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On one-factorizations of replacement products

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Abstract. Let G be an (n, m)-graph (n vertices and m-regular) and H be an (m, d)-graph. Randomly number the edges around each vertex of G by $\{1, \ldots, m\}$ and fix it. Then the replacement product $G \circledR H$ of graphs G and H (with respect to the numbering) has vertex set $V(G \circledR H) = V(G) \times V(H)$ and there is an edge between (v, k) and (w, l) if $v = w$ and $kl \in E(H)$ or $vw \in E(G)$ and kth edge incident on vertex v in G is connected to the vertex w and this edge is the lth edge incident on w in G, where the numberings k and l refers to the random numberings of edges adjacent to any vertex of G. If the set of edges of a graph can be partitioned to a set of complete matchings, then the graph is called 1-factorizable and any such partition is called a 1-factorization. In this paper, 1-factorizability of the replacement product $G \circledR H$ of graphs G and H is studied. As an application we show that fullerene C_{60} and C_4C_8 nanotorus are 1-factorizable.

1. Introduction

Graphs considered in this paper are finite, simple and undirected. Let G be a simple graph. The vertex set, edge set, maximum degree and minimum degree of G will be denoted by $V(G)$, $E(G)$, $\Delta(G)$, $\delta(G)$ respectively. The order of G is $|V(G)|$, i.e. the number of vertices of G. A graph G is called regular if $\delta(G) = \Delta(G)$ and the latter integer is called the degree of G and denoted by $d(G)$. A d-regular graph G on n vertices is called an (n, d)-graph. A graph is called cubic if it is 3-regular. We denote by K_n and C_n the complete graph of order n and the cycle of order n, respectively. We denote by $N_G(x)$ the neighborhoods of $x \in V(G)$, that is, the set of vertices adjacent to x.

A spanning subgraph of a graph G is a subgraph H of G such that $V(H) = V(G)$. An r-regular spanning subgraph of G is called an r-factor. A 1-factorization of G is a set of edge-disjoint 1-factors of G whose union is $E(G)$. The graph G is said to be 1-factorizable if it has a 1-factorization. A necessary condition for a graph G to be 1-factorizable is that G is a regular graph of even order. The concept of 1-factorization can be expressed by the concept of edge coloring. An edge coloring of a graph G is a map $\theta : E(G) \to C$, where C is a set, called the color set, and $\theta(e) \neq \theta(f)$ for any pair e and f of adjacent edges of G. If α is a color, the set $\theta^{-1}(\{\alpha\})$, i.e. the set of edges of G colored α, is called the α-color class. If $|C| = m$ we say that θ is an m-edge coloring. The least integer m for which an m-edge coloring of G exists is called the edge chromatic index of G and denoted by $\chi'(G)$. It is easily seen that $\chi'(G) \geq \Delta(G)$ for any graph G. Vizing [9] proved that

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Theorem 1.1 (Vizing's Theorem). *$\chi^{'}(G) \leq \Delta(G) + 1$ for any graph G.*

If in a regular graph G, we have $\chi^{'}(G) = \Delta(G)$ then edges of each color class consists a 1-factor and G is 1-factorizable. We call a graph Hamiltonian if it contains a spanning cycle. Such a cycle is called a Hamiltonian cycle. Any Hamiltonian graph of even order has a 1-factor, for if C is a Hamiltonian cycle of G and we select alternate edges of C, we eventually end up with a 1-factor of G. A 2-factor all of whose components are even is called an even 2-factor.

We shall also need the following well known sufficient condition for the existence of a Hamilton cycle in a graph due to Dirac [4].

Theorem 1.2 (Dirac's Theorem). *Let G be a graph of order at least three such that $\delta(G) \geq \frac{|V(G)|}{2}$. Then G is Hamiltonian.*

Let A and B be finite groups. Assume that B acts on A, namely we are fixing a homomorphism ϕ from B to the automorphism group of A and for elements $a \in A, b \in B$ we denote by a^b the element a^{b^ϕ} the action of b on a. We also use a^B to denote the orbit of a under this action.

The Cayley graph $C(H, S)$ of a group H and a generating set $S = S^{-1} := \{s^{-1} \mid s \in S\}$, is an undirected graph whose vertices are the elements of H, and where $\{g, h\}$ is an edge if $g^{-1}h \in S$. The Cayley graph $C(H, S)$ is $|S|$-regular.

2. 1-Factorizations of replacement products

In this section we describe the replacement product and investigate 1-factorizability of replacement product.

Let G be any (n, k)-graph and let $[k] = \{1, \ldots, k\}$. By a random numbering of G we mean a random numbering of the edges around each vertex of G by the numbers in $\{1, \ldots, k\}$. More precisely, a random numbering of G is a set φ_G consisting of bijection maps $\varphi_G^x : N_G(x) \to [k]$ for any $x \in V(G)$. Thus the graph G has $(k!)^n$ random numberings.

Example 2.1. Suppose $G = C(A, S)$ is a Cayley graph. Then the edges around each vertex of G are naturally labeled by the elements of S: if $\{x, y\} \in E(G)$ then $\varphi_G^x(y) = f(x^{-1}y)$, where f is a bijection map from S to $[|S|]$.

Definition 2.2. *Let G be an (n, k)-graph and let H be a (k, k')-graph with $V(H) = [k] = \{1, \ldots, k\}$ and fix a random numbering φ_G of G. The replacement product $G®_{\varphi_G}H$ is the graph whose vertex set is $V(G) \times V(H)$ and there is an edge between vertices (v, k) and (w, l) whenever $v = w$ and $kl \in E(H)$ or $vw \in E(G)$, $\varphi_G^v(w) = k$ and $\varphi_G^w(v) = l$.*

Note that the definition of $G®_{\varphi_G}H$ clearly depends on φ_G. Thus for given any two regular graphs G and H as above, there are $(|V(H)|!)^{|V(G)|}$ replacement products which are not necessarily isomorphic.
It follows from the definition that $G®_{\varphi_G}H$ is a regular graph and in fact it is a $(nk, k' + 1)$-graph (see [1, 7, 8]).

Example 2.3. Let $G = K_5$ and $H = C_4$ and let φ_G, φ'_G be as shown in Figure 1. Then the replacement product $G®_{\varphi_G}H$ and $G®_{\varphi'_G}H$ are not isomorphic, (see Figure 2) since the determinants of the adjacency matrices of $G®_{\varphi_G}H$ and $G®_{\varphi'_G}H$ are 9 and 12, respectively.

Lemma 2.4. *Let G be an (n, k)-graph. Then G is 1-factorizable if and only if there exists a random numbering φ_G such that for any edge xx' of G we have $\varphi_G^x(x') = \varphi_G^{x'}(x)$.*

Proof. ⇒: Let G have a 1-factorization. Then, since G is k-regular, there exists an edge coloring $\theta : E(G) \to [k]$ for G. Now define the map φ_G^x from $N_G(x)$ to $[k]$ for any $x \in V(G)$ as $\varphi_G^x(y) = \theta(xy)$ for all $y \in N_G(x)$. Clearly φ_G^x is a bijection and we have

$$\varphi_G^x(x') = \theta(xx') = \theta(x'x) = \varphi_G^{x'}(x)$$

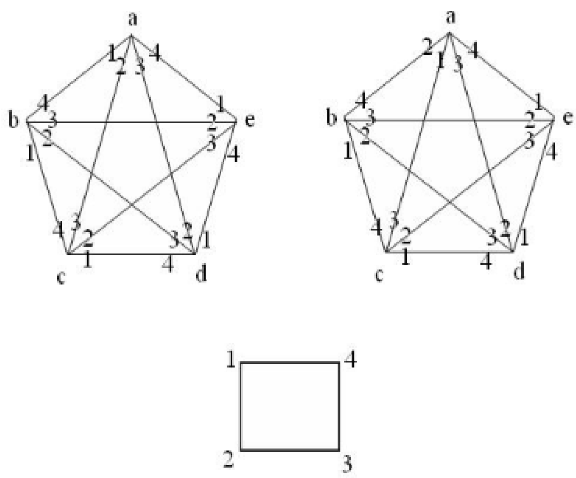

Figure 1: $G = K_5$ and $H = C_4$

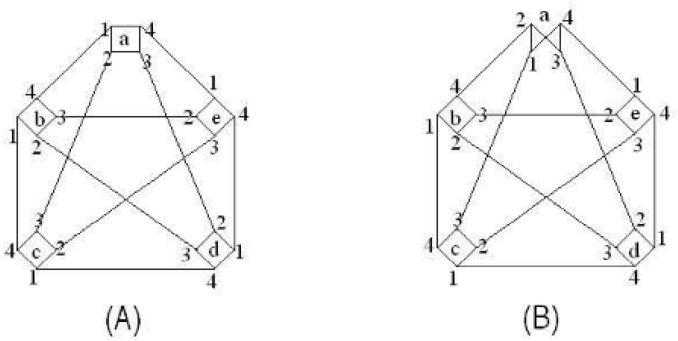

Figure 2: (A) is $G \circledR_{\varphi_G} H$ and (B) is $G \circledR_{\varphi'_G} H$

for any $xx' \in E(G)$. Thus $\varphi_G = \{\varphi_G^x \mid x \in V(G)\}$ is the desired random numbering of G.

$\Leftarrow$: Suppose that G has a random numbering φ_G with the mentioned property in the lemma. Let β be the map from $E(G)$ to $[k]$ defined by $\beta(xy) = \varphi_G^x(y)$ for any $xy \in E(G)$. Since $\varphi_G^x(y) = \varphi_G^y(x)$ ($\beta(xy) = \beta(yx)$), then β is well-defined. Now we show that β is an edge coloring for G. To do this, suppose e, f be two adjacent edges of $E(G)$. Then there exist $x, y, y' \in V(G)$ with $e = xy$, $f = xy'$ and $y \neq y'$. Hence $\varphi_G^x(y) \neq \varphi_G^x(y')$, as φ_G^x is one-to-one. This implies that $\beta(e) \neq \beta(f)$, as required. □

If G is a 1-factorizable graph, a random numbering φ_G is called a 1-factorizable numbering whenever for any edge xx' of G we have $\varphi_G^x(x') = \varphi_G^{x'}(x)$. Thus it follows from Lemma 2.4 that every 1-factorizable graph has a 1-factorizable numbering.

We now consider the case when the two components of the product graph are Cayley graphs $G = C(A, S_A)$ and $H = C(B, S_B)$. Furthermore, suppose that B acts on A in such a way that $S_A = \alpha^B$ for some $\alpha \in S_A$. So the edges around each vertex of G are naturally labeled by the elements of B. This enables us to define the replacement product of G and H.

The following result is well-known and we give a proof by using Lemma 2.4.

Lemma 2.5. *Let G be a cubic graph. Then the following properties are equivalent:*
(i) G is 1-factorizable.
(ii) G has an even 2-factor.
(iii) $G®_{\varphi_G}C_3$ is 1-factorizable for any 1-factorizable numbering φ_G.

Proof. (i) ⇔ (ii) Suppose that G is 1-factorizable. Then, since G is 3-regular, there exists an edge coloring $\theta : E(G) \to [3]$. For any two colors α, β, the subgraph generated by the union of α- and β-color classes is a spanning subgraph of G which is the union of some cycle subgraphs $C_1, \dots, C_s$ of G with disjoint vertices and even length.

If G has an even 2-factor then G contains a spanning subgraph $\bigcup_{i=1}^{s} C_i$ where C_i are disjoint cycles of even length. Thus C_i are 1-factorizable and by Lemma 2.4, there exists a random numbering φ_{C_i} such that for any edge ab of C_i we have $\varphi^a_{C_i}(b) = \varphi^b_{C_i}(a)$. Now define φ^x_G for each $x \in V(G)$ as follows:

$$\varphi^x_G(y) = \begin{cases} \varphi^x_{C_i}(y) & xy \in E(C_i) \\ 3 & \text{otherwise.} \end{cases}$$

It is easy to see that φ_G is a random numbering of G satisfying the conditions of Lemma 2.4.

(i) ⇔ (iii) We know G is 1-factorizable if only if there exists a 1-factorizable numbering φ_G such that for any $xy \in E(G)$ we have $\varphi^x_G(y) = \varphi^y_G(x)$. Assume G is 1-factorizable and φ_G is a 1-factorizable numbering. If $V(C_3) = [3] = \{1, 2, 3\}$ then we define $\varphi^{(x,i)}_{G®_{\varphi_G}C_3}$ for $(x, i) \in V(G®_{\varphi_G}C_3)$ as follows:

$$\varphi^{(x,i)}_{G®_{\varphi_G}C_3}((y, j)) = \begin{cases} [3]\text{-}\{i,j\} & x = y, ij \in E(C_3) \\ \varphi^x_G(y) & xy \in E(G) \end{cases},$$

where $(y, j) \in N_{G®_{\varphi_G}C_3}((x, i))$. It is obvious that $\varphi^{(x,i)}_{G®H}((y, j)) = \varphi^{(y,j)}_{G®H}((x, i))$. Now Lemma 2.4 completes the proof. □

Corollary 2.6. *Let G be a 1-factorizable graph and φ_G is 1-factorizable numbering of G. Assume also that G^1 denotes $G®_{\varphi_G}C_3$ and recursively let $G^r = G^{r-1}®_{\varphi_{G^{r-1}}}C_3$. Then $\{G^r\}_{r\geq 1}$ is an infinite family of 1-factorizable cubic graphs.*

Theorem 2.7. *Let G be an (n, k)-graph and let H be a (k, k')-graph with $V(H) = [k]$. If H has a 1-factorization, then the replacement product $G®_{\varphi_G}H$ is 1-factorizable for any random numbering φ_G of G.*

Proof. Let $G®H = G®_{\varphi_G}H$. By Lemma 2.4, there exists a set $\varphi_H = \{\varphi^a_H : N_H(a) \to [k'] \mid a \in V(H)\}$ such that for any $ab \in E(H)$ we have $\varphi^a_H(b) = \varphi^b_H(a)$. Now define $\varphi^{(x,a)}_{G®H}$ for any $(x, a) \in V(G®H)$ and $(y, b) \in N_{G®H}(x, a)$ as follows:

$$\varphi^{(x,a)}_{G®H}((y, b)) = \begin{cases} \varphi^a_H(b) & x = y \\ k' + 1 & \text{otherwise.} \end{cases}$$

It is obvious that $\varphi^{(x,a)}_{G®H}((y, b)) = \varphi^{(y,b)}_{G®H}((x, a))$. Now Lemma 2.4 completes the proof. □

Example 2.8. *Let C_n be a cycle of length n. Then for any random numbering φ_{C_n}, we have*

$$C_n®_{\varphi_{C_n}}K_2 \cong C_{2n}$$

If n is odd then C_n is not 1-factorizable but $C_n®_{\varphi_{C_n}}K_2 \cong C_{2n}$ is 1-factorizable. If n is even then C_{n-1} is not 1-factorizable. Therefore for any random numbering φ_{K_n}, the graph $K_n®_{\varphi_{K_n}}C_{n-1}$ is cubic having an even 2-factor. Then by Lemma 2.5, $K_n®_{\varphi_{K_n}}C_{n-1}$ is a 1-factorizable graph.

Theorem 2.9. *Let G be an (n, m)-graph. Then the replacement product $G®_{\varphi_G}K_m$ is 1-factorizable whenever one of the following conditions holds:*
(a) m is even and φ_G is any random numbering of G.
(b) G is 1-factorizable and φ_G is any 1-factorizable numbering of G.

Proof. The complete graph K_m is 1-factorizable whenever m is even. Then by Theorem 2.7, G®$_{\varphi_G}K_m$ is 1-factorizable for any random numbering φ_G of G. If m is odd and G is 1-factorizable then by Lemma 2.4, there exists a random numbering φ_G such that for $x \in V(G)$ and $y \in N_G(x)$, we have $\varphi_G^x(y) = \varphi_G^y(x)$. For any $(x, i) \in V(G$®$_{\varphi_G}K_m)$ and $(y, j) \in N_{G®_{\varphi_G}K_m}((x, i))$, where $i, j \in V(K_m) = [m]$, we define $\varphi^{(x,i)}_{G®_{\varphi_G}K_m}$ as follows:

$$\varphi^{(x,i)}_{G®_{\varphi_G}K_m}((y, j)) = \begin{cases} \varphi_G^x(y) & xy \in E(G) \\ \frac{\overline{i+j}}{2} & x = y \text{ and } \overline{i+j} \text{ is non}-\text{zero and even} \\ m & x = y \text{ and } \overline{i+j} = 0 \\ \frac{\overline{i+j}+m}{2} & x = y \text{ and } \overline{i+j} \text{ is odd} \end{cases},$$

where $\overline{i+j}$ is the remainder when $i + j$ is divided by m. It is easy to see that $\varphi^{(x,i)}_{G®K_m}((y, j)) = \varphi^{(y,j)}_{G®K_m}((x, i))$. This completes the proof. □

For the replacement product G®C_k, $k \geq 4$, we derive the following result.

Theorem 2.10. *Let G be an (n, k)-graph and C_k a cycle of length $k \geq 4$. Then the replacement product G®$_{\varphi_G}C_k$ is 1-factorizable if one of the following conditions holds:*

(a) k is even and φ_G is any random numbering of G

(b) G has an even 2-factor $H = \bigcup_{i=1}^{s} C_i$ and $\varphi_G |_{C_i} = \varphi_{C_i}$, where φ_{C_i} are 1-factorizable numbering of C_i.

Proof. If k is even then $C_k = (123 \ldots k)$ has 1-factorization and by Theorem 2.7, G®$_{\varphi_G}C_k$ is 1-factorizable. Let G have an even 2-factor. Thus there exist disjoint cycles $C_i (1 \leq i \leq s)$ with even lengths, whose $\bigcup_{i=1}^{s} C_i$ is a spanning subgraph of the graph G. Cycles $C_i = (a_{i1}a_{i2} \ldots a_{in_i})$, $1 \leq i \leq s$, are 1-factorizable and $n = \sum_{i=1}^{s} n_i$. Because each n_i is even then by Lemma 2.4, there are φ_{C_i} where for any edge a_{il}, a_{im} of C_i we have $\varphi_{C_i}^{a_{il}}(a_{im}) = \varphi_{C_i}^{a_{im}}(a_{il})$. Let $\varphi_{C_i}^{a_{i1}}(a_{i2}) = \varphi_{C_i}^{a_{i2}}(a_{i1}) = 1$ thus we define cycle T_i as follows:

$$\begin{aligned} T_i \ = \ & ((a_{i1}, 1)(a_{i2}, 1)(a_{i2}, k)(a_{i2}, k-1) \ldots (a_{i2}, 3)(a_{i2}, 2)(a_{i3}, 2)(a_{i3}, 3) \\ & \ldots (a_{i3}, k-1)(a_{i3}, k)(a_{i3}, 1) \ldots (a_{in_i}, 1)(a_{in_i}, k)(a_{in_i}, k-1) \\ & \ldots (a_{in_i}, 3)(a_{in_i}, 2)(a_{i1}, 2)(a_{i1}, 3) \ldots (a_{i1}, k-1)(a_{i1}, k)) \end{aligned}$$

T_i are cycles of length kn_i and $\bigcup_{i=1}^{s} T_i$ is a spanning subgraph of the replacement product G®$_{\varphi_G}C_k$, where $\varphi_G |_{C_i} = \varphi_{C_i}$. Then by Lemma 2.5, G®$_{\varphi_G}C_k$ has 1-factorization. □

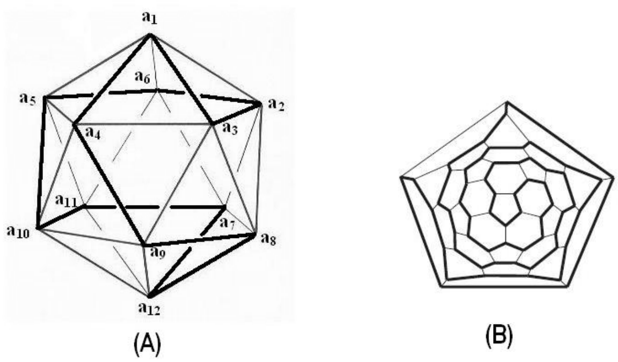

Figure 3: (A) is Icosahedron and (B) is Fullerene C_{60}

Corollary 2.11. *Let G be an (n,k)-graph and $2k \geq n$. Then by Dirac's Theorem, the replacement product $G Ⓡ C_k$ is 1-factorizable.*

Let us end the paper with two applications of the above result.

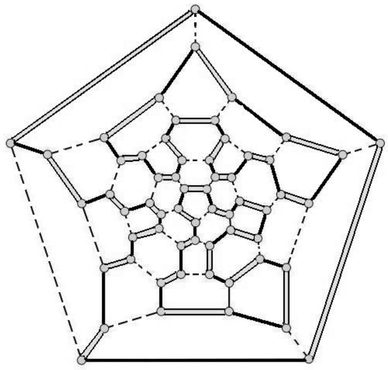

Figure 4: Coloring the edges of C_{60}.

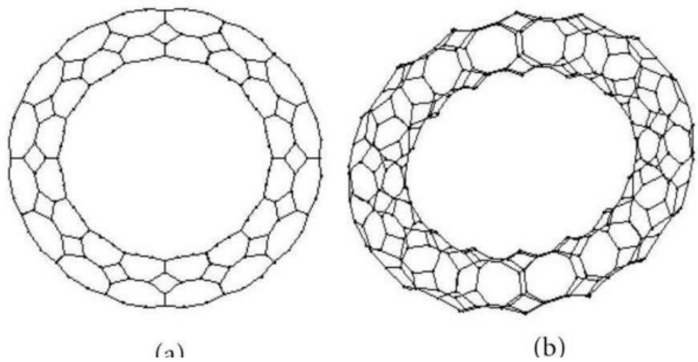

Figure 5: A $TC_4C_8(R)$ tori (a) Top view (b) Side view

Example 2.12. *A fullerene graph (in short a fullerene) is a 3-connected cubic planar graph, all of whose faces are pentagons and hexagons. By Euler formula the number of pentagons equals 12. The first fullerenes, C_{60} and C_{70}, were isolated in 1990. The smaller version, C_{60}, is in the shape of a soccer ball. Graph-theoretic observations on structural properties of fullerenes are important in this respect [3, 5, 6]. Suppose F is a fullerene C_{60} and A is the icosahedron ((12, 5)-graph). By Figure 3(A), $C = (a_1a_3a_2a_6a_5a_{10}a_{11}a_7a_{12}a_8a_9a_4)$ is a Hamiltonian cycle of A and A is hamiltonian. If φ_A for any edges a_ia_j are defined in Table I, then $A Ⓡ_{\varphi_A} C_5 = F$ is Hamiltonian, see Figure 3(B). By Lemma 2.5, F is 1-factorizable. There are many different ways to color the edges of the graph C_{60} (for example see Figure 4).*

Example 2.13. *Consider C_4 nanotorus S. Then $S = C_k \times C_m$, where C_n is the cycle of order n and S be an $(km, 4)$-graph (S is the Cartesian product of cycles). Suppose T is a $T = TC_4C_8[k, m](R)$ nanotorus (which k is the number of squares in every row and m is the number of squares in every column, Figure 5), [2]. Then T is an $(4km, 3)$-graph. It is easy to see that $T = S Ⓡ_{\varphi_S} C_4$, for some of random numbering φ_S of graph S. Because C_4 is 1-factorizable then by Theorem 2.10, nanotorus T is 1-factorizable.*

1	2	3	4	5
$\varphi_A^{a_1}(a_4)$	$\varphi_A^{a_1}(a_3)$	$\varphi_A^{a_1}(a_2)$	$\varphi_A^{a_1}(a_6)$	$\varphi_A^{a_1}(a_5)$
$\varphi_A^{a_4}(a_{10})$	$\varphi_A^{a_4}(a_9)$	$\varphi_A^{a_4}(a_3)$	$\varphi_A^{a_4}(a_1)$	$\varphi_A^{a_4}(a_5)$
$\varphi_A^{a_3}(a_9)$	$\varphi_A^{a_3}(a_8)$	$\varphi_A^{a_3}(a_2)$	$\varphi_A^{a_3}(a_1)$	$\varphi_A^{a_3}(a_4)$
$\varphi_A^{a_2}(a_8)$	$\varphi_A^{a_2}(a_7)$	$\varphi_A^{a_2}(a_6)$	$\varphi_A^{a_2}(a_1)$	$\varphi_A^{a_2}(a_3)$
$\varphi_A^{a_6}(a_7)$	$\varphi_A^{a_6}(a_{11})$	$\varphi_A^{a_6}(a_5)$	$\varphi_A^{a_6}(a_1)$	$\varphi_A^{a_6}(a_2)$
$\varphi_A^{a_5}(a_{11})$	$\varphi_A^{a_5}(a_{10})$	$\varphi_A^{a_5}(a_4)$	$\varphi_A^{a_5}(a_1)$	$\varphi_A^{a_5}(a_6)$
$\varphi_A^{a_{12}}(a_{11})$	$\varphi_A^{a_{12}}(a_7)$	$\varphi_A^{a_{12}}(a_8)$	$\varphi_A^{a_{12}}(a_9)$	$\varphi_A^{a_{12}}(a_{10})$
$\varphi_A^{a_{11}}(a_6)$	$\varphi_A^{a_{11}}(a_6)$	$\varphi_A^{a_{11}}(a_7)$	$\varphi_A^{a_{11}}(a_{12})$	$\varphi_A^{a_{11}}(a_{10})$
$\varphi_A^{a_7}(a_5)$	$\varphi_A^{a_7}(a_2)$	$\varphi_A^{a_7}(a_8)$	$\varphi_A^{a_7}(a_{12})$	$\varphi_A^{a_7}(a_{11})$
$\varphi_A^{a_8}(a_4)$	$\varphi_A^{a_8}(a_3)$	$\varphi_A^{a_8}(a_9)$	$\varphi_A^{a_8}(a_{12})$	$\varphi_A^{a_8}(a_7)$
$\varphi_A^{a_9}(a_3)$	$\varphi_A^{a_9}(a_4)$	$\varphi_A^{a_9}(a_{10})$	$\varphi_A^{a_9}(a_{12})$	$\varphi_A^{a_9}(a_8)$
$\varphi_A^{a_{10}}(a_2)$	$\varphi_A^{a_{10}}(a_5)$	$\varphi_A^{a_{10}}(a_{11})$	$\varphi_A^{a_{10}}(a_{12})$	$\varphi_A^{a_{10}}(a_9)$

Table 1: Table I

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