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A New Characterization of Line-to-line Maps in the Upper Plane

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Abstract. The characterization of typical maps in a domain of a given space is a much harder problem than that in the whole space. In this paper, by using methods of hyperbolic and affine geometry, we give a new characterization of line-to-line maps in the upper plane. We show that a line-to-line surjection is either an affine transformation, or a composition of an affine transformation and a g−reflection. Moreover, we prove that the composition of two g−reflections with the same boundary is an affine transformation.

1. Introduction and The Main Result

Suppose that $\mathcal{D} \subset \mathbb{R}^2$ is a domain. We say that ℓ is a *line* in $\mathcal{D}$ if there is a straight line $s \subset \mathbb{R}^2$ such that $\ell = s \cap \mathcal{D}$, and a map

$$f : \mathcal{D} \mapsto \mathcal{D}$$

is *line-to-line* if the image of each line in $\mathcal{D}$ is contained in a line of $\mathcal{D}$.

The line-to-line maps have been investigated for a long time and there are many papers in literature. Among them, the following results are due to Artin and Jeffers, respectively [2, 10].

Theorem A. *[2] Suppose that $f : \mathbb{R}^n \mapsto \mathbb{R}^n$ $(n > 1)$ is a bijection and preserves lines, and suppose that the images of any two parallel lines under f are still parallel lines, then f is an affine transformation.*

Here, f is said to preserve lines if the image of each line is still a line.

Theorem B. *[10, Theorem 4.5] Suppose that $f : \mathbb{R}^n \mapsto \mathbb{R}^n$ $(n > 1)$ is a bijection and preserves lines. Then f is an affine transformation.*

In [5], Chubarev and Pinelis show that the condition "f being injective" in Theorems A and B can be removed, and the condition "f preserving lines" can be replaced by the one "f being line-to-line". Precisely, we have the following.

Theorem C. *[5] Suppose that $f : \mathbb{R}^n \mapsto \mathbb{R}^n$ $(n > 1)$ is a line-to-line surjection. Then f is an affine transformation.*

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In [11], the authors proved

Theorem D. *[11, Theorem 3] Suppose that* $f : \mathbb{R}^n \mapsto \mathbb{R}^n$ $(n > 1)$ *preserves lines. Then* f *is an affine transformation if and only if it is non-degenerate.*

Here, a line-preserving map $f : \mathbb{R}^n \mapsto \mathbb{R}^n$ $(n > 1)$ is *non-degenerate* if the image $f(\mathbb{R}^n)$ is not contained in a line.

In [12], the authors introduce a new class of line to line map on $\mathbb{R}^2\backslash\mathcal{L}_b$, g-(triangle)-reflection ϕ, which is affinely conjugated to the following form:

$$\eta : (x, y) \mapsto (-\frac{x}{1 + Kx}, \frac{y}{1 + Kx}), (K \neq 0).$$

Any g−reflection has a fixed-point line $\mathcal{L}_a$, called ***axis***, an only isolated fixed point $\mathcal{P}_0$, called ***base*** point, an undefined line $\mathcal{L}_b$, called ***boundary*** and two invariant domains Ω_1, Ω_2, and $\mathbb{R}^2\backslash\mathcal{L}_b = \Omega_1 \cup \Omega_2$.

The characterization of typical maps in a domain of a given space is a much harder problem than that in the whole space. In this paper, we mainly consider the transformations that preserves the upper plane. The g-reflection with the boundary $\mathcal{L}_b = \mathcal{X} = \{((x, y) \in \mathbb{R}^2 | y = 0)\}$, would have the following form

$$\eta_{(K,a)} : (x, y) \mapsto (\frac{x - a}{Ky} + a, \frac{1}{K^2 y}), \quad (K \neq 0, a \in \mathbb{R})$$

which is determined by the base point $(a, -\frac{1}{K})$.

By using methods of hyperbolic and affine geometry, we shall prove the following

Theorem 1.1. *Suppose that* $f : \mathbb{H} \mapsto \mathbb{H}$ *is a line-to-line surjection. Then either* f *is an affine transformation, or* $f = A \circ \eta$*, where* $A : \mathbb{H} \mapsto \mathbb{H}$ *is an affine transformation, and* $\eta : \mathbb{H} \mapsto \mathbb{H}$ *is a* g*-reflection.*

Theorem 1.2. *Suppose that* η_1, η_2 *are two* g*-reflections with the same boundary. Then the composition* $\eta_1 \circ \eta_2$ *is an affine transformation.*

2. The Proof of Theorem 1.1

For the sake of convenience, in the following, we always use A, B, $\cdots$ to denote the points in $\mathbb{H}^2$, l or L a line in $\mathbb{H}$, L_{AB} the line determined by A and B, AB the segment with the endpoints A and B. Primes will denote images under the function we consider.

As in [12], we say a line l in $\mathbb{H}$ is complete, if l is a complete line in $\mathbb{R}^2$, denoted by $L_{y=k}$ for some $k > 0$. Obviously, there exists only one complete line passing through a given point P, denoted by L_P. For any line l in $\mathbb{H}$, either it is complete, or it crosses any complete lines in $\mathbb{H}$.

In this section, we shall prove Theorem 1.1. If f is an affine transformation, it is obvious. So we suppose that f is not an affine transformation in the following.

Lemma 2.1. *f is line onto line.*

Proof. Suppose that f is not line onto line. There exists a line l, such that $f(l) \subset l'$, and $l' \backslash f(l) \neq \emptyset$. Since f is onto. One can choose $Q \notin l$, such that $Q' \in l' \backslash f(l)$.

If l is not a complete line, $L_Q \cap l \neq \emptyset$.

If l is a complete line, choose any point $P \in l$, and get L_{PQ}.

All above, we can get a complete line and a non-complete line, all of their images are contained in l'.

On the other hand, choose $A', B' \in \mathbb{H}$, such that $L_{A'B'} \cap l' = \emptyset$. Let A, B be their inverse images. One can find L_{AB} will cross at least one of the complete line and the non-complete line, which is impossible. This contradiction complete the proof. □

Lemma 2.2. *For any complete line, the image is complete. Moreover, the image of any non-complete line is non-complete.*

Proof. Suppose that l is a complete line, and $l' = f(l)$ is non-complete. There exist three non-collinear points A, B, C, such that $l_{A'B'} \cap l' = l_{A'C'} \cap l' = \emptyset$. Their inverse images A, B, C are non-collinear, and $l_{AB} \cap l = l_{AC} \cap l = \emptyset$, which is impossible since l is complete.

Suppose that l is not complete, and l' is complete. For any complete line L, L' is complete, and $L \cap l \neq \emptyset$, denoting the cross point by P. One can find that $L' = L_{P'}$, $P' \in l'$, so $L' = l'$. From which we can obtain that $f(\mathbb{H}) = l'$, this is a contradiction.

Above all, we complete the proof. □

Lemma 2.3. *f is injection.*

Proof. Suppose that f is not injection. There exist P_1, P_2, such that $f(P_1) = f(P_2) = P'$.

Case I. $L_{P_1P_2}$ is not complete. By Lemma 2.2, $f(L_{P_1P_2})$ is not complete. Choose $Q \in L_{P_1}$, such that $P' \neq Q'$. Then $L_{P'Q'} = f(L_{P_1Q})$ is a complete line. One the other hand, L_{P_2Q} is non-complete, and $L_{P'Q'} = f(L_{P_2Q})$ is complete, this is a contradiction.

Case II. $L_{P_1P_2}$ is complete. $L_{P'} = f(L_{P_1P_2})$ is a complete line. Choose Q, such that $Q' \notin L_{P'}$.

Choose Q_2, such that $Q_2' \notin L_{P'Q'} \cup L_{P'}$. Then $L_{P_1Q_2} \cap L_{QP_2}$ or $L_{P_1Q} \cap L_{P_2Q_2}$ must be not all empty set, denoting the cross point by P_3. Obviously, $P_3 \notin L_{P_1P_2}$ and $f(P_3) = P'$. $L_{P_1P_3}$ is not a complete line. By case I, this is a contradiction.

Therefore we complete the proof. □

Lemma 2.4. *f is order-preserving.*

Proof. Suppose that f is not order-preserving. There are some line l and three points $A, B, C \in l$, B is between A, C, while B' is not between A', C'. As in Figure I, we suppose that A' is between B', C'. Then one can choose three parallel non-complete lines, passing through A', B', C', respectively. Denoted by l_1, l_2, l_3. Then we can get $E \in l_2$, $F \in l_3$, such that $l_{EF} \cap l_1 = \emptyset$. On the other hand, $l_{E'F'} \cap l_1 \neq \emptyset$.

This contraction completes the Lemma. □

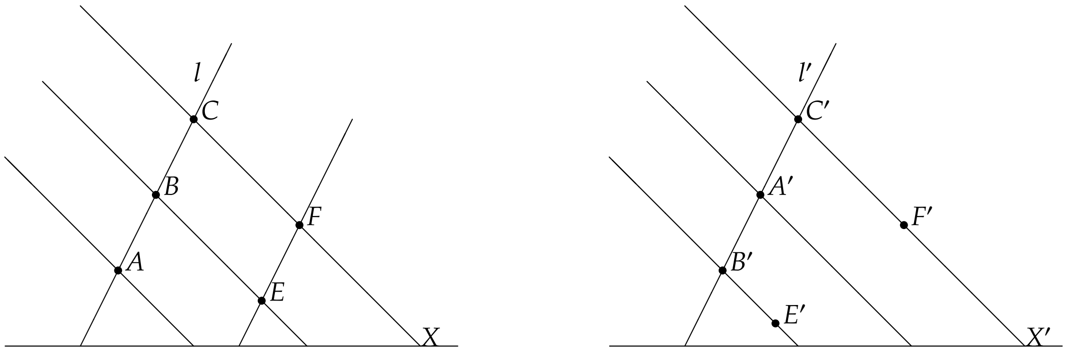

Figure I

By composing some suit affine transformation(preserving $\mathbb{H}$), we always suppose that f fixes $(0, 1), (1, 1)$ and the line $L_{x=0} = \{(x, y) \in \mathbb{H} | x = 0\}$ in the following. Denote the image of $(0, 2)$ by $(0, a)$. We also suppose that $a > 1$. Otherwise, we can compose the $g-$reflection:

$$\eta_{(1,0)} : (x, y) \mapsto (\frac{x}{y}, \frac{1}{y}).$$

Lemma 2.5. *f is parallel-preserving. That is, for any two parallel lines l_1, l_2, the image lines l'_1, l'_2 are parallel.*

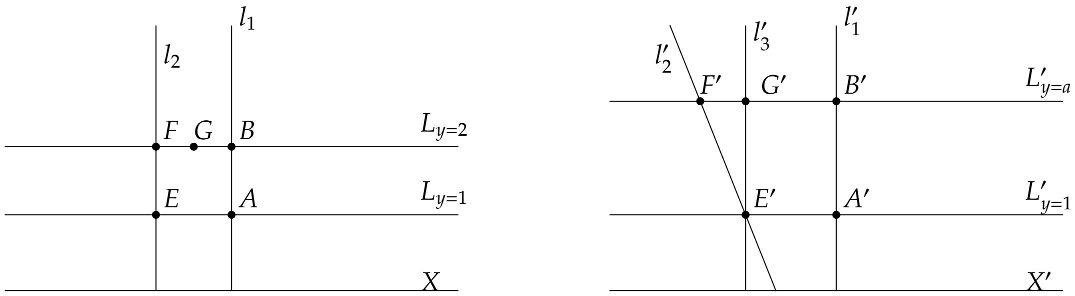

Figure II

Proof. By Lemma 2.2, if l_1, l_2 are complete, their image lines are complete. They are parallel obviously. So we can assume that l_1, l_2 are not complete. For the contradiction, we suppose that l_1 and l_2 are parallel to each other, and their image l'_1 and l'_2 are not(As in Figure II). Let A, B, E, F denote the intersection points of l_1, l_2, and $L_{y=1}, L_{y=2}$. Since $l'_1 \cap l'_2 = \emptyset$. So there exists $G' \in B'F'$, such that $L_{E'G'}$ is parallel to l'_1. On the other hand, $G \in BF$, $L_{EG} \cap l_1 \neq \emptyset$. This is a contradiction, which complete the proof. □

As in Figure *III*, denote $A(0, 1), B(0, 2), P(1, 1), A'(0, 1), B'(0, a), P'(1, 1)$. Denote $\tau = a - 1 > 0$. Since f is parallel-preserving, we can get the image $Q'(1, 1 + \tau)$ of $Q(1, 2)$ by $L_{PQ} \parallel L_{AB}$. $L_{BE} \parallel L_{AQ}$ and $L_{BE} \cap L_{AP} = E(-1, 1)$. So $L_{B'E'} \parallel L_{A'Q'}$ and $L_{B'E'} \cap L_{A'P'} = E'(-1, 1)$. $L_{BE} \cap L_{PQ} = R(1, 3)$ and $L_{B'E'} \cap L_{P'Q'} = R'(1, 1 + 2\tau)$. And so it goes on, we can find the following proposition.

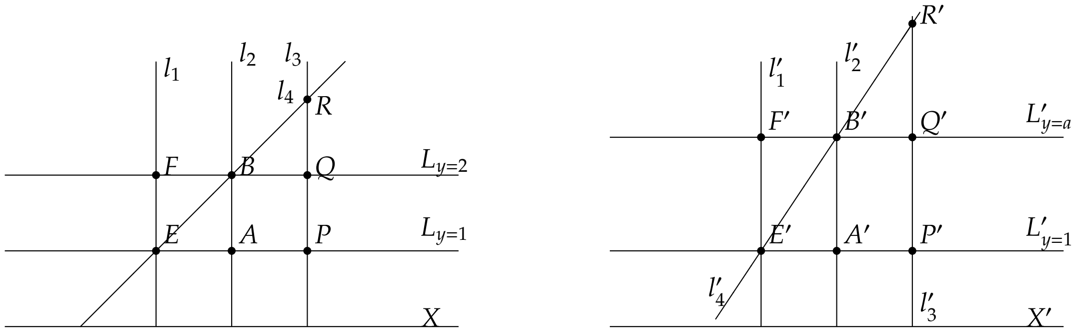

Figure III

Proposition 2.6. *For any whole number n_1 and positive whole number n_2, the point $P(n_1, 1+n_2)$, $f(P) = (n_1, 1+n_2\tau)$.*

Obviously, both of f and f^{-1} are line-onto-line bijections. So we can suppose that $a > 2$, that is $\tau > 1$. Otherwise, if $1 < a < 2$, we can consider f^{-1} instead of f. One can find two positive whole numbers n_1, n_2, such that $1 < \frac{n_2}{n_1} < \tau$. Then the line passing through $P_1(0, 1)$ and $P_2(n_1, 1 + n_2)$ will cross the line $L_{x=-1}$. While the line passing through $P'_1(0, 1)$ and $P'_2(n_1, 1 + n_2\tau)$ will not cross the line $L_{x=-1}$. This is the desired contradiction. That is $a = 2$. Moreover, f fixes any points in $L_{x=0}$. So we can obtain

Lemma 2.7. *Suppose that $f : \mathbb{H} \to \mathbb{H}$ is a line-to-line surjection, fixes $P_1(0, 1)$, $P_2(1, 1)$, and $f(P_3(0, 2)) = P'_3(0, a)$. If $a > 1$, then $f = id$.*

Lemma 2.7 show that Theorem 1.1 holds, and the following results can be got from Thoerem 1.1.

Corollary 2.8. *Suppose that $f : \mathbb{H} \mapsto \mathbb{H}$ is a line to line surjection, and $f = A \cdot \eta$, where $A : \mathbb{H} \mapsto \mathbb{H}$ is an affine transformation, and $\eta : \mathbb{H} \mapsto \mathbb{H}$ is a g-reflection. Moreover, there exist A' and η', such that $f = \eta' \cdot A'$.*

Corollary 2.9. *Suppose that $f : \mathbb{H} \to \mathbb{H}$ is a line-to-line surjection. If there exist two fixed points P, Q of f, then for any point $E \in L_{PQ}$, $f(E) = E$.*

Corollary 2.10. *Suppose that $f : \mathbb{H} \to \mathbb{H}$ is a line-to-line surjection. If there exist three non-collinear fixed points of f, then $f = id$.*

Corollary 2.11. *Suppose that $f : \mathbb{H} \to \mathbb{H}$ is a line-to-line surjection. If there exists some parallelogram $\mathfrak{P}$, such that $f(\mathfrak{P})$ is a parallelogram. f is an affine transformation.*

3. The Proof of Theorem 1.2

In this section, we shall prove Theorem 1.2 by computation. Conjugated by some suitable affine transformation, we suppose that g−reflections have the same boundary $\mathcal{L}_b = \mathcal{X}$. Then all of them preserve the half upper plane $\mathbb{H}^2$. In fact, any g−reflection preserving the half upper plane is determined by the base point $\mathcal{P}_0 = \{(a, -\frac{1}{K})\}$ for any real number $K \neq 0$ and a, which has the form

$$\eta_{(K,a)} : (x, y) \mapsto (\frac{x-a}{Ky} + a, \frac{1}{K^2 y}).$$

Denote the other g−reflection $\eta_{(K',a')}$,

$$\eta_{(K',a')} : (x, y) \mapsto (\frac{x-a'}{K'y} + a', \frac{1}{K'^2 y}).$$

The composition:

$$\eta_{(K',a')} \circ \eta_{(K,a)} : (x, y) \mapsto (\frac{K}{K'}x + \frac{a-a'}{K'}K^2 y + a' - \frac{aK}{K'}, \frac{K^2}{K'^2}y)$$

is an affine transformation. Therefore we complete the proof of Theorem 1.2.

Moreover, we have the following propositions

Proposition 3.1. *The affine transformation $\eta_{(K',a')} \circ \eta_{(K,a)}$ fixes some point P in the boundary of $\mathbb{H}$ in $\mathbb{R}^2$, if and only if the base points $\mathcal{P}_0^1(a, -\frac{1}{K})$ and $\mathcal{P}_0^2(a', -\frac{1}{K'})$ are collinear with P. That is $K \neq K'$. Moreover, if $K = -K'$, $\eta_{(K',a')} \circ \eta_{(K,a)}$ fixes any point in the line $L_{\mathcal{P}_0^1\mathcal{P}_0^2}$. If $K \neq \pm K'$, P is the only fixed point of $\eta_{(K',a')} \circ \eta_{(K,a)}$.*

Proposition 3.2. *If $K = K'$, the affine transformation $\eta_{(K',a')} \circ \eta_{(K,a)}$ fixes any point in the line $L = \{(x, y) \in \mathbb{R}^2 | y = \frac{1}{K}\}$.*

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