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Some fixed point results on weak partial metric spaces

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Abstract. The concept of partial metric p on a nonempty set X was introduced by Matthews [13] and it was slightly modified by Heckmann [11] as weak partial metric. In [12], the authors studied fixed point result of new extension of Banach's contraction principle to partial metric space and give some generalized versions of the fixed point theorem of Matthews. In the present paper, we extend and generalize the previous results to weak partial metric spaces.

1. Introduction

One of the simplest and most useful result in the fixed point theory is the Banach fixed point theorem: Let (X, d) be a complete metric space and T be self mapping of X satisfying

$$d(Tx, Ty) \leq \lambda d(x, y) \text{ for all } x, y \in X, \tag{1}$$

where $\lambda \in [0, 1)$, then T has a unique fixed point. A mapping satisfying the condition (1) is called contraction mapping. As well as, there are a lot of extensions of this famous fixed point theorem in metric space which are obtained generalizing contractive condition, there are a lot of generalizations of it in different space which has metric type structure. For example, generalized metric space, fuzzy metric space and uniform space. One of the most interesting is partial metric space, which was introduced by Matthews [13] as a part of the study of denotational semantics of data flow networks. It is widely recognized that partial metric spaces play an important role in constructing models in the theory of computation. In a partial metric spaces, the distance of a point in the self may not be zero. After the definition of partial metric space, Matthews proved the partial metric version of Banach fixed point theorem. Recently, fixed point theory studies on partial metric space have been rapidly developed. Valero [18], Oltra and Valero [14], Altun et al [3, 5, 8], Ciric et al [9] and Romaguera [15, 16] gave some generalizations of the result of Matthews. Also, in [1, 2, 6] some Caristi type fixed point theorems and characterization of completeness of partial metric space are given.

First, we recall some definitions of partial metric space and some properties of theirs. See [7, 11, 13, 14, 17, 18] for details.

A partial metric on a nonempty set X is a function $p : X \times X \to \mathbb{R}^+$ (nonnegative real numbers) such that for all $x, y, z \in X$:

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(i) $x = y \iff p(x,x) = p(x,y) = p(y,y)$ (T_0-separation axiom), (ii) $p(x,x) \leq p(x,y)$ (small self-distance axiom), (iii) $p(x,y) = p(y,x)$ (symmetry) and (iv) $p(x,y) \leq p(x,z) + p(z,y) - p(z,z)$ (modified triangular inequality).

A partial metric space (for short PMS) is a pair (X,p) such that X is a nonempty set and p is a partial metric on X. One of the most interesting properties of a partial metric is that $p(x,x)$ may not be zero for $x \in X$. A basic example of a PMS is the pair $(\mathbb{R}^+, p)$, where $p(x,y) = \max\{x,y\}$ for all $x, y \in \mathbb{R}^+$ (nonnegative real numbers). For another example, let I denote the set of all intervals $[a,b]$ for any real numbers $a \leq b$. Let $p : I \times I \to \mathbb{R}^+$ be the function such that $p([a,b],[c,d]) = \max\{b,d\} - \min\{a,c\}$. Then (I,p) is a PMS. Other examples of PMS which are interesting from a computational point of view may be found in [10, 13].

The other interesting properties of partial metric space is that each partial metric p on a nonempty set X generates a T_0 topology τ_p on X which has as a base the family open p-balls

$$\{B_p(x,\varepsilon) : x \in X, \varepsilon > 0\},$$

where

$$B_p(x,\varepsilon) = \{y \in X : p(x,y) < p(x,x) + \varepsilon\}$$

for all $x \in X$ and $\varepsilon > 0$.

If p is a partial metric on X, then the functions $d_p, d_w : X \times X \to \mathbb{R}^+$ given by

$$d_p(x,y) = 2p(x,y) - p(x,x) - p(y,y) \tag{2}$$

and

$$\begin{aligned} d_w(x,y) &= \max\{p(x,y) - p(x,x), p(x,y) - p(y,y)\} \\ &= p(x,y) - \min\{p(x,x), p(y,y)\} \end{aligned} \tag{3}$$

are ordinary metrics on X. It is easy to see that d_p and d_w are equivalent metrics on X.

According to [13], a sequence $\{x_n\}$ in a partial metric (X,p) converges, with respect to τ_{p^s}, to a point $x \in X$ if and only if

$$\lim_{n,m\to\infty} p(x_n, x_m) = \lim_{n\to\infty} p(x_n, x) = p(x,x).$$

A sequence $\{x_n\}$ in a partial metric (X,p) is called a Cauchy sequence if there exists (and is finite) $\lim_{n,m\to\infty} p(x_n, x_m)$. (X,p) is called complete if every Cauchy sequence $\{x_n\}_{n\in\omega}$ in X converges, with respect to τ_p, to a point $x \in X$ such that $p(x,x) = \lim_{n,m\to\infty} p(x_n, x_m)$.

Finally, the following crucial facts are shown in [13]:

(a) $\{x_n\}$ is a Cauchy sequence in (X,p) if and only if it is a Cauchy sequence in the metric space (X,p^s).

(b) (X,p) is complete if and only if (X,p^s) is complete.

Now we recall Matthews' fixed point theorem: Let (X,p) be a complete partial metric space and $T : X \to X$ be a map such that

$$p(Tx, Ty) \leq \alpha p(x,y)$$

for all $x, y \in X$, where $\alpha \in [0,1)$, then T has a unique fixed point.

A very nice generalizations of fixed point result of Matthews was given by Ilic et al.

Theorem 1.1 ([12]). *Let (X,p) be a complete partial metric space, $\alpha \in [0,1)$ and $T : X \to X$ be a map. Suppose for each $x, y \in X$ the following condition holds*

$$p(Tx, Ty) \leq \max\{\alpha p(x,y), p(x,x), p(y,y)\}. \tag{4}$$

Then

(i) the set $X_p = \{x \in X : p(x,x) = \rho_p\}$ *is nonempty, where* $\rho_p = \inf\{p(x,y) : x, y \in X\}$,

(ii) there is unique $u \in X_p$ *such that* $Tu = u$,

(iii) for each $x \in X_p$*, the sequence* $\{T^n x\}$ *converges with respect to the metric* d_p *to* u.

By omitting the small self distance axiom, Heckmann [11] introduced the concept of weak partial metric space (for short WPMS), which is generalized version of Matthews' partial metric space. That is, the function $p : X \times X \to \mathbb{R}^+$ is called weak partial metric on X if it satisfies T_0 separation axiom, symmetry and modified triangular inequality. Heckmann also shows that, if p is weak partial metric on X, then for all $x, y \in X$ we have the following weak small self-distance property

$$p(x,y) \geq \frac{p(x,x) + p(y,y)}{2}. \tag{5}$$

Weak small self-distance property shows that WPMS are not far from small self-distance axiom. It is clear that PMS is a WPMS, but the converse may not be true. A basic example of a WPMS but not a PMS is the pair $(\mathbb{R}^+, p)$, where $p(x,y) = \frac{x+y}{2}$ for all $x, y \in \mathbb{R}^+$. For another example, let I denote the set of all intervals $[a,b]$ for any real numbers $a \leq b$. Let $p : I \times I \to \mathbb{R}^+$ be the function such that $p([a,b],[c,d]) = \frac{b+d-a-c}{2}$. Then (I,p) is a WPMS but not PMS. Again, for $x, y \in \mathbb{R}$ the function $p(x,y) = \frac{e^x + e^y}{2}$ is a non partial metric weak partial metric on $\mathbb{R}$.

Remark 1.2. *As mentioned in [4], if* (X,p) *be a WPMS, but not PMS, then the function* d_p *as in (2) may not be an ordinary metric on X, but* d_w *as in (3) is still an ordinary metric on X.*

The concepts of convergence of a sequence, Cauchy sequence and completeness in WPMS are defined as in PMS. The following lemma, which is very important for fixed point theory on WPMS, was given in [4] without small self-distance axiom.

Lemma 1.3. *Let* (X,p) *be a WPMS.*

(a) $\{x_n\}$ *is a Cauchy sequence in* (X,p) *if and only if it is a Cauchy sequence in the metric space* (X,d_w).

(b) (X,p) *is complete if and only if* (X,d_w) *is complete. Furthermore*

$$\lim_{n\to\infty} d_w(x_n,x) = 0$$

if and only if

$$p(x,x) = \lim_{n\to\infty} p(x_n,x) = \lim_{n,m\to\infty} p(x_n,x_m).$$

2. Fixed Point Results

In this section we give fixed point result of new extensions of Banach's contraction principle on weak partial metric spaces and we prove fixed point result for single map satisfying (ψ,φ)-weakly contractive condition.

Theorem 2.1. *Let* (X,p) *be a complete weak partial metric space,* $\alpha \in [0,1)$ *and* $T : X \to X$ *a mapping. Suppose that for each* $x, y \in X$ *the following condition holds:*

$$p(Tx,Ty) \leq \max\{\alpha p(x,y), \min\{p(x,x), p(y,y)\}\} \tag{6}$$

Then:

(1) *the set* X_p *is nonempty*

(2) *there is a unique* $u \in X_p$ *such that* $u = Tu$

(3) *for each* $x \in X_p$ *the sequence* $\{T^n x\}$ *converges with respect to the metric* d_w *to* u.

Proof. Let $x \in X$. Clearly, (6) implies

$$p(Tx, Tx) \leq \max\{\alpha p(x,x), \min\{p(x,x), p(x,x)\}\} = p(x,x)$$

So $\{p(T^n x, T^n x)\}$ is a nonincreasing sequence and

$$\begin{aligned} p(T^n x, T^m x) &\leq \max\left\{ \begin{array}{l} \alpha p(T^{n-1}x, T^{m-1}x), \\ \min\left\{p(T^{n-1}x, T^{n-1}x), p(T^{m-1}x, T^{m-1}x)\right\} \end{array} \right\} \\ &\leq \max\left\{\alpha p(T^{n-1}x, T^{m-1}x), p(T^{m-1}x, T^{m-1}x)\right\}. \end{aligned}$$

for all $m > n \geq 1$.

Set

$$r_x := \lim_{n\to\infty} p(T^n x, T^n x) = \inf_{n\in\mathbb{N}} p(T^n x, T^n x) \geq 0$$

and

$$M_x := \frac{1}{1-\alpha} p(x, Tx) + p(x,x).$$

Let us prove that, for any $n \geq 0$

$$p(x, T^n x) \leq M_x \tag{7}$$

Clearly, (7) is true for $n = 0, 1$. Suppose that (7) is true for each $n \leq n_0 - 1$, and let us prove it for $n = n_0 \geq 2$. Here, we have

$$\begin{aligned} p(x, T^{n_0}x) &\leq p(x, Tx) + p(Tx, T^{n_0}x) \\ &\leq p(x, Tx) + \max\left\{\alpha p(x, T^{n_0-1}x), \min\left\{p(x,x), p(T^{n_0-1}x, T^{n_0-1}x)\right\}\right\} \\ &\leq p(x, Tx) + \frac{\alpha}{1-\alpha} p(x, Tx) + p(x,x) = M_x. \end{aligned}$$

Thus by induction we obtain (7). Now, we shall prove

$$\lim_{n,m\to\infty} p(T^n x, T^m x) = r_x. \tag{8}$$

Using the weak small self distance property (5), we have for all $n, m \in \mathbb{N}$

$$2p(T^n x, T^m x) \geq p(T^n x, T^n x) + p(T^m x, T^m x) \geq 2r_x.$$

Given any $\epsilon > 0$ find $n_0 \in \mathbb{N}$ such that $p(T^{n_0}x, T^{n_0}x) < r_x + \epsilon$ and $2M_x \alpha^{n_0} < r_x + \epsilon$. Now, for any $n, m \geq 2n_0$

$$\begin{aligned} r_x &\leq p(T^n x, T^m x) \\ &\leq \max\left\{ \begin{array}{l} \alpha p(T^{n-1}x, T^{m-1}x), \\ \min\left\{p(T^{n-1}x, T^{n-1}x), p(T^{m-1}x, T^{m-1}x)\right\} \end{array} \right\} \\ &\leq \max\left\{ \begin{array}{l} \alpha^2 p(T^{n-2}x, T^{m-2}x) \\ \min\left\{p(T^{n-2}x, T^{n-2}x), p(T^{m-2}x, T^{m-2}x)\right\} \end{array} \right\} \\ &\;\;\vdots \\ &\leq \max\left\{ \begin{array}{l} \alpha^{n_0} p(T^{n-n_0}x, T^{m-n_0}x) \\ \min\{p(T^{n-n_0}x, T^{n-n_0}x), p(T^{m-n_0}x, T^{m-n_0}x)\} \end{array} \right\} \\ &< r_x + \epsilon. \end{aligned}$$

Thus we obtain (8), it follows that $\{T^n x\}$ is Cauchy sequence. Since (X, p) is a complete there is $z_x \in X$ such that

$$r_x = p(z_x, z_x) = \lim_{n\to\infty} p(z_x, T^n x) = \lim_{n,m\to\infty} p(T^n x, T^m x). \tag{9}$$

Let us prove

$$p(z_x, Tz_x) \leq p(z_x, z_x). \tag{10}$$

For each $n \in \mathbb{N}$ we have

$$p(z_z, Tz_x) \leq p(z_z, T^n x) + p(T^n x, Tz_x) - p(T^n x, T^n x) \tag{11}$$

From (6) it follows that there is a subsequence $\{n_k\}_{k\geq 1}$ of positive integers such that $p(Tz_x, T^{n_k}x) \leq \alpha p(z_x, T^{n_k-1}x)$ or $p(Tz_x, T^{n_k}x) \leq p(T^{n_k-1}x, T^{n_k-1}x)$ or $p(Tz_x, T^{n_k}x) \leq p(z_x, z_x), k \geq 1$. In each of these cases from (11) taking the limit as $k \to \infty$ it follows $p(z_x, Tz_x) \leq p(z_x, z_x)$. Thus we obtain (10).

Let us prove that X_p is nonempty. For each $k \in \mathbb{N}$ choose $x_k \in X$ with $p(x_k, x_k) < \rho_p + \frac{1}{k}$. Let us show

$$\lim_{n,m\to\infty} p(z_{x_n}, z_{x_m}) = \rho_p \tag{12}$$

Given $\epsilon > 0$ find $n_0 := \left[\frac{3}{\varepsilon(1-\alpha)}\right] + 1$. If $k \geq n_0$ then we have

$$\begin{aligned} \rho_p &\leq p(Tz_{x_k}, Tz_{x_k}) \leq p(z_{x_k}, z_{x_k}) = r_{x_k} \leq p(x_k, x_k) \\ &< \rho_p + \frac{1}{k} \leq \rho_p + \frac{1}{n_0} < \rho_p + \frac{\varepsilon(1-\alpha)}{3}. \end{aligned}$$

Hence we conclude

$$U_k := p(z_{x_k}, z_{x_k}) - p(Tz_{x_k}, Tz_{x_k}) < \frac{\varepsilon(1-\alpha)}{3}, \text{ for } k \geq n_0. \tag{13}$$

Also, if $k \geq n_0$ then $p(z_{x_k}, z_{x_k}) = r_{x_k} \leq p(x_k, x_k) \leq \rho_p + \frac{1}{n_0}$ implies

$$p(z_{x_k}, z_{x_k}) \leq \rho_p + \frac{\varepsilon}{3}(1-\alpha) \ \text{ for all } k \geq n_0. \tag{14}$$

Now if $n, m \geq n_0$ then

$$\begin{aligned} p(z_{x_n}, z_{x_m}) \leq\ & p(z_{x_n}, Tz_{x_n}) + p(Tz_{x_n}, Tz_{x_m}) + p(Tz_{x_m}, z_{x_m}) \\ & - p(Tz_{x_n}, Tz_{x_n}) - p(Tz_{x_m}, Tz_{x_m}) \end{aligned}$$

and (10) imply

$$\begin{aligned} p(z_{x_n}, z_{x_m}) &\leq U_n + U_m + p(Tz_{x_n}, Tz_{x_m}) \\ &< U_n + U_m + \max\{\alpha p(z_{x_n}, z_{x_m}), \min\{p(z_{x_n}, z_{x_n}), p(z_{x_m}, z_{x_m})\}\}. \end{aligned}$$

Whence using (13) and (14) we obtain

$$\rho_p \leq p(z_{x_n}, z_{x_m}) \leq \max\left\{\frac{2}{3}\varepsilon, \frac{2}{3}\varepsilon(1-\alpha) + p(z_{x_n}, z_{x_n}), \frac{2}{3}\varepsilon(1-\alpha) + p(z_{x_m}, z_{x_m})\right\}$$

$$\leq \max\left\{\frac{2}{3}\varepsilon, \rho_p + \varepsilon(1-\alpha)\right\} < \rho_p + \varepsilon$$

This shows (12). Now by completeness of the weak partial metric space (X, p) there is $y \in X$ such that

$$p(y, y) = \lim_{n\to\infty} p(y, z_{x_n}) = \lim_{n,m\to\infty} p(z_{x_n}, z_{x_m}) = \rho_p$$

In particular $y \in X_p$ so $X_p \neq \emptyset$.

Now let $x \in X_p$ be arbitrary. Then, by (9) we have

$$\rho_p \leq p(z_x, Tz_x) \leq p(z_x, z_x) = r_x = \rho_p$$

so $Tz_x = z_x \in X_p$. From (9) and Lemma 1.3, $\{T^n x\}$ converges with respect to the metric d_w to z_x.

If $u, v \in X_p$ are both fixed points of T then from

$$p(u, v) = p(Tu, Tv) \leq \max\{\alpha p(u, v), \min\{p(u, u), p(v, v)\}\}$$

and so we have either $(1 - \alpha)p(u, v) \leq 0$, i.e. $p(u, v) = 0$ or $p(u, v) \leq p(u, u)$ or $p(u, v) \leq p(v, v)$. Since $u, v \in X_p$ then $p(u, u) = p(v, v) = \rho_p$ and so from (5) we have $p(u, v) = p(u, u) = p(v, v)$ in all cases. Therefore, $u = v$. □

Now we give an illustrative example.

Example 2.2. *Let $X = [0, 1] \cup [2, 3]$ and $p : X \times X \to \mathbb{R}$*

$$p(x, y) = \begin{cases} \dfrac{x + y}{2} &, \quad \{x, y\} \cap [2, 3] \neq \emptyset \\ \left|x - y\right| &, \quad \{x, y\} \cap [2, 3] = \emptyset \end{cases}$$

for all $x, y \in X$. Then (X, p) is a complete WPMS. Define $T : X \to X$ by

$$Tx = \begin{cases} 0 &, \quad x \in [0, 1] \\ x - 2 &, \quad x \in [2, 3] \end{cases}.$$

Then, we claim that (6) holds. Indeed,

If $x, y \in [0, 1]$, then

$$\begin{aligned} p(Tx, Ty) &= p(0, 0) \\ &\leq \max\{\alpha p(x, y), \min\{p(x, x), p(y, y)\}\} \end{aligned}$$

for any $\alpha \geq 0$.

If $x, y \in [2, 3]$, then

$$\begin{aligned} p(Tx, Ty) &= p(x - 2, y - 2) \\ &= \left|x - y\right| \\ &\leq \max\left\{\alpha \frac{x + y}{2}, \min\{x, y\}\right\} \\ &= \max\{\alpha p(x, y), \min\{p(x, x), p(x, x)\}\} \end{aligned}$$

for $\alpha \geq \frac{1}{2}$.

If $x \in [0, 1]$ and $y \in [2, 3]$, then

$$\begin{aligned} p(Tx, Ty) &= p(0, y - 2) \\ &= \left|y - 2\right| \\ &\leq \alpha \frac{x + y}{2} \\ &= \max\{\alpha p(x, y), \min\{p(x, x), p(x, x)\}\} \end{aligned}$$

for $\alpha \geq \frac{2}{3}$. Therefore 6 holds for $\alpha \in [\frac{2}{3}, 1)$. Thus from Theorem 2.1, X_p is nonempty, there is a unique $u \in X_p$ such that $u = Tu$ and for each $x \in X_p$ the sequence $\{T^n x\}$ converges with respect to the metric d_w to u. It is clear that $X_p = [0, 1]$ and $T0 = 0 \in X_p$. Note that, if X endowed the usual metric $d(x, y) = \left|x - y\right|$, then the Banach contraction can not be applicable to this example.

Now we prove fixed point result for single map satisfying (ψ, φ)-weakly contractive condition. At first, we recall the definition of altering distance function that will be used later.

Definition 2.3. *$\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ is called an altering distance function if the following properties are satisfied:*

(a) φ is continuous and nondecreasing,
(b) $\varphi(t) = 0 \Longleftrightarrow t = 0$.

Theorem 2.4. *Let (X, p) be a complete WPMS and suppose $T : X \to X$ be a mapping such that*

$$\psi(p(Tx, Ty)) \leq \psi(p(x, y)) - \varphi(p(x, y)) \tag{15}$$

for all $x, y \in X$, where ψ and φ are altering distance functions. Then T has a unique fixed point.

Proof. Let $x_0 \in X$ be an arbitrary point and construct a sequence $\{x_n\}$ in X as $x_{n+1} = Tx_n$ for $n \in \{0, 1, 2, \cdots\}$. Then for $n \geq 1$, from (15) we get

$$\begin{aligned} \psi(p(x_n, x_{n+1})) &= \psi(p(Tx_{n-1}, Tx_n)) \\ &\leq \psi(p(x_{n-1}, x_n)) - \varphi(p(x_{n-1}, x_n)) \\ &\leq \psi(p(x_{n-1}, x_n)). \end{aligned} \tag{16}$$

Using the fact that ψ is nondecreasing, we have

$$p(x_n, x_{n+1}) \leq p(x_{n-1}, x_n) \tag{17}$$

so the sequence$\{p(x_n, x_{n+1})\}$ is nonincreasing and bounded below. If there exist $n_0 \in \mathbb{N}$ such that $p(x_{n_0}, x_{n_0+1}) = 0$ then $x_{n_0} = x_{n_0+1} = Tx_{n_0}$ and x_{n_0} is a fixed point. In other case, suppose that $p(x_n, x_{n+1}) \neq 0$ for all $n \in \mathbb{N}$. Since the sequence$\{p(x_n, x_{n+1})\}$ is nonincreasing and bounded below, there exists $r \geq 0$ such that

$$\lim_{n\to\infty} p(x_n, x_{n+1}) = r. \tag{18}$$

Now suppose that $r > 0$. Therefore, taking $n \to \infty$ in (16) we get

$$\psi(r) \leq \psi(r) - \varphi(r),$$

which is a contradiction because $\varphi(r) > 0$. Therefore it must be $r = 0$ and so

$$\lim_{n\to\infty} p(x_n, x_{n+1}) = 0. \tag{19}$$

By weak small self-distance property $\lim_{n\to\infty} p(x_n, x_n) = 0$.

Now we claim that $\{x_n\}$ is a Cauchy sequence in the metric space (X, d_w). Suppose to the contrary. Then there exists $\varepsilon > 0$ for which we can find two subsequences $\{x_{n_k}\}$ and $\{x_{m_k}\}$ of $\{x_n\}$ such that n_k is the smallest index for which

$$n_k > m_k > k\ ;\ d_w(x_{m_k}, x_{n_k}) \geq \varepsilon. \tag{20}$$

This means that

$$d_w(x_{m_k}, x_{n_k-1}) < \varepsilon. \tag{21}$$

Using (20), (21) and the triangular inequality, we have

$$\begin{aligned}
\varepsilon &\leq d_w(x_{m_k},x_{n_k}) \\
&\leq d_w(x_{m_k},x_{m_k+1}) + d_w(x_{m_k+1},x_{n_k-1}) + d_w(x_{n_k-1},x_{n_k}) \\
&\leq d_w(x_{m_k},x_{m_k+1}) + d_w(x_{m_k+1},x_{n_k}) + 2d_w(x_{n_k-1},x_{n_k}) \\
&\leq d_w(x_{m_k},x_{m_k+1}) + d_w(x_{m_k+1},x_{m_k}) + d_w(x_{m_k},x_{n_k}) + 2d_w(x_{n_k-1},x_{n_k}) \\
&\leq 2d_w(x_{m_k},x_{m_k+1}) + d_w(x_{m_k},x_{m_k+1}) + d_w(x_{m_k+1},x_{n_k}) + 2d_w(x_{n_k-1},x_{n_k}) \\
&= 3d_w(x_{m_k},x_{m_k+1}) + d_w(x_{m_k+1},x_{n_k}) + 2d_w(x_{n_k-1},x_{n_k}) \\
&\leq 3d_w(x_{m_k},x_{m_k+1}) + d_w(x_{m_k+1},x_{n_k-1}) + d_w(x_{n_k-1},x_{n_k}) + 2d_w(x_{n_k-1},x_{n_k}) \\
&= 3d_w(x_{m_k},x_{m_k+1}) + d_w(x_{m_k+1},x_{n_k-1}) + 3d_w(x_{n_k-1},x_{n_k}) \\
&\leq 3d_w(x_{m_k},x_{m_k+1}) + d_w(x_{m_k+1},x_{m_k}) + d_w(x_{m_k},x_{n_k-1}) + 3d_w(x_{n_k-1},x_{n_k}) \\
&= 4d_w(x_{m_k},x_{m_k+1}) + d_w(x_{m_k},x_{n_k-1}) + 3d_w(x_{n_k-1},x_{n_k}) \\
&< 4d_w(x_{m_k},x_{m_k+1}) + \varepsilon + 3d_w(x_{n_k-1},x_{n_k})
\end{aligned}$$

Letting $k \to \infty$ we get that

$$\begin{aligned}
\lim_{k\to\infty} d_w(x_{m_k},x_{n_k}) &= \lim_{k\to\infty} d_w(x_{m_k+1},x_{n_k-1}) \\
&= \lim_{k\to\infty} d_w(x_{m_k+1},x_{n_k}) \\
&= \lim_{k\to\infty} d_w(x_{m_k},x_{n_k-1}) \\
&= \varepsilon.
\end{aligned}$$

Since $d_w(x,y) = p(x,y) - \min\{p(x,x),p(y,y)\}$ for all $x,y \in X$, then by using by $\lim_{n\to\infty} p(x_n,x_n) = 0$ we conclude that

$$\begin{aligned}
\lim_{k\to\infty} p(x_{m_k},x_{n_k}) &= \lim_{k\to\infty} p(x_{m_k+1},x_{n_k-1}) \\
&= \lim_{k\to\infty} p(x_{m_k+1},x_{n_k}) \\
&= \lim_{k\to\infty} p(x_{m_k},x_{n_k-1}) \\
&= \varepsilon.
\end{aligned}$$

Now we can use the inequality 15 for x_{m_k},x_{n_k-1} then we have

$$\begin{aligned}
\psi(p(x_{m_k+1},x_{n_k})) &= \psi(p(Tx_{m_k},Tx_{n_k-1})) \\
&\leq \psi(p(x_{m_k},x_{n_k-1})) - \varphi(p(x_{m_k},x_{n_k-1}))
\end{aligned}$$

Letting $k \to \infty$ and using the continuity of ψ and φ, we get

$$\psi(\varepsilon) \leq \psi(\varepsilon) - \varphi(\varepsilon),$$

which is a contradiction because $\varphi(\varepsilon) > 0$. Thus $\{x_n\}$ is a Cauchy sequence in (X,d_w). Since (X,p) is complete then from Lemma 1.3, the sequence $\{x_n\}$ converges in the metric space (X,d_w), thus there exists $z \in X$ such that

$$\lim_{n\to\infty} d_w(x_n,z) = 0$$

and then

$$p(z,z) = \lim_{n\to\infty} p(x_n,z) = \lim_{n,m\to\infty} p(x_n,x_m) \tag{22}$$

Moreover, since $\lim_{n,m\to\infty} d_w(x_n, x_m) = 0$ and $\lim_{n\to\infty} p(x_n, x_n) = 0$, then from the definition d_w, we have $\lim_{n,m\to\infty} p(x_n, x_m) = 0$. Therefore from (22) we have

$$p(z, z) = \lim_{n\to\infty} p(x_n, z) = \lim_{n,m\to\infty} p(x_n, x_m) = 0.$$

Now we claim that $Tz = z$. From 15 we have

$$\begin{aligned}\psi(p(x_{n+1}, Tz)) &= \psi(p(Tx_n, Tz)) \\ &\leq \psi(p(x_n, z)) - \varphi(p(x_n, z))\end{aligned}$$

and letting $n \to \infty$ we get (note that $p(z, z) = 0$)

$$\psi(p(z, Tz)) \leq \psi(p(z, z)) - \varphi(p(z, z)) = 0.$$

Therefore $p(z, Tz) = 0$ and so from weak small self distance property and T_0-separation axiom we have $z = Tz$. The uniqueness of fixed point follows from 15. □

Now we give an illustrative example.

Example 2.5. *Let $X = [0, \frac{1}{2}]$ and $p : X \times X \to \mathbb{R}$, $p(x, y) = \dfrac{x + y}{2}$. Then (X, p) is a complete WPMS. Define $T : X \to X$, $Tx = x^2$. Then for all $x, y \in X$, we obtain*

$$\begin{aligned}p(Tx, Ty) &\leq p(x^2, y^2) \\ &= \frac{x^2 + y^2}{2} \\ &\leq \frac{\frac{1}{2}x + \frac{1}{2}y}{2} = \frac{1}{2}(\frac{x + y}{2}) \\ &= \frac{1}{2}p(x, y) \\ &= p(x, y) - \frac{1}{2}p(x, y).\end{aligned}$$

Therefore the condition 15 is satisfied for the altering distance function $\psi(t) = t$ and $\varphi(t) = \frac{t}{2}$ and thus T has a unique fixed point in X.

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