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Some fixed point results for generalized quasi-contractive multifunctions on graphs

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Abstract. Over the past few decades, there have been a lot of activity about combining fixed point theory and another branches in mathematics such differential equations, geometry and algebraic topology. In 2005, Echenique started combining fixed point theory and graph theory by giving a short constructive proof for the Tarski fixed point theorem by using graphs. In 2006, Espinola and Kirk started combining fixed point theory and graph theory. Recently, this field have been of great interest for fixed point theorists. In this paper, we give some fixed point results for generalized quasi-contractive multifunctions on graphs.

1. Introduction

In 2005, Echenique started combining fixed point theory and graph theory by giving a short constructive proof for the Tarski fixed point theorem by using graphs ([8]). In 2006, Espinola and Kirk applied fixed point results in graph theory ([9]). Later, some notable contributions were made by Jachymski in 2008 ([14]), O'Regan and Petrusel in 2008 ([19]), Jachymski in 2009 ([11]), Bojor in 2010 ([5]), Beg, Butt and Radojevic in 2010 ([4]), Espinola, Lorenzo and Nicolae in 2011 ([10]), Nicolae, O'Regan and Petrusel in 2011 ([18]), Bojor in 2012 ([6]) and Aleomraninejad, Rezapour and Shahzad ([1]). On the other hand, the notion of quasi-contractions provided by Ciric in 1974 ([7]). It has been published some papers about quasi-contractions, but we like to review here some recent published works. After providing some results on fixed points of quasi-contractions on normal cone metric spaces by Ilic and Rakocevic in 2009 ([13]), Kadelburg, Radonevic and Rakocevic generalized the results by considering an additional assumption and deleting the assumption on normality ([16]). In 2011, Haghi, Rezapour and Shahzad proved same results without the additional assumption and for $\lambda \in (0, 1)$ ([20]). Then, Amini-Harandi proved a result on existence of fixed points of set-valued quasi-contraction maps in metric spaces by using the technique of [20] (see [2]). But similar to ([16]), he could prove it only for $\lambda \in (0, \frac{1}{2})$ (see [2]). In 2012, Haghi, Rezapour and Shahzad the main result of ([2]) by using a simple method ([12]). Also, they introduce quasi-contraction type multifunctions and show that the main result of ([2]) holds for quasi-contraction type multifunctions. They raised an open problem about difference of quasi-contraction and quasi-contraction type multifunctions ([12]). Recently, Mohammadi, Rezapour and Shahzad gave a positive answer to the question ([17]). In this paper, by combining

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obtained idea from the listed papers, we give some fixed point results for generalized quasi-contractive multifunctions on graphs.

Let (X, d) be a metric space and G an undirected graph such that the set $V(G)$ of its vertices coincides with X and the set $E(G)$ of its edges contains all loops. We assume that G has no parallel edges. Hereafter, we denote it for abbreviation by (G, d) and call it graph-metric space. We say that a self-map f on the graph G preserves edges of G whenever $(x, y) \in E(G)$ implies $(fx, fy) \in E(G)$ for all $x, y \in X$. A graph G is connected if there is a path $x_0 = x, \cdots, x_n = y$ between any two vertices x and y. In this case, we say that the length of this path is n. If there is a path between x and y, we denote it by $y \in [x]_G$. It is clear that there is path via minimum length between each two vertices when G is a finite graph. But, we deal infinite graphs (see [1]) and so we assume in this paper that there is path via minimum length between each two vertices and we denote it by $o(x, y) = n$. We say that G is a (C)-graph whenever for each sequence $\{x_n\}_{n\geq 0}$ in X with $x_n \to x$ and $(x_n, x_{n+1}) \in E(G)$ for all $n \geq 0$, there is a subsequence $\{x_{n_k}\}_{k\geq 0}$ such that $(x_{n_k}, x) \in E(G)$ for all $k \geq 0$ (see [1] and [14]). We say that a multifunction $T : G \to 2^G$ has comparable approximative valued property whenever for each $x \in G$ there exists $y \in Tx$ such that $(x, y) \in E(G)$ and $d(x, Tx) = d(x, y)$. Denote by Φ the set of all continuous and nondecreasing mappings $\phi : [0, \infty) \to [0, \infty)$ such that $\phi(t) < t$ for all $t > 0$. Let ϕ be a function in Φ. The multifunction $T : G \to 2^G$ is called weakly generalized ϕ-quasi-contractive whenever $H(Tx, Ty) \leq \phi(M(x, y))$ for all $x, y \in G$ with $y \in [x]_G$, where

$$M(x, y) = max\{d(x, y), d(x, Tx), d(y, Ty), \frac{(d(x, Ty) + d(y, Tx))}{2}\}$$

and H is the Hausdorff metric.

2. Main Results

Now, we are ready to state and prove our main results.

Theorem 2.1. *Let (G, d) be a complete graph-metric space, $\phi \in \Phi$ and also $T : G \to 2^G$ a weakly generalized ϕ-quasi-contractive multifunction which has comparable approximative valued property. If G is a (C)-graph, then T has a fixed point.*

Proof. Let $x_0 \in X$. If $x_0 \in Tx_0$, then we have nothing to prove. Suppose that $x_0 \notin Tx_0$. Since T has comparable approximative valued property, there exists $x_1 \in Tx_0$ such that $(x_0, x_1) \in E(G)$ and $d(x_0, Tx_0) = d(x_0, x_1)$. It is clear that $x_1 \neq x_0$. If $x_1 \in Tx_1$, then we have nothing to prove. Suppose that $x_1 \notin Tx_1$. Now, there exists $x_2 \in Tx_1$ such that $(x_0, x_1) \in E(G)$ and $d(x_1, Tx_1) = d(x_1, x_2)$. It is clear that $x_2 \neq x_1$. By continuing this process, we obtain a sequence $\{x_n\}$ in X such that $x_n \in Gx_{n-1}$, $(x_{n-1}, x_n) \in E(G)$ and $d(x_{n-1}, x_n) = d(x_{n-1}, Tx_{n-1})$ for all n. Now, we have

$$\begin{aligned} d(x_{n-1}, x_n) = d(x_{n-1}, Tx_{n-1}) &\leq H(Tx_{n-2}, Tx_{n-1}) \leq \phi(M(x_{n-2}, x_{n-1})) \\ &\leq \phi(max\{d(x_{n-2}, x_{n-1}), d(x_{n-2}, Tx_{n-2}), d(x_{n-1}, Tx_{n-1}), \\ &\qquad \frac{1}{2}[d(x_{n-1}, Tx_{n-2}) + d(x_{n-2}, Tx_{n-1})]\}) \\ &\leq \phi(max\{d(x_{n-2}, x_{n-1}), d(x_{n-1}, x_n), 1/2d(x_{n-2}, x_n)\}) \\ &\leq \phi(max\{d(x_{n-2}, x_{n-1}), d(x_{n-1}, x_n)\}) \end{aligned}$$

for all n and so we get $d(x_{n-1}, x_n) \leq \phi(d(x_{n-2}, x_{n-1}))$ for all n. Thus,

$$d(x_{n-1}, x_n) \leq \phi(d(x_{n-2}, x_{n-1})) \leq \phi^2(d(x_{n-3}, x_{n-2}))... \leq \phi^{n-1}(d(x_0, x_1))$$

for all n and so $d(x_{n-1}, x_n) \to 0$. If $\{x_n\}$ is not a Cauchy sequence, then there exists $\varepsilon > 0$ and subsequences $\{x_{n_i}\}$ and $\{x_{m_i}\}$ of $\{x_n\}$ with $n_i < m_i$ such that $d(x_{n_i}, x_{m_i}) > \varepsilon$ for all i. For each n_i, put $k_i = \min\{m_i \mid d(x_{n_i}, x_{m_i}) > \varepsilon\}$. Then, we have

$$\varepsilon < d(x_{n_i}, x_{k_i}) \leq d(x_{n_i}, x_{k_i-1}) + d(x_{k_i-1}, x_{k_i}) \leq \varepsilon + d(x_{k_i-1}, x_{k_i})$$

and so $d(x_{n_i}, x_{k_i}) \to \varepsilon$. But, we have

$$d(x_{n_i}, x_{k_i}) - d(x_{n_i}, x_{n_i+1}) - d(x_{k_i}, x_{k_i+1})$$

$$\leq d(x_{n_i+1}, x_{k_i+1}) \leq d(x_{n_i+1}, x_{k_i+1}) + d(x_{n_i}, x_{n_i+1}) + d(x_{k_i}, x_{k_i+1})$$

and so $d(x_{n_i+1}, x_{k_i+1}) \to \varepsilon$. Thus, we get

$$\max\{d(x_{n_i}, x_{k_i}), d(x_{n_i}, Tx_{n_i}), d(x_{k_i}, Tx_{k_i}), \frac{1}{2}[d(x_{n_i}, Tx_{k_i}) + d(x_{k_i}, Tx_{n_i})]\}$$

$$\leq d(x_{n_i}, x_{k_i}) + \frac{3}{2} d(x_{k_i+1}, x_{k_i}) + d(x_{n_i+1}, x_{n_i})$$

and so $M(x_{n_i}, x_{k_i}) \to \varepsilon$. On the other hand, we have

$$d(x_{n_i+1}, x_{k_i+1}) \leq d(x_{n_i+1}, Tx_{n_i}) + H(Tx_{n_i}, Tx_{k_i})$$

$$= H(Tx_{n_i}, Tx_{k_i}) \leq \phi(M(x_{n_i}, x_{k_i})$$

and so $\varepsilon \leq \phi(\varepsilon)$. This contradiction shows that $\{x_n\}$ is a Cauchy sequence. Since G is complete, there exists $x \in G$ such that $x_n \to x$. Since G is a (C)-graph, there is a subsequence $\{x_{n_k}\}_{k\geq 0}$ such that $(x_{n_k}, x) \in E(G)$ for all $k \geq 0$. Thus, we obtain

$$d(x, Tx) = \lim_{n\to\infty} d(x_{n_k+1}, Tx) \leq \lim_{n\to\infty} H(Tx_{n_k}, Tx) \leq \lim_{n\to\infty} \phi(M(x_{n_k}, x))$$

$$\leq \lim_{n\to\infty} \phi(\max\{d(x_{n_k}, x), d(x_{n_k}, Tx_{n_k}), d(x, Tx), \frac{d(x_{n_k}, Tx) + d(x, Tx_{n_k})}{2}\})$$

$$\leq \lim_{n\to\infty} \phi(\max\{d(x_{n_k}, x), (d(x_{n_k}, x_{n_k+1}), d(x, Tx), \frac{d(x_{n_k}, Tx) + d(x, x_{n_k+1})}{2}\})$$

$$= \phi(d(x, Tx))$$

and so $d(x, Tx) = 0$. Since T has comparable approximative valued property, there exists $y \in Tx$ such that $(x, y) \in E(G)$ and $d(x, Tx) = d(x, y)$. Hence, $d(x, y) = 0$ and so $x = y \in Tx$. □

Corollary 2.2. *Let (G, d) be a complete graph-metric space, $\phi \in \Phi$ and also $T : G \to 2^G$ a weakly generalized ϕ-quasi-contractive and compact valued multifunction such that for each $x \in G$ and $y \in Tx$ we have $(x, y) \in E(G)$. If G is a (C)-graph, then T has a fixed point.*

Corollary 2.3. *Let (G, d) be a complete graph-metric space, $\phi \in \Phi$ and f a selfmap on G which preserves edges of G and $d(fx, fy) \leq \phi(M(x, y))$ for all $x, y \in X$ with $y \in [x]_G$. If G is a (C)-graph, then f has a fixed point.*

Proof. It is sufficient we define $T : G \to 2^G$ by $Tx = \{fx\}$ for all $x \in G$. □

Finally by using [15], we can find also some equivalent conditions for some presented results. Note that, one can present more similar results via interesting corollaries by considering above ones.

Proposition 2.4. *Let $\psi : [0, \infty) \to [0, \infty)$ a lower semi-continuous function, $\eta : [0, \infty) \to [0, \infty)$ a map such that $\eta^{-1}(\{0\}) = \{0\}$ and $\liminf_{t\to\infty} \eta(t) > 0$ for all $t > 0$, (G, d) be a complete graph-metric space and $T : G \to 2^G$ a multifunction such that has comparable approximative valued property and*

$$\psi(H(Tx, Ty)) \leq \psi(M(x, y)) - \eta(M(x, y))$$

for all $x, y \in X$ with $y \in [x]_G$. If G is a (C)-graph, then T has a fixed point.

Denote by Ψ the family of nondecreasing functions $\psi : [0, +\infty) \to [0, +\infty)$ such that $\sum_{n=1}^{+\infty} \psi^n(t) < +\infty$ for each $t > 0$. It is well known that $\psi(t) < t$ for all $t > 0$.

Proposition 2.5. *Let (G, d) be a complete graph-metric space, $\psi \in \Psi$ a nondecreasing map which is continuous from right at each point, $\phi : [0, \infty) \to [0, \infty)$ a map such that $\phi(t) < t$ for all $t > 0$ and $T : G \to 2^G$ a multifunction such that has comparable approximative valued property and*

$$\psi(H(Tx, Ty)) \leq \phi(\psi(M(x, y)))$$

for all $x, y \in X$ with $y \in [x]_G$. If G is a (C)-graph, then T has a fixed point.

Let (G, d) be a graph-metric space, $\alpha : G \times G \to [0, \infty)$ a mapping and $T : G \to CB(G)$ a multifunction. We say that T is α-admissible whenever for each $x \in G$ and $y \in Tx$ with $\alpha(x, y) \geq 1$ we have $\alpha(y, z) \geq 1$ for all $z \in Ty$. Finally, recall that T is continuous whenever $H(Tx_n, Tx) \to 0$ for all sequence $\{x_n\}$ in G with $x_n \to x$. Finally, we say that G is a (C_α)-graph whenever for each sequence $\{x_n\}_{n\geq 0}$ in X with $x_n \to x$, there is a subsequence $\{x_{n_k}\}_{k\geq 0}$ such that $\alpha(x_{n_k}, x) \geq 1$ for all $k \geq 0$. By following a similar proof of Theorem 2.1 in [3], we provide next result.

Theorem 2.6. *Let (G, d) be a complete graph-metric space, $\psi \in \Psi$ a strictly increasing map, $\alpha : G \times G \to [0, \infty)$ a function and $T : G \to CB(G)$ an α-admissible multifunction such that $\alpha(x, y)H(Tx, Ty) \leq \psi(d(x, y))$ for all $x, y \in G$ and there exist $x_0 \in G$ and $x_1 \in Tx_0$ with $\alpha(x_0, x_1) \geq 1$. If T is continuous or G is a (C_α)-graph, then T has a fixed point.*

Proof. If $x_1 = x_0$, then we have nothing to prove. Let $x_1 \neq x_0$. If $x_1 \in Tx_1$, then x_1 is a fixed point of T. Let $x_1 \notin Tx_1$ and $q > 1$ be given. Then

$$0 < d(x_1, Tx_1) \leq \alpha(x_0, x_1)H(Tx_0, Tx_1) < q\alpha(x_0, x_1)H(Tx_0, Tx_1).$$

Hence, there exists $x_2 \in Tx_1$ such that

$$d(x_1, x_2) < q\alpha(x_0, x_1)H(Tx_0, Tx_1) \leq q\psi(d(x_0, x_1)).$$

It is clear that $x_2 \neq x_1$. Put $t_0 = d(x_0, x_1) > 0$. Then, $d(x_1, x_2) < q\psi(t_0)$. since ψ is strictly increasing, $\psi(d(x_1, x_2)) < \psi(q\psi(t_0))$. Put $q_1 = \frac{\psi(q\psi(t_0))}{\psi(d(x_1,x_2))}$. Then $q_1 > 1$. If $x_2 \in Tx_2$, then x_2 is a fixed point of T. Assume that $x_2 \notin Tx_2$. Then,

$$0 < d(x_2, Tx_2) \leq \alpha(x_1, x_2)H(Tx_1, Tx_2) < q_1\alpha(x_1, x_2)H(Tx_1, Tx_2).$$

Hence, there exists $x_3 \in Tx_2$ such that

$$d(x_2, x_3) < q_1\alpha(x_1, x_2)H(Tx_1, Tx_2) \leq q_1\psi(d(x_1, x_2)) = \psi(q\psi(t_0)).$$

It is clear that $x_3 \neq x_2$ and $\psi(d(x_2, x_3)) < \psi^2(q\psi(t_0))$. Put $q_2 = \frac{\psi^2(q\psi(t_0))}{\psi(d(x_2,x_3))}$. Then $q_2 > 1$. If $x_3 \in Tx_3$, then x_3 is a fixed point of T. Assume that $x_3 \notin Tx_3$. Then,

$$0 < d(x_3, Tx_3) \leq \alpha(x_2, x_3)H(Tx_2, Tx_3) < q_2\alpha(x_2, x_3)H(Tx_2, Tx_3).$$

Thus, there exists $x_4 \in Tx_3$ such that

$$d(x_3, x_4) < q_1\alpha(x_2, x_3)H(Tx_2, Tx_3) \leq q_2\psi(d(x_2, x_3)) = \psi^2(q\psi(t_0)).$$

By continuing this process we obtain a sequence $\{x_n\}$ in G such that $x_n \in Tx_{n-1}$, $x_n \neq x_{n-1}$ and $d(x_n, x_{n+1}) \leq \psi^{n-1}(q\psi(t_0))$ for all n. Now for each $m > n$ we have

$$d(x_n, x_m) \leq \Sigma_{i=n}^{m-1} d(x_i, x_{i+1}) \leq \Sigma_{i=n}^{m-1} \psi^{i-1}(q\psi(t_0)).$$

Hence, $\{x_n\}$ is a Cauchy sequence in G. Since G is complete, there exists $x^\star \in G$ such that $x_n \to x^\star$. Now if T is continuous, then

$$d(x^\star, Tx^\star) = \lim_{n\to\infty} d(x_{n+1}, Tx^\star) \leq \lim_{n\to\infty} H(Tx_n, Tx^\star) = 0$$

and so $x^\star \in Tx^\star$. If G is a (C_α)-graph, then

$$d(x^\star, Tx^\star) = \lim_{k\to\infty} d(x_{n_k+1}, Tx^\star) \leq \lim_{k\to\infty} H(Tx_{n_k}, Tx^\star)$$

$$\leq \lim_{k\to\infty} \alpha(x_{n_k}, x^\star) H(Tx_{n_k}, Tx^\star) \leq \lim_{k\to\infty} \psi(d(x_{n_k}, x^\star)) = 0.$$

Hence, $x^\star \in Tx^\star$. □

Corollary 2.7. *Let (G, d) be a complete graph-metric space, $\psi \in \Psi$ a strictly increasing map and $T : G \to CB(G)$ a multifunction such that*

$$H(Tx, Ty) \leq \psi(d(x, y))$$

for all $x, y \in G$ with $o(x, y) \geq 3$. Assume that for each $x \in G$ and $y \in Tx$ with $o(x, y) \geq 3$ we have $o(y, z) \geq 3$ for all $z \in Ty$. Suppose that there exist $x_0 \in G$ and $x_1 \in Tx_0$ such that $o(x_0, x_1) \geq 3$. If T is continuous or G has this property that for each sequence $\{x_n\}_{n\geq 0}$ in X with $x_n \to x$, there is a subsequence $\{x_{n_k}\}_{k\geq 0}$ such that $o(x_{n_k}, x) \geq 3$ for all $k \geq 0$, then T has a fixed point.

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