = f(x)$$

uniformly with respect to $x \in [0, b]$. □

In the present paper, by defining the two parameter family of Lipschitz-type space, we study the approximation properties of the operators T_n in this space. Local approximation behavior of the operators T_n with the help of the first and the second modulus of smoothness is also studied. Finally in the last section we give applications of the main results.

2. Main Results

In this section, we first define two parameter family of Lipschitz-type space.

Let $a, b > 0$ be fixed. We consider the following Lipschitz-type space

$$Lip_M^{(a,b)}(\alpha) := \left\{ f \in C[0, \infty) : \left|f(y) - f(x)\right| \le M \frac{\left|y - x\right|^{\alpha}}{(y + ax^2 + bx)^{\alpha/2}};\ x, y \in (0, \infty) \right\},$$

where M is any positive constant and $0 < \alpha \le 1$.

We should note that the space $Lip_M^{(0,1)}(\alpha)$ coincides with the space $Lip_M^*(\alpha)$ considered by Otto Szász [14].

We have the following local approximation result.

Theorem 2.1. *For any $f \in Lip_M^{(a,b)}(\alpha)$, $\alpha \in (0, 1]$, and for every $x \in (0, \infty)$, $n \in \mathbb{N}$, we have*

$$\left|T_n(f; x) - f(x)\right| \le \frac{M}{(ax^2 + bx)^{\alpha/2}} \left\{ (x - u_n(x))^2 + \frac{a u_n^2(x) + b u_n(x)}{\lambda_n} \right\}^{\alpha/2}.$$

Proof. Assume that $\alpha = 1$. Then, for $f \in Lip_M^{(a,b)}(1)$ and $x \in (0,\infty)$, we have

$$\begin{aligned} \left|T_n(f;x) - f(x)\right| &\leq T_n(\left|f(y) - f(x)\right|;x) \\ &\leq MT_n\left(\frac{\left|y-x\right|}{(y+ax^2+bx)^{1/2}};x\right) \\ &\leq \frac{M}{(ax^2+bx)^{1/2}}T_n(\left|y-x\right|;x). \end{aligned}$$

Applying Cauchy-Schwarz inequality, we get by (3) that

$$\begin{aligned} \left|T_n(f;x) - f(x)\right| &\leq \frac{M}{(ax^2+bx)^{1/2}}\sqrt{T_n((y-x)^2;x)} \\ &= \frac{M}{(ax^2+bx)^{1/2}}\left\{(x-u_n(x))^2 + \frac{au_n^2(x)+bu_n(x)}{\lambda_n}\right\}^{1/2}. \end{aligned}$$

Now assume that $\alpha \in (0,1)$. Then, for $f \in Lip_M^{(a,b)}(\alpha)$ and $x \in (0,\infty)$, we have

$$\begin{aligned} \left|T_n(f;x) - f(x)\right| &\leq T_n(\left|f(y) - f(x)\right|;x) \\ &\leq MT_n\left(\frac{\left|y-x\right|^\alpha}{(y+ax^2+bx)^{\alpha/2}};x\right) \\ &\leq \frac{M}{(ax^2+bx)^{\alpha/2}}T_n(\left|y-x\right|^\alpha;x). \end{aligned}$$

Taking $p = \frac{1}{\alpha}$ and $q = \frac{1}{1-\alpha}$, for any $f \in Lip_M^{(a,b)}(\alpha)$, and applying the Hölder inequality, we have

$$\left|T_n(f;x) - f(x)\right| \leq \frac{M}{(ax^2+bx)^{\alpha/2}}\left[T_n(\left|y-x\right|;x)\right]^\alpha.$$

Finally, by Cauchy-Schwarz inequality, we get from (3)

$$\begin{aligned} \left|T_n(f;x) - f(x)\right| &\leq \frac{M}{(ax^2+bx)^{\alpha/2}}\left[\sqrt{T_n((y-x)^2;x)}\right]^\alpha \\ &\leq \frac{M}{(ax^2+bx)^{\alpha/2}}\left\{(x-u_n(x))^2 + \frac{au_n^2(x)+bu_n(x)}{\lambda_n}\right\}^{\alpha/2}. \end{aligned}$$

Whence the result. □

Taking $u_n(x) := x$ in the above theorem we can state the following corollary which gives global result:

Corollary 2.2. *For any $f \in Lip_M^{(a,b)}(\alpha)$, $\alpha \in (0,1]$, we have*

$$\left|L_n(f;x) - f(x)\right| \leq \frac{M}{\lambda_n^{\alpha/2}}$$

uniformly for $x \in I \cap (0,\infty)$ as $n \to \infty$.

By $C_B^2[0,\infty)$ we denote the space of all functions $f \in C_B[0,\infty)$ such that $f', f'' \in C_B[0,\infty)$. Let $\|f\|$ denote the usual supremum norm of a bounded function f. Then, the classical Peetre's K-functional and the second modulus of smoothness of a function $f \in C_B[0,\infty)$ are defined respectively by

$$K(f,\delta) := \inf_{g \in C_B^2[0,\infty)} \left\{ \|f-g\| + \delta \|g''\| \right\}$$

and

$$\omega_2(f,\delta) := \sup_{0<h\le\delta,\, x\in[0,\infty)} \left| f(x+2h) - 2f(x+h) + f(x) \right|,$$

where $\delta > 0$. Then, by Theorem 2.4 of [3, p. 177], there exists a constant $C > 0$ such that

$$K(f,\delta) \le C\omega_2\left(f, \sqrt{\delta}\right). \tag{5}$$

We now get the following local approximation result.

Theorem 2.3. *For any $f \in C_B[0,\infty)$ and for every $x \in [0,\infty)$, $n \in \mathbb{N}$, we have*

$$\left|T_n(f;x) - f(x)\right| \le C\omega_2\left(f, \sqrt{(x-u_n(x))^2 + \frac{au_n^2(x) + bu_n(x)}{\lambda_n}}\right) + \omega(f, x-u_n(x))$$

for some constant $C := C(x) > 0$, where ω denotes the usual modulus of continuity of f.

Proof. *Using the operator T_n given by (1), define a new operator $T_n^* : C_B[0,\infty) \to C_B[0,\infty)$ as*

$$T_n^*(f;x) = T_n(f;x) - f(u_n(x)) + f(x), \tag{6}$$

where (u_n) satisfies (2). Then

$$T_n^*(e_1 - e_0x; x) = u_n(x) - u_n(x) + x - x = 0. \tag{7}$$

Let $g \in C_B^2[0,\infty)$ and $x \in [0,\infty)$. Using Taylor's formula, we have

$$g(t) - g(x) = g'(x)(t-x) + \int_x^t (t-u)g''(u)du, \quad t \in [0,\infty).$$

Thus, from (6) and (7), we obtain

$$\begin{aligned}
\left|T_n^*(g;x) - g(x)\right| &= \left| g'(x)T_n^*(e_1 - e_0x;x) + T_n^*\left(\int_x^{e_1(t)} (e_1(t)-u)g''(u)du; x\right)\right| \\
&= \left| T_n^*\left(\int_x^{e_1(t)} (e_1(t)-u)g''(u)du; x\right)\right| \\
&= \left| T_n\left(\int_x^{e_1(t)} (e_1(t)-u)g''(u)du; x\right) - \int_x^{u_n(x)} (u_n(x)-u)\, g''(u)du \right| \\
&\le T_n\left(\left|\int_x^{e_1(t)} (e_1(t)-u)g''(u)du\right|; x\right) + \left|\int_{u_n(x)}^x (u_n(x)-u)\, g''(u)du\right| \\
&\le \frac{\|g''\|}{2} T_n\left((e_1 - e_0x)^2; x\right) + \frac{\|g''\|}{2}(x-u_n(x))^2 .
\end{aligned}$$

By (3), we have

$$\left|T_n^*(g;x)-g(x)\right|\leq\frac{\left\|g''\right\|}{2}\left(2\,(x-u_n(x))^2+\frac{au_n^2(x)+bu_n(x)}{\lambda_n}\right).\tag{8}$$

Then, for any $f\in C_B[0,\infty)$, it follows from (8) that

$$\begin{aligned}\left|T_n(f;x)-f(x)\right| &\leq \left|T_n^*(f-g;x)-(f-g)(x)\right|\\ &+\left|T_n^*(g;x)-g(x)\right|+\left|f(u_n(x)-f(x)\right|\\ &\leq 4\left\|f-g\right\|+2\left((x-u_n(x))^2+\frac{au_n^2(x)+bu_n(x)}{\lambda_n}\right)\left\|g''\right\|\\ &+\left|f(u_n(x))-f(x)\right|.\end{aligned}$$

Finally, by (5), we have

$$\begin{aligned}\left|T_n(f;x)-f(x)\right| &\leq 4\left\{\left\|f-g\right\|+\left((x-u_n(x))^2+\frac{au_n^2(x)+bu_n(x)}{\lambda_n}\right)\left\|g''\right\|\right\}\\ &+\omega(f,x-u_n(x))\\ &\leq 4K\left(f,(x-u_n(x))^2+\frac{au_n^2(x)+bu_n(x)}{\lambda_n}\right)+\omega(f,x-u_n(x))\\ &\leq C\omega_2\left(f,\sqrt{(x-u_n(x))^2+\frac{au_n^2(x)+bu_n(x)}{\lambda_n}}\right)+\omega(f,x-u_n(x)).\end{aligned}$$

□

3. Applications of the main results

As it is mentioned in introduction section, the operators $L_n(f;x)$ includes many well known operators such as Szász- Mirakjan ($a=0,b=1,\lambda_n=n$), Baskakov ($a=1,b=1,\lambda_n=n$), Post-Widder ($a=1,b=0,\lambda_n=n$) and Stancu operators ($a=1,b=1,\lambda_n=n-1$). Now, we list the applications of the main results for the above mentioned operators:

3.1. Szász- Mirakjan operators

For every $x\in(0,\infty)$, $n\in\mathbb{N}$, the Szász- Mirakjan operators $S_n(f;x)$ are defined by

$$S_n(f;x)=e^{-nx}\sum_{k=0}^{\infty}f\left(\frac{k}{n}\right)\frac{(nx)^k}{k!}.$$

Now considering the family of Szász- Mirakjan operators defined by

$$S_n^*(f;x)=S_n(f,u_n(x)),\quad x\in I,\ n\geq n_0$$

where (u_n) satisfies (2). As a consequence of Theorem 2.1, Corollary 2.2 and Theorem 2.3, we have:

Corollary 3.1. *For any $f\in Lip_M^{(0,1)}(\alpha)$, $\alpha\in(0,1]$, and for every $x\in(0,\infty)$, $n\in\mathbb{N}$, we have*

$$\left|S_n^*(f;x)-f(x)\right|\leq\frac{M}{x^{\alpha/2}}\left\{(x-u_n(x))^2+\frac{u_n(x)}{n}\right\}^{\alpha/2}.$$

The following global result holds true:

Corollary 3.2. *For any $f \in Lip_M^{(0,1)}(\alpha)$, $\alpha \in (0,1]$, we have*

$$\left|S_n(f;x) - f(x)\right| \leq \frac{M}{n^{\alpha/2}}$$

uniformly for $x \in (0,\infty)$ as $n \to \infty$. Note that this result was obtained by Otto Szász [14].

Corollary 3.3. *For any $f \in C_B[0,\infty)$ and for every $x \in (0,\infty)$, $n \in \mathbb{N}$, we have*

$$\left|S_n^*(f;x) - f(x)\right| \leq C\omega_2\left(f, \sqrt{(x-u_n(x))^2 + \frac{u_n(x)}{n}}\right) + \omega(f, x-u_n(x))$$

for some constant $C := C(x) > 0$, where ω denotes the usual modulus of continuity of f.

Note that Theorems 1.1 and 1.2 of [10] are special cases of Corollaries 5 and 7, respectively.

3.2. Baskakov operators

The Baskakov operators $V_n(f;x)$ are defined by

$$V_n(f;x) = (1+x)^{-n}\sum_{k=0}^{\infty} f\left(\frac{k}{n}\right)\binom{n+k-1}{k}\left(\frac{x}{x+1}\right)^k,$$

where $x \in (0,\infty)$, $n \in \mathbb{N}$. Consider the family of Baskakov operators defined by

$$V_n^*(f;x) = V_n(f, u_n(x)), \quad x \in I, \ n \geq n_0$$

where (u_n) satisfies (2). We have:

Corollary 3.4. *For any $f \in Lip_M^{(1,1)}(\alpha)$, $\alpha \in (0,1]$, and for every $x \in (0,\infty)$, $n \in \mathbb{N}$, we have*

$$\left|V_n^*(f;x) - f(x)\right| \leq \frac{M}{(x^2+x)^{\alpha/2}}\left\{(x-u_n(x))^2 + \frac{u_n^2(x)+u_n(x)}{n}\right\}^{\alpha/2}.$$

The following global result holds true:

Corollary 3.5. *For any $f \in Lip_M^{(1,1)}(\alpha)$, $\alpha \in (0,1]$, we have*

$$\left|V_n(f;x) - f(x)\right| \leq \frac{M}{n^{\alpha/2}},$$

uniformly for $x \in (0,\infty)$ as $n \to \infty$.

Corollary 3.6. *[12, Theorem 4.1] For any $f \in C_B[0,\infty)$ and for every $x \in [0,\infty)$, $n \in \mathbb{N}$, we have*

$$\left|V_n^*(f;x) - f(x)\right| \leq C\omega_2\left(f, \sqrt{(x-u_n(x))^2 + \frac{u_n^2(x)+u_n(x)}{n}}\right) + \omega(f, x-u_n(x))$$

for some constant $C := C(x) > 0$, where ω denotes the usual modulus of continuity of f.

3.3. Post-Widder operators

The Post-Widder operators are defined, for every $x \in (0,\infty)$, $n \in \mathbb{N}$, by

$$P_n(f;x) = \int_0^\infty f(t)\frac{(n/x)^n t^{n-1}}{(n-1)!}\exp\left(\frac{-nt}{x}\right)dt.$$

Consider the family of Post-Widder operators defined by

$$P_n^*(f;x) = P_n(f,u_n(x)), \quad x \in I,\ n \geq n_0$$

where (u_n) satisfies (2).

Corollary 3.7. *For any $f \in Lip_M^{(1,0)}(\alpha)$, $\alpha \in (0,1]$, and for every $x \in (0,\infty)$, $n \in \mathbb{N}$, we have*

$$\left|P_n^*(f;x) - f(x)\right| \leq \frac{M}{x^\alpha}\left\{(x-u_n(x))^2 + \frac{u_n^2(x)}{n}\right\}^{\alpha/2}.$$

The following global result holds true:

Corollary 3.8. *For any $f \in Lip_M^{(1,0)}(\alpha)$, $\alpha \in (0,1]$, we have*

$$\left|P_n(f;x) - f(x)\right| \leq \frac{M}{n^{\alpha/2}},$$

uniformly for $x \in (0,\infty)$ as $n \to \infty$.

Corollary 3.9. *For any $f \in C_B[0,\infty)$ and for every $x \in [0,\infty)$, $n \in \mathbb{N}$, we have*

$$\left|P_n^*(f;x) - f(x)\right| \leq C\omega_2\left(f, \sqrt{(x-u_n(x))^2 + \frac{u_n^2(x)}{n}}\right) + \omega(f, x-u_n(x))$$

for some constant $C := C(x) > 0$, where ω denotes the usual modulus of continuity of f.

3.4. Stancu operators

For every $x \in (0,\infty)$, $n \in \{2,3,4,...\}$, Stancu operators are defined by

$$L_n(f;x) = \int_0^\infty f(t)\frac{t^{nx-1}}{B(nx,n+1)\,(1+t)^{nx+n-1}}dt,$$

with the Euler beta function B. Introduce the family of Stancu operators by

$$L_n^*(f;x) = L_n(f,u_n(x)), \quad x \in I,\ n \geq n_0$$

where (u_n) satisfies (2).

Corollary 3.10. *For any $f \in Lip_M^{(1,1)}(\alpha)$, $\alpha \in (0,1]$, and for every $x \in (0,\infty)$, $n \in \{2,3,4,...\}$, we have*

$$\left|L_n^*(f;x) - f(x)\right| \leq \frac{M}{(x^2+x)^{\alpha/2}}\left\{(x-u_n(x))^2 + \frac{u_n^2(x)+u_n(x)}{n-1}\right\}^{\alpha/2}.$$

The following global result holds true:

Corollary 3.11. *For any $f \in Lip_M^{(1,1)}(\alpha)$, $\alpha \in (0,1]$, we have*

$$\left|L_n(f;x) - f(x)\right| \leq \frac{M}{(n-1)^{\alpha/2}},$$

uniformly for $x \in (0,\infty)$ as $n \to \infty$.

Corollary 3.12. *For any $f \in C_B[0,\infty)$ and for every $x \in [0,\infty)$, $n \in \mathbb{N}$, we have*

$$\left|L_n^*(f;x) - f(x)\right| \leq C\omega_2\left(f, \sqrt{(x-u_n(x))^2 + \frac{u_n^2(x)+u_n(x)}{n-1}}\right) + \omega(f, x-u_n(x))$$

for some constant $C := C(x) > 0$, where ω denotes the usual modulus of continuity of f.

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