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Relative n-isoclinism Classes and Relative n-th Nilpotency Degree of Finite Groups

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Abstract. We correlate the notion of n-isoclinism of finite groups, introduced by J. C. Bioch in 1976, with the relative n-th nilpotency degree, recently studied in literature. We characterize also all the pairs which are isoclinic with (C, D_8) via the relative commutativity degree $d(C, D_8)$, where C is a cyclic maximal subgroup of D_8. A final conjecture is opened for the groups with few nontrivial values of $d(C, G)$.

1. Introduction and main results

All groups are supposed to be finite. After the initial work [9] of W. Gustafson, several contributions appeared on the probability that two randomly chosen elements x and y of a group G commute. If H is a subgroup of G, it was introduced in [5] the *relative n-th nilpotency degree of H in G*,

$$d^{(n)}(H,G) = \frac{|\{(x_1, x_2, ..., x_n, g) \in H^n \times G : [x_1, x_2, ..., x_n, g] = 1\}|}{|H|^n|G|}.$$

In particular, $d(G, G) = d(G)$ is the *commutativity degree*, largely exploited in [1, 4, 6–9, 15–17]. Two isomorphic groups have of course the same commutativity degree, but this is also true if the two groups are *isoclinic* in the sense of P. Hall [11]. The reader may find a proof of this statement in [6, Theorem 3.8] in very weak hypotheses. The original ideas of P. Hall on isoclinic groups in [11, 12] were successively modified in [2, 3, 5, 6, 13, 15, 16] and adapted to the classification of p-groups, where p is a given prime.

Definition 1.1. *Let G_1 and G_2 be two groups, H_1 be a subgroup of G_1 and H_2 be a subgroup of G_2. A pair (α, β) is said to be a relative n-isoclinism from (H_1, G_1) to (H_2, G_2) if we have the following conditions:*

(i) α is an isomorphism from $G_1/Z_n(G_1)$ to $G_2/Z_n(G_2)$ such that the restriction of α to $H_1/(Z_n(G_1) \cap H_1)$ is an isomorphism from $H_1/(Z_n(G_1) \cap H_1)$ to $H_2/(Z_n(G_2) \cap H_2)$;

(ii) β is an isomorphism from $[_nH_1, G_1]$ to $[_nH_2, G_2]$;

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(iii) the following diagram is commutative:

$$\begin{array}{ccc} \frac{H_1}{Z_n(G_1)\cap H_1} \times \ldots \times \frac{H_1}{Z_n(G_1)\cap H_1} \times \frac{G_1}{Z_n(G_1)} & \xrightarrow{\alpha^{n+1}} & \frac{H_2}{Z_n(G_2)\cap H_2} \times \ldots \times \frac{H_2}{Z_n(G_2)\cap H_2} \times \frac{G_2}{Z_n(G_2)} \\ \gamma(n,H_1,G_1)\Big\downarrow & & \gamma(n,H_2,G_2)\Big\downarrow \\ [_nH_1, G_1] & \xrightarrow{\beta} & [_nH_2, G_2]. \end{array}$$

where

$$\gamma(n, H_1, G_1)((h_1(Z_n(G_1) \cap H_1), \ldots, h_n(Z_n(G_1) \cap H_1), g_1Z_n(G_1))) = [h_1, \ldots, h_n, g_1]$$

and

$$\gamma(n, H_2, G_2)((k_1(Z_n(G_2) \cap H_2), \ldots, k_n(Z_n(G_2) \cap H_2), g_2Z_n(G_2))) = [k_1, \ldots, k_n, g_2],$$

for each $h_1, \ldots, h_n \in H_1$, $k_1, \ldots, k_n \in H_2$, $g_1 \in G_1$, $g_2 \in G_2$.

It is easy to check that the maps $\gamma(n, H_1, G_1)$ and $\gamma(n, H_2, G_2)$ are well-posed. If Definition 1.1 is satisfied, we say that (H_1, G_1) and (H_2, G_2) are *relative n-isoclinic*, briefly $(H_1, G_1) \underset{n}{\sim} (H_2, G_2)$. In particular, G_1 and G_2 are called *n-isoclinic*, briefly $G_1 \underset{n}{\sim} G_2$, if $(G_1, G_1) \underset{n}{\sim} (G_2, G_2)$. In particular, G_1 and G_2 are *isoclinic* if they are 1-isoclinic. It is straightforward to check that $\underset{n}{\sim}$ is an equivalence relation in the class of all groups (see also [2, 3, 13]). $(H_1, G_1) \underset{n}{\sim} (H_2, G_2)$ does not imply in general that G_1 and G_2 are n-isoclinic (while the converse is obviously true). For instance, assume that $SL(2, 5)$ is the special linear group of order 120 and $PSL(2, 5)$ is the projective special linear group of order 60. They are relative 1-isoclinic but not isoclinic. For instance, $(Z(SL(2, 5)), SL(2, 5)) \underset{1}{\sim} (1, PSL(2, 5))$, but $|[SL(2, 5), SL(2, 5)]| = 120$ and $|[PSL(2, 5), PSL(2, 5)]| = 60$. In order to illustrate the importance of considering two isoclinic groups, we note that two abelian groups fall into the same equivalence class with respect to isoclinisms (see [2, Theorem 1.4]), while this is no longer true with respect to the notion of isomorphism. J.C. Bioch and R.W. van der Waall [3] proved the invariance under isoclinism of the following hierarchy of classes of groups: abelian < nilpotent < supersoluble < strongly-monomial < monomial < soluble.

Our main results are the following.

Theorem 1.2. *Let G be a group and H, N be subgroups of G such that* $N \triangleleft G$ *and* $N \subseteq H$. *Then for all* $n \geq 0$,

$$\left(\frac{H}{N}, \frac{G}{N}\right) \underset{n}{\sim} \left(\frac{H}{N \cap \gamma_{n+1}(G)}, \frac{G}{N \cap \gamma_{n+1}(G)}\right).$$

In particular, if $N \cap \gamma_{n+1}(G) = 1$, *then* $(H, G) \underset{n}{\sim} (H/N, G/N)$.

Theorem 1.3. *Let H be a subgroup of a group G.*

(i) If $G = HZ_n(G)$, *then* $(H, H) \underset{n}{\sim} (H, G) \underset{n}{\sim} (G, G)$ *and* $d^{(n)}(H) = d^{(n)}(H, G) = d^{(n)}(G)$.

(ii) $d^{(n)}(H, G) = d^{(n)}(\varphi(H), G)$ *for every* $\varphi \in Aut(G)$.

Theorem 1.4. *Let H be a subgroup of a group G such that* $Z(G) \subseteq H$. *Then* $d(H, G) = \frac{3}{4}$ *if and only if* (H, G) *and* $(\langle a \rangle, D_8)$ *are relative 1-isoclinic, where* $\langle a \rangle$ *is a subgroup of order* 4 *of the dihedral group* D_8 *of order 8.*

2. Proofs

Roughly speaking, two groups H and K are isoclinic if their central quotients $H/Z(H)$, $K/Z(K)$ are isomorphic and if their commutator subgroups H', K' are isomorphic. If we look at the construction of the finite extra-special 2-groups (see [14, pp.145–147]) and at the construction of the quaternion groups (see [14, pp.140–141]), then we will find such groups in the situation which has been just described. For instance, we may think at the dihedral group D_8 of order 8 and at the quaternion group Q_8 of order 8. We note that both $D_8/Z(D_8)$, $Q_8/Z(Q_8)$ are isomorphic and D_8', Q_8' are isomorphic. Situations as we just mentioned have been largely studied in literature under the point of view of the relative n-th nilpotency degree in [5, 6, 15, 16]. The following result follows from [16, Theorem 1.1] when we deal with a group having the counting measure.

Proposition 2.1. *Let G_1 and G_2 be two n-isoclinic groups. For every subgroup H_1 of G_1, there exists a subgroup H_2 of G_2 such that H_1 and H_2 are n-isoclinic.*

We will use the following lemma.

Lemma 2.2. *$(H_1, G_1) \underset{n}{\sim} (H_2, G_2)$ if and only if there exist two isomorphisms α and β such that $\alpha : G_1/Z_n(G_1) \to G_2/Z_n(G_2)$, $\beta : [_nH_1, G_1] \to [_nH_2, G_2]$, $\alpha(H_1/(Z_n(G_1) \cap H_1)) = H_2/(Z_n(G_2) \cap H_2)$ and for all $g_1 \in G_1$ and $h_i \in H_1$, $\beta([h_1, ..., h_n, g_1]) = [k_1, ..., k_n, g_2]$, where $g_2 \in \alpha(g_1 Z_n(G_1))$, $k_i \in \alpha(h_i(Z_n(G_1) \cap H_1))$ and $1 \le i \le n$.*

Proof. It is clear by Definition 1.1. □

The proofs of the following two facts can be deduced from [16, Theorem 1.2], when we have the counting measure on a finite group.

Proposition 2.3. *Let G_1 and G_2 be two groups, H_1 be a subgroup of G_1 and H_2 be a subgroup of G_2. If $(H_1, G_1) \underset{n}{\sim} (H_2, G_2)$, then $d^{(n)}(H_1, G_1) = d^{(n)}(H_2, G_2)$.*

Proposition 2.4. *If $(H_1, G_1) \underset{n}{\sim} (H_2, G_2)$, then $(H_1, G_1) \underset{n+1}{\sim} (H_2, G_2)$.*

Theorem 1.2 generalizes [2, Lemma 1.3] and is proved below.

Proof. [Proof of Theorem 1.2] Put $\bar{G} = G/N$ and $\widetilde{G} = G/(N \cap [_nH, G])$. Since $\bar{g} \in Z_n(\bar{G})$ if and only if $\widetilde{g} \in Z_n(\widetilde{G})$, the map α from $\bar{G}/Z_n(\bar{G})$ onto $\widetilde{G}/Z_n(\widetilde{G})$ given by $\alpha(\bar{g}Z_n(\bar{G})) = \widetilde{g}Z_n(\widetilde{G})$ is an isomorphism and $\alpha(\bar{H}/(Z_n(\bar{G}) \cap \bar{H})) = \widetilde{H}/(Z_n(\widetilde{G}) \cap \widetilde{H})$. Also one can see that $\beta : [_n\bar{H}, \bar{G}] \to [_n\widetilde{H}, \widetilde{G}]$ by the rule $\beta(\bar{x}) = \widetilde{x}$ is an isomorphism. By Lemma 2.2, (α, β) is a relative n-isoclinism from $(\bar{H}, \bar{G})$ to $(\widetilde{H}, \widetilde{G})$. □

Theorem 1.3 is proved below.

Proof. [Proof of Theorem 1.3] (i). We claim that $(H, H) \underset{n}{\sim} (H, G) \underset{n}{\sim} (G, G)$. First, we prove $(H, H) \underset{n}{\sim} (H, G)$. Let $G = HZ_n(G)$. We may easily see that $Z_n(H) = Z_n(G) \cap H$. Thus $H/Z_n(H) = H/(Z_n(G) \cap H)$ is isomorphic to $HZ_n(G)/Z_n(G) = G/Z_n(G)$. Therefore $\alpha : H/Z_n(H) \to G/Z_n(G)$ is an isomorphism which is induced by the inclusion $i : H \to G$. Furthermore, we can consider α as isomorphism from $H/Z_n(H)$ to $H/Z_n(G) \cap H$. On the other hand, $[_nH, G] = [_nH, HZ_n(G)] = \gamma_{n+1}(H)$. By Lemma 2.2, the pair $(\alpha, 1_{\gamma_{n+1}(H)})$ allows us to state that $(H, H) \underset{n}{\sim} (H, G)$. The remaining cases $(H, H) \underset{n}{\sim} (G, G)$ and $(H, H) \underset{n}{\sim} (H, G)$ follow by a similar argument. Now the result follows from this claim and Proposition 2.3.

(ii). Assume $\varphi \in Aut(G)$. Then φ induces the isomorphisms α from $G/Z_n(G)$ to $G/Z_n(G)$ by the rule $\alpha(gZ_n(G)) = \varphi(g)Z_n(G)$ and β from $[_nH, G]$ to $[_n\varphi(H), G]$ by the rule $\beta([h_1, ..., h_n, x]) = \varphi([h_1, ..., h_n, x])$. Note that $\alpha(H/Z_n(G) \cap H) = \varphi(H)/(Z_n(G) \cap \varphi(H))$. On the other hand, for every $g \in G$ and $h_i \in H$, $1 \le i \le n$, we have $\varphi(g) \in \alpha(gZ_n(G))$, $\varphi(h_i) \in \alpha(h_i(Z_n(G) \cap H))$ and $\beta([h_1, ..., h_n, g]) = [\varphi(h_1), ..., \varphi(h_n), \varphi(g)]$. By Lemma 2.2, the pair (α, β) implies that $(H, G) \underset{n}{\sim} (\varphi(H), G)$ and so $d^{(n)}(H, G) = d^{(n)}(\varphi(H), G)$. □

Theorem 1.2 has two useful consequences, as we see in the next statements.

Corollary 2.5. *Let H be subgroup of a group G. Then there exists a group G_1 and a normal subgroup H_1 of G_1 such that $(H, G) \underset{1}{\sim} (H_1, G_1)$ and $Z(G_1) \cap H_1 \subseteq H_1 \cap G_1'$.*

Proof. Let $1 \to R \to F \to G \to 1$ be a free presentation of G, S be a subgroup of F, H be a group isomorphic to S/R. If $\bar{F} = F/(R \cap \bar{F}')$ and $\bar{S} = S/(R \cap \bar{F}')$, then Theorem 1.2 with $n = 1$ implies $(H, G) \underset{1}{\sim} (\bar{S}, \bar{F})$. On another hand, $(Z(\bar{F}) \cap \bar{S})/(Z(\bar{F}) \cap \bar{S} \cap \bar{F}')$ is isomorphic to $((Z(\bar{F}) \cap \bar{S})\bar{F}')/\bar{F}'$, which is a subgroup of $\bar{F}/\bar{F}'$. Therefore, for a normal subgroup $\bar{B}$ of $\bar{F}$, $Z(\bar{F}) \cap \bar{S} = (Z(\bar{F}) \cap \bar{S} \cap \bar{F}') \times \bar{B}$. Now $\bar{B} \cap \bar{F}' = 1$ and we have $(H, G) \underset{1}{\sim} (H_1, G_1)$ again by Theorem 1.2 with $n = 1$, where $G_1 = \bar{F}/\bar{B}$ and $H_1 = \bar{S}/\bar{B}$. Now $Z(G_1) \cap H_1 \simeq Z(\bar{F}/\bar{B}) \cap \bar{S}/\bar{B} = (Z(\bar{F}) \cap \bar{S})/\bar{B}$, which is a subgroup of $(\bar{S} \cap \bar{F}')\bar{B}/\bar{B} = H_1 \cap G_1'$. □

Corollary 2.6. *Assume that H is a subgroup of a finite group G. Then there exists a group G_1 and a normal subgroup H_1 of G_1 such that $d(H, G) = d(H_1, G_1)$ and $Z(G_1) \cap H_1 \subseteq G_1' \cap H_1$.*

Proof. By Proposition 2.3 and Corollary 2.5, the result follows. □

We know that $D_8 = \langle a, b \mid a^4 = b^2 = (ab)^2 = 1\rangle$. It is easy to check that $(D_8, \langle a\rangle) \sim_1 (D_8, \langle a^2, b\rangle) \sim_1 (D_8, \langle a^2, ab\rangle)$. and that $d(D_8, \langle a\rangle) = d(D_8, \langle a^2, b\rangle) = d(D_8, \langle a^2, ab\rangle) = \frac{3}{4}$. We will see that all pairs of groups with the relative commutativity degree $\frac{3}{4}$ belong to the class of relative 1-isoclinism of $(\langle a\rangle, D_8)$.

The following lemma gives an upper bound for $d(H, G)$ which will be used in the proof of Theorem 1.4.

Lemma 2.7. *For every subgroup H of a group G,*

$$d(H, G) \le \frac{1}{2}\left(1 + \frac{|Z(G) \cup Z(H)|}{|G|}\right).$$

Proof. We have

$$d(H, G) = \frac{1}{|G||H|}|\{(h, g) \in H \times G : [h, g] = 1\}| = \frac{1}{|G|}\sum_{g \in G}\frac{|C_H(g)|}{|H|} = \frac{1}{|G|}\left(\sum_{g \in Z(G)\cup Z(H)}\frac{|C_H(g)|}{|H|} + \sum_{g \notin Z(G)\cup Z(H)}\frac{|C_H(g)|}{|H|}\right)$$

$$\le \frac{1}{|G|}\left(|Z(G) \cup Z(H)| + \frac{1}{2}(|G| - |Z(G) \cup Z(H)|)\right) = \frac{1}{2}\left(1 + \frac{|Z(G) \cup Z(H)|}{|G|}\right).$$

□

Proof. [Proof of Theorem 1.4] Assume $d(H, G) = \frac{3}{4}$. Then H is abelian by [5, Theorems 2.2 and 3.3] and $|G : H| \le 2$ by Lemma 2.7. Moreover, $|H/Z(G)| = 2$ by [5, Theorem 3.10] and so $|G : Z(G)| = 4$. Therefore $G/Z(G)$ is a 2-elementary abelian group of rank 2 so we may define the isomorphism α from $G/Z(G)$ to $D_8/Z(D_8)$ by $\alpha(\bar{x}) = \bar{a}$ and $\alpha(\bar{y}) = \bar{b}$. Since $Z(G) \subseteq H$, $H/Z(G)$ is either $\langle\bar{x}\rangle$ or $\langle\bar{y}\rangle$ or $\langle\bar{x}\bar{y}\rangle$.

Assume that $H/Z(G) = \langle\bar{x}\rangle$. Then $\alpha(H/Z(G)) = \langle a\rangle/\langle a^2\rangle$ and $[H, G] = \langle x, y\rangle$. Therefore $\beta : [H, G] \to \langle a^2\rangle$ by $\beta([x, y]) = [a, b]$ is an isomorphism. Hence (α, β) is a relative isoclinism from (H, G) to $(\langle a\rangle, D_8)$ by Lemma 2.2. Now we have the remaining cases $H/Z(G) = \langle\bar{y}\rangle$ and $H/Z(G) = \langle\bar{x}\bar{y}\rangle$. If $H/Z(G) = \langle\bar{y}\rangle$, then a similar argument shows that $(H, G) \sim_1 (\langle a^2, b, D_8\rangle)$. If $H/Z(G) = \langle\bar{x}\bar{y}\rangle$, then a similar argument shows that $(H, G) \sim_1 (\langle a^2, ab\rangle, D_8)$. There are no other cases so we deduce that $(H, G) \sim_1 (\langle a\rangle, D_8)$, as claimed.

Conversely, if $(H, G) \sim_1 (\langle a\rangle, D_8)$, then $d(H, G) = d(\langle a\rangle, D_8) = \frac{3}{4}$ and the result follows from [5, Theorem 3.10]. □

A final question originates from computations and evidences of GAP [18]. We have written in fact a program in GAP which allows us to do some qualitative considerations on the values which we found for the case of groups of order ≤ 30. We list what we have found. If $|G| = 2, 3, 4, 5, 7, 9, 11, 13, 15, 17, 19, 25, 29, 31$, then $d(Z, G) = 1$ for all cyclic maximal subgroups Z of G. Now assume that $|G| \ge 6$, G is nonabelian and $d(C, G) \neq 1$. If $|G| = 2 \cdot 3 = 6$, then there exist a cyclic maximal subgroup C of G such that $d(C, G) = \frac{2}{3}$. If $|G| = 2^3 = 8$, the same is true and $d(C, G) = \frac{3}{4}$. If $|G| = 2 \cdot 5 = 10$, then $d(C, G) = \frac{3}{5}$. If $|G| = 2^2 \cdot 3 = 12$, then $d(C, G) \in \{\frac{2}{3}, \frac{1}{2}\}$. If $|G| = 2 \cdot 7 = 14$, then $d(C, G) = \frac{4}{7}$. If $|G| = 16 = 2^4$, then $d(C, G) \in \{\frac{3}{4}, \frac{5}{8}\}$. If $|G| = 2 \cdot 3^2 = 18$, then $d(C, G) \in \{\frac{2}{3}, \frac{5}{9}\}$. If $|G| = 2^2 \cdot 5 = 20$, then $d(C, G) \in \{\frac{3}{5}, \frac{2}{5}\}$. If $|G| = 3 \cdot 7 = 21$, then $d(C, G) = \frac{3}{7}$. If $|G| = 2 \cdot 11 = 22$, then $d(C, G) = \frac{6}{11}$. If $|G| = 2^3 \cdot 3 = 24$, then $d(C, G) \in \{\frac{2}{3}, \frac{1}{2}, \frac{3}{4}, \frac{7}{12}\}$. If $|G| = 2 \cdot 13 = 26$, then $d(C, G) = \frac{7}{13}$. If $|G| = 3^3 = 27$, then $d(C, G) = \frac{5}{9}$. If $|G| = 2^2 \cdot 7 = 28$, then $d(C, G) = \frac{4}{7}$. If $G = 2 \cdot 3 \cdot 5 = 30$, then $d(C, G) \in \{\frac{2}{3}, \frac{3}{5}, \frac{8}{15}\}$.

We note that for a nonabelian group G of $|G| = 6, 8, 10, 14, 21, 22, 26, 27, 28$ we have just one nontrivial value of $d(C, G)$ in correspondence of a cyclic maximal subgroup C of G. We note that for a nonabelian group G of $|G| = 12, 16, 18, 20$ we have just two nontrivial values of $d(C, G)$ in correspondence of a cyclic maximal subgroup C of G. The remaining cases show nontrivial values of $d(C, G)$, which are either 3 or 4.

At this point, it is useful to introduce the set

$$\mathcal{D} = \{d(C, G) \neq 1 \mid C \text{ is a cyclic maximal subgroup of } G\},$$

where G is supposed to be nonabelian. We summarize some interesting evidences. For a nonabelian group G of order $|G| = pq$ for two distinct primes p and q, the previous computations show that $|\mathcal{D}| = 1$. The following question comes naturally.

Conjecture 2.8. *What is the structure of G if* $|\mathcal{D}| = 1$ *? And if* $|\mathcal{D}|$ *is small (i.e.: 2 or 3) ? Can we find restrictions on G ?*

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