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Mapping problems for quasiregular mappings

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Abstract. We study images of the unit ball under certain special classes of quasiregular mappings. For homeomorphic, i.e., quasiconformal mappings problems of this type have been studied extensively in the literature. In this paper we also consider non-homeomorphic quasiregular mappings. In particular, we study (topologically) closed quasiregular mappings originating from the work of J. Väisälä and M. Vuorinen in 1970's. Such mappings need not be one-to-one but they still share many properties of quasiconformal mappings. The global behavior of closed quasiregular mappings is similar to the local behavior of quasiregular mappings restricted to a so-called normal domain.

1. Introduction

We consider quasiregular mappings in the n-dimensional Euclidean space $\mathbb{R}^n$. Quasiconformal and quasiregular mappings in $\mathbb{R}^n$, $n \geq 3$ are natural generalizations of conformal and analytic functions of one complex variable, respectively. For basic properties of these classes of mappings, we refer to [15, 18, 22]. In the complex plane, it follows from the Riemann mapping theorem that any simply connected domain is the image of the unit disk in a conformal, and thus quasiconformal, mapping. The so-called measurable Riemann mapping theorem further generalizes this result by allowing one to find a quasiconformal mapping of given dilatation. However, the problem of characterizing the quasidisks, i.e., quasiconformal images of the unit disk in the quasiconformal mappings of the the whole plane onto itself is interesting (see e.g. [2, 6]). For $n \geq 3$ the question of characterizing the quasiconformal images of the unit ball $\mathbb{B}^n$ is highly non-trivial, and it has been studied by many authors [7, 8, 16]. In this paper, we present several examples related to this topic, and new results concerning the so-called closed quasiregular mappings.

The topological properties of quasiregular mappings are similar to those of analytic functions. It is well-known that a nonconstant quasiregular mapping is discrete (i.e. sets $f^{-1}(y)$ are discrete) and open (see e.g. [15, I.4.1]). We study a subclass of the quasiregular mappings which are characterized by the property that they preserve closed sets. This class of mappings is more general than the quasiconformal mappings,

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as closed mappings need not be homeomorphic. The class of closed quasiregular mappings originates from the work of J. Väisälä [17] and M. Vuorinen [19–21].

The global behavior of closed quasiregular mappings is similar to the behavior of quasiregular mappings restricted to the so-called normal domains. The existence of such neighborhoods is well-known, but usually nothing is known of their diameter. The importance of the assumption that mappings are closed arises from the fact that it allows us to extend local estimates which are based on the conformal modulus to global ones.

2. Preliminaries

We shall follow standard notation and terminology adopted from [18], [22] and [15]. For $x \in \mathbb{R}^n$, $n \geq 2$, and $r > 0$, let $\mathbb{B}^n(x,r) = \{z \in \mathbb{R}^n : |z-x| < r\}$, $\mathbb{S}^{n-1}(x,r) = \partial\mathbb{B}^n(x,r)$, $\mathbb{B}^n(r) = \mathbb{B}^n(0,r)$, $\mathbb{S}^{n-1}(r) = \partial\mathbb{B}^n(r)$, $\mathbb{B}^n = \mathbb{B}^n(1)$ and $\mathbb{S}^{n-1} = \partial\mathbb{B}^n$. The space $\overline{\mathbb{R}}^n = \mathbb{R}^n \cup \{\infty\}$ is the one-point compactification of $\mathbb{R}^n$. The surface area of $\mathbb{S}^{n-1}$ is denoted by ω_{n-1} and Ω_n is the volume of $\mathbb{B}^n$. It is well-known that $\omega_{n-1} = n\Omega_n$ and that

$$\Omega_n = \frac{\pi^{n/2}}{\Gamma(1+n/2)}$$

for $n = 2, 3, \ldots$, where Γ is Euler's gamma function. The standard coordinate unit vectors are denoted by $e_1, \ldots, e_n$. The Lebesgue measure on $\mathbb{R}^n$ is denoted by m.

Quasiregular mappings

A continuous mapping $f\colon G \to \mathbb{R}^n$, $n \geq 2$, of a domain G in $\mathbb{R}^n$ is called *quasiregular* if f is in the Sobolev space $W^{1,n}_{\mathrm{loc}}(G)$, and there exists a constant K, $1 \leq K < \infty$, such that the inequality

$$|f'(x)|^n \leq K J_f(x)$$

holds a.e. in G, where $f'(x)$ is the formal derivative of f, and $|f'(x)| = \max_{|h|=1} |f'(x)h|$. The smallest $K \geq 1$ for which this inequality is true is called the outer dilatation of f and denoted by $K_O(f)$. If f is quasiregular, then the smallest $K \geq 1$ for which the inequality

$$J_f(x) \leq K l(f'(x))^n$$

holds a.e. in G is called the inner dilatation of f and denoted by $K_I(f)$, where $l(f'(x)) = \min_{|h|=1} |f'(x)h|$. The maximal dilatation of f is the number $K(f) = \max\{K_I(f), K_O(f)\}$. If $K(f) \leq K$, f is said to be K-quasiregular. A quasiregular homeomorphism $f\colon G \to fG$ is called *quasiconformal*.

By generalized Liouville's theorem for $n \geq 3$, every 1-quasiregular mapping in $\mathbb{R}^n$ is a restriction of a Möbius transformation or a constant. The Möbius transformations are very useful in the study of quasiregular mappings. In particular, we make use of the mapping T_a, $a \in \mathbb{B}^n$, which is the Möbius transformation with $T_a(\mathbb{B}^n) = \mathbb{B}^n$, $T_a(a) = 0$ and for $e_a = a/|a|$, $T_a(e_a) = e_a$ and $T_a(-e_a) = -e_a$. For $a = 0$, we set $T_0 = \mathrm{id}$ (see [22, p. 11] or [1, II 2.6]).

Modulus of a path family

Let Γ be a path family in $\mathbb{R}^n$, $n \geq 2$. Let $\mathcal{F}(\Gamma)$ be the set of all Borel functions $\rho\colon \mathbb{R}^n \to [0,\infty]$ such that

$$\int_\gamma \rho\, ds \geq 1$$

for every locally rectifiable path $\gamma \in \Gamma$. The functions in $\mathcal{F}(\Gamma)$ are called *admissible* for Γ. For $1 \leq n < \infty$, we define

$$\mathrm{M}(\Gamma) = \inf_{\rho \in \mathcal{F}(\Gamma)} \int_{\mathbb{R}^n} \rho^n\, dm \tag{1}$$

and call $M(\Gamma)$ the (*conformal*) *modulus* of Γ. If $\mathcal{F}(\Gamma) = \emptyset$, which is true only if Γ contains constant paths, we set $M(\Gamma) = \infty$. If Γ_1, Γ_2 are path families in $\mathbb{R}^n$, and every $\gamma \in \Gamma_2$ has a subcurve in Γ_1, we say that Γ_2 is *minorized* by Γ_1 and write $\Gamma_2 > \Gamma_1$. If $\Gamma_1 < \Gamma_2$, then $M(\Gamma_1) \geq M(\Gamma_2)$. For the basic properties of the modulus of the path family, we refer to [15, 18, 22]. It is well-known that the modulus of a path family is invariant under conformal mappings. We denote by $\Delta(A, B; G)$ the family of paths joining A and B in G.

We use the following well-known identity of the modulus of the spherical annulus: Let $0 < a < b$. Then,

$$M\Big(\Delta(\mathbb{B}^n(a), \mathbb{S}^{n-1}(b); \mathbb{B}^n(b))\Big) = \omega_{n-1}\Big(\log \frac{b}{a}\Big)^{1-n}. \tag{2}$$

Canonical ring domains

The complementary components of the *Grötzsch ring* $R_{G,n}(s)$ in $\overline{\mathbb{R}}^n$ are $\overline{\mathbb{B}}^n$ and $[se_1, \infty]$, $s > 1$, and those of the *Teichmüller ring* $R_{T,n}(s)$ are $[-e_1, 0]$ and $[se_1, \infty]$, $s > 0$. We define two special functions $\gamma_n(s)$, $s > 1$, and $\tau_n(s)$, $s > 0$, by

$$\begin{cases} \gamma_n(s) = M\Big(\Delta(\overline{\mathbb{B}}^n, [se_1, \infty])\Big) = \gamma(s), \\ \tau_n(s) = M\Big(\Delta([-e_1, 0], [se_1, \infty])\Big) = \tau(s), \end{cases}$$

respectively. The subscript n is omitted if there is no danger of confusion. We shall refer to these functions as the *Grötzsch capacity* and the *Teichmüller capacity*. It is well-known that for all $s > 1$

$$\gamma_n(s) = 2^{n-1}\tau_n(s^2 - 1),$$

and that $\tau_n \colon (0, \infty) \to (0, \infty)$ is a decreasing homeomorphism. For $s > 1$ we have the following inequalities (see e.g. [22, 7.24]):

$$\omega_{n-1}\Big(\log \lambda_n s\Big)^{1-n} \leq \gamma(s) \leq \omega_{n-1}\Big(\log s\Big)^{1-n}, \tag{3}$$

where λ_n is the Grötzsch ring constant depending only on n. The value of λ_n is known only for $n = 2$, namely $\lambda_2 = 4$. For $n \geq 3$ it is known that $2e^{0.76(n-1)} < \lambda_n \leq 2e^{n-1}$. For more information on the constant λ_n, see [3, Chapter 12].

We will use the following estimate from [5] (see also [11, 2.11]). Suppose that $G = A \setminus C$ is a ring domain such that $A \subset \mathbb{B}^n$ and C is a connected set with $0, x \in C$. Then

$$M(\Delta(C, \partial A; G)) \geq \gamma(1/|x|). \tag{4}$$

K_I- and K_O-inequalities

Next we give two very useful inequalities, known as K_I- and K_O-inequalities, respectively. The K_I-inequality is also known as Väisälä's inequality.

Theorem 2.1. *([15, Theorem II.9.1]) Let $f \colon G \to \mathbb{R}^n$ be a nonconstant quasiregular mapping, Γ be a path family in G, Γ' be a path family in $\mathbb{R}^n$, and m be a positive integer such that the following is true. For every path $\beta \colon I \to \mathbb{R}^n$ in Γ' there are paths $\alpha_1, \ldots, \alpha_m$ in Γ such that $f \circ \alpha_j \subset \beta$ for all j and such that for every $x \in G$ and $t \in I$ the equality $\alpha_j(t) = x$ holds for at most $i(x, f)$ indices j. Then*

$$M(\Gamma') \leq \frac{K_I(f)}{m} M(\Gamma).$$

In particular, we have the Poleckiĭ inequality:

Theorem 2.2. *([15, Theorem 8.1]) Let $f \colon G \to \mathbb{R}^n$ be a nonconstant quasiregular mapping and let Γ be a path family in G. Then*

$$M(f\Gamma) \leq K_I(f) M(\Gamma).$$

Theorem 2.3. *([15, Theorem II.2.4]) Let $f \colon G \to \mathbb{R}^n$ be a nonconstant K-quasiregular mapping. Let $A \subset G$ be a Borel set with $N(f, A) < \infty$, and let Γ be a family of paths in A. Then*

$$M(\Gamma) \leq K_O(f) N(f, A) M(f\Gamma).$$

3. Topological properties

Next we recall some topological properties of quasiregular mappings.

Discrete and open mappings.

It is well-known that a nonconstant quasiregular mapping is discrete and open. We denote by B_f the branch set of f, i.e. the set of points where f fails to be a local homeomorphism. A result by V. A. Chernavskii states that $\dim B_f \le n-2$ for a discrete and open $f\colon G \to \mathbb{R}^n$. The properties of discrete and open mappings were further studied in by J. Väisälä in [17], where also the multiplicity of discrete, open and closed mappings was studied.

Normal domains.

Let $f\colon G \to \mathbb{R}^n$ be a discrete and open mapping. A domain $D \subset\subset G$ is called a *normal domain* for f if $f\partial D = \partial f D$. A normal neighborhood of x is a normal domain D such that $D \cap f^{-1}(f(x)) = \{x\}$.

Multiplicity and normal domains

Let $f\colon G \to \mathbb{R}^n$ be a discrete and open mapping. We denote

$$i(x,f) = \inf_U \sup_y \operatorname{card} f^{-1}(y) \cap U,$$

where U runs through the neighborhoods of x. The number $i(x,f)$ is called the local (topological) index of f at x. Let $C \subset G$. The minimal multiplicity $M(f,C)$ and the maximal multiplicity $N(f,C)$ are defined by

$$M(f,C) = \inf_{y \in fC} \sum_{x \in f^{-1}(y) \cap C} i(x,f), \tag{5}$$

$$N(f,C) = \sup_{y \in fC} \sum_{x \in f^{-1}(y) \cap C} i(x,f), \tag{6}$$

respectively.

The following result holds for discrete, open and sense-preserving mappings:

Lemma 3.1. ([15, Corollary II.3.4]) *Let $f\colon G \to \mathbb{R}^n$ be discrete, open and sense-preserving, $D \subset G$ a normal domain for f, $\beta\colon [a,b) \to fD$ a path and $m = N(f,D)$. Then there exist paths $\alpha_j\colon [a,b) \to D$, $1 \le j \le m$, such that*

(1) $f \circ \alpha_j = \beta$,

(2) $\operatorname{card}\{j : \alpha_j(t) = x\} = i(x,f)$ *for* $x \in D \cap f^{-1}\beta(t)$,

(3) $|\alpha_1| \cup \ldots \cup |\alpha_m| = D \cup f^{-1}|\beta|$,

where $|\alpha|$ stands for the locus of α, i.e. the image set $\alpha[a,b)$, and $a \le t < b$.

Cluster sets

The cluster set of $f\colon G \to \mathbb{R}^n$ at a point $b \in \partial G$ is the set $C(f,b)$ of all points $z \in \overline{\mathbb{R}}^n$ for which there exists a sequence (b_k) in G such that $b_k \to b$, and $f(b_k) \to z$. Let

$$C(f,E) = \bigcup_{b \in E} C(f,b)$$

for a non-empty set $E \subset \partial G$, and $C(f) = C(f,\partial G)$. A mapping f is *closed* if fA is closed in fG whenever A is closed in G and *proper* if $f^{-1}Q$ is compact in G, where Q is compact in fG. If $C(f) \subset \partial fG$, f is said to be *boundary-preserving*.

Discrete, open and closed mappings

Next we recall some useful topological results for discrete, open, and closed mappings.

Theorem 3.2. (*See [17, 5.5], [12, 3.3] and [19, 3.2–3.3]*) *Let $f\colon G \to \mathbb{R}^n$ be discrete and open. Then the following conditions are equivalent:*

(1) f is proper.

(2) f is closed.

(3) f is boundary-preserving.

(4) Each sequence of points of G converging to a point of ∂G is transformed by f onto a sequence no subsequence of which converges to a point of fG.

(5) $N(f, G) = p < \infty$ and for all $y \in fG$, we have

$$p = \sum_{j=1}^{k} i(x_j, f), \ \ \{x_1, \ldots, x_k\} = f^{-1}(y).$$

Corollary 3.3. *If $f\colon G \to \mathbb{R}^n$ is discrete, open, and closed, then $C(f) = \partial fG$.*

Lemma 3.4. *[19, Lemma 3.6] Let $f\colon G \to \mathbb{R}^n$ be discrete, open, and closed, let $U \subset fG$ be a domain, and let D be a component of $f^{-1}U$. Then $fD = U$ and $f|D$ is closed. Moreover, $C(f|D) = \partial U$. If f has a continuous extension $\overline{f}$ to $\overline{D}$, then $\overline{f}\partial D = \partial U$.*

Remark 3.5. *In the plane each closed quasiregular mapping $f\colon \mathbb{B}^2 \to \mathbb{B}^2$ has a representation*

$$f = g \circ h,$$

where $h\colon \mathbb{B}^2 \to \mathbb{B}^2$ is a quasiconformal mapping and $g\colon \mathbb{B}^2 \to \mathbb{B}^n$ is a finite Blaschke product or a constant (see [19, Theorem 4.1]). This result follows immediately from the Stoïlov decomposition and the fact that each closed analytic function is a finite Blaschke product.

4. Unions of balls

In this section, we prove a result which shows that a domain which is a union of a finite number of balls is always a K-quasiconformal image of a ball. The proof of this result also gives an explicit upper bound for the dilatation K.

We say that a domain $G \subset \overline{\mathbb{R}}^n$ is a *K-quasiball*, or simply *quasiball*, if there exists a K-quasiconformal mapping f of $\overline{\mathbb{R}}^n$ onto itself such that $G = f(\mathbb{B}^n)$, where $\overline{\mathbb{R}}^n = \mathbb{R}^n \cup \{\infty\}$.

Theorem 4.1. *Let $B_1, B_2, \ldots, B_m$ be balls in $\mathbb{R}^n$ such that for $1 \le j < m$, $|r_{j+1} - r_j| < |x_{j+1} - x_j| < r_j + r_{j+1}$ and $\overline{B_j} \cap \overline{B_k} = \emptyset$ for $|j - k| > 1$. Then $D = B_1 \cup B_2 \cup \ldots \cup B_m$ is a quasiball.*

Wedge-shaped domains

Let (r, φ, z) be the cylindrical coordinates of a point $x \in \mathbb{R}^n$, $n \ge 3$. For $r \ge 0$, $0 \le \varphi < 2\pi$ (or $-\pi \le \varphi < \pi$) and $z \in \mathbb{R}^{n-2} = \{(0, 0, z_3, \cdots, z_n) : \ z_i \in \mathbb{R}, \ i = 3, \cdots, n\}$ we define

$$\left\{ \begin{array}{lcl} x_1 & = & r\cos\varphi, \\ x_2 & = & r\sin\varphi, \\ x_i & = & z_i \ \text{ for } \ 3 \le i \le n. \end{array} \right.$$

The domain $W_{(\gamma,\gamma+\alpha)}$, defined by $\gamma < \varphi < \gamma + \alpha$, is called a *wedge* of angle α, where $0 \le \gamma < 2\pi$, $0 < \alpha < 2\pi$ and $0 < \gamma + \alpha \le 2\pi$ (or $-\pi \le \gamma < \pi$, $0 < \alpha < 2\pi$ and $-\pi < \gamma + \alpha \le \pi$). We also say that the domain $W_{(\gamma,\gamma+\alpha)}$ is a *wedge* of angle α with the starting angle γ. For any rotation σ around the subspace $\mathbb{R}^{n-2}$, $\sigma(W_{\gamma,\gamma+\alpha})$ is still a wedge of angle α. In particular, $W_{(\gamma,\gamma+\pi)}$ is a half-space in $\mathbb{R}^n$ for any γ.

Given two wedges $W_{(\gamma_1,\gamma_1+\alpha)}$ and $W_{(\gamma_2,\gamma_2+\beta)}$, the quasiconformal diffeomorphism $f\colon W_{(\gamma_1,\gamma_1+\alpha)} \to W_{(\gamma_2,\gamma_2+\beta)}$, defined by

$$f(r,\varphi,z) = (r, \beta\varphi/\alpha + \gamma_2 - \gamma_1, z),$$

is called a *folding*. Assuming that $0 < \alpha \le \beta < 2\pi$ we have

$$K_I(f) = \beta/\alpha, \qquad K_O(f) = (\beta/\alpha)^{n-1}.$$

Then f is a $(\beta/\alpha)^{n-1}$-quasiconformal mapping. See [18, 16.3] for more details.

In what follows, we always denote by $\mathbb{B}^n(x_i, r_i)$ the ball in $\mathbb{R}^n$ with the center x_i and the radius r_i.

Lemma 4.2. *Suppose that B_1 and B_2 are two balls which satisfy $|r_2 - r_1| < |x_1 - x_2| < r_1 + r_2$ in $\mathbb{R}^n$. Then there exists $\alpha \in (\pi, 2\pi)$ such that the domain $D = B_1 \cup B_2$ can be mapped onto a wedge $W_{(\gamma,\gamma+\alpha)}$ by a Möbius transformation.*

Proof. Choose three distinct points $y_1, y_2, y_3 \in S = \partial B_1 \cap \partial B_2$. Then there exists (see e.g. [3, 7.21]) a Möbius transformation g such that $g(y_1) = 0$, $g(y_2) = e_n$ and $g(y_3) = \infty$. It follows that $H_1 = g(B_1)$ and $H_2 = g(B_2)$ are half spaces in $\mathbb{R}^n$ and $0 \in S' = \partial H_1 \cap \partial H_2$. We may assume that S' is orthogonal to the x_1x_2-plane. Clearly $g(D)$ is a wedge $W_{(\gamma,\gamma+\alpha)}$ for some $\alpha \in (\pi, 2\pi)$. □

Angle of intersection

Suppose that B_1, B_2 are two balls in $\mathbb{R}^n$ with $|r_2 - r_1| < |x_1 - x_2| < r_1 + r_2$. Then the angle of intersection, $\alpha(B_1, B_2)$, of B_1 and B_2 is the number $\alpha \in (\pi, 2\pi)$ such that there exists a Möbius transformation g such that $g(B_1 \cup B_2)$ is a wedge $W_{(\gamma,\gamma+\alpha)}$ of angle α.

Corollary 4.3. *Suppose that B_1, B_2 are balls in $\mathbb{R}^n$ with $|r_2 - r_1| < |x_1 - x_2| < r_1 + r_2$. Then $D = B_1 \cup B_2$ is a K-quasiball, where $K < \infty$ is a constant depending only on $\alpha(B_1, B_2)$ and n.*

Proof. By Lemma 4.2, it is sufficient to prove that for any $\alpha \in (\pi, 2\pi)$, the wedge $W_{(\gamma,\gamma+\alpha)}$ of angle α is a quasiball. Without loss of generality, we may assume that $W_{(\gamma,\gamma+\alpha)} = W_{(0,\alpha)}$. Then the interior of $\mathbb{R}^n \backslash W_{(0,\alpha)}$ is the wedge $W_{(\alpha,2\pi)}$. Let

$$f(r,\varphi,z) = \begin{cases} (r, \pi\varphi/\alpha, z) & \text{for } 0 \le \varphi \le \alpha, \\ (r, \pi(1 + (\varphi - \alpha)/(2\pi - \alpha)), z) & \text{for } \alpha < \varphi < 2\pi, \end{cases}$$

and let $f(\infty) = \infty$. Then f is clearly a homeomorphism of $\overline{\mathbb{R}}^n$ onto itself. It follows that $f\colon \overline{\mathbb{R}}^n \to \overline{\mathbb{R}}^n$ is quasiconformal with $K(f) = \max\{K(f_1), K(f_2)\}$, where the mappings $f_1\colon W_{(0,\alpha)} \to W_\pi = f_1(W_{(0,\alpha)})$ and $f_2\colon W_{(\alpha,2\pi)} \to W_{(\pi,2\pi)} = f_2(W_{(\alpha,2\pi)})$, are foldings. □

Proof. [Proof of Theorem 4.1] By Lemma 4.2 we may find a Möbius transformation g taking $B_1 \cup B_2$ onto a wedge $W_{(\gamma,\gamma+\alpha_1)}$ of angle α_1 for some $\alpha_1 \in (\pi, 2\pi)$. Further, we assume that $g(B_2) = W_{(0,\pi)}$ and $g(B_1) = W_{(\pi-\alpha_1, 2\pi-\alpha_1)}$, i.e., $W_{(\gamma,\gamma+\alpha_1)} = W_{(\pi-\alpha_1,\pi)}$. Let $D_{m-i+1} = B_i \cup B_{i+1} \cup \cdots \cup B_m$ and $D'_{m-i+1} = g(D_{m-i+1})$ ($i = 1, 2, \cdots, m$). Define

$$\varphi_0 = \min\{\varphi : D'_{m-2} \backslash \overline{W}_{(\pi-\alpha_1,\pi)} \subset W_{(\pi,\pi+\varphi)}\}.$$

Obviously, $0 < \varphi_0 < 2\pi - \alpha_1$. We define a function $f_0\colon \overline{\mathbb{R}}^n \to \overline{\mathbb{R}}^n$ by

$$f_0(r,\varphi,z) = \begin{cases} (r, \frac{\pi}{\alpha_1}(\varphi + (\alpha_1 - \pi)), z) & \text{for } \pi - \alpha_1 < \varphi \le \pi, \\ (r, \varphi, z) & \text{for } \pi < \varphi \le \pi + \varphi_0, \\ (r, \frac{(\pi-\varphi_0)\varphi + (\pi+\varphi_0)(\pi-\alpha_1)}{(\pi-\alpha_1)+(\pi-\varphi_0)}, z) & \text{for } \pi + \varphi_0 < \varphi \le 3\pi - \alpha_1. \end{cases}$$

Then f_0 is a K_1-quasiconformal mapping, and $K_1 < \infty$ depends only on α_1, φ_0 and n. Let $f_1 = g^{-1} \circ f_0 \circ g$. Then $f_1 \colon \overline{\mathbb{R}}^n \to \overline{\mathbb{R}}^n$ is K_1-quasiconformal and $f_1(D_m) = D_{m-1}$.

Similarly, for $j = 2, \ldots, m-1$ we may define a K_j-quasiconformal mapping $f_j \colon \overline{\mathbb{R}}^n \to \overline{\mathbb{R}}^n$ with $f_j(D_{m-j+1}) = D_{m-j}$. Then $f = f_{m-1} \circ f_{m-2} \circ \cdots \circ f_1 \circ h$ is a K-quasiconformal mapping of the whole space onto itself and $f(D_m) = \mathbb{B}^n$, where h is a suitable Möbius transformation and $K = \prod_{j=1}^{m-1} K_j$. The claim follows. $\square$

5. Closed quasiregular mappings

In this section, we study some of boundary regularity conditions, introduced by J. Väisälä, under closed quasiregular mappings. These conditions are closely related to the boundary the mapping problems. We show that under certain assumptions boundary regularity conditions are preserved under closed quasiregular mappings. Indeed, without additional assumptions, the mapping properties of quasiregular mappings can be very different from quasiconformal ones, as illustrated by the following simple example.

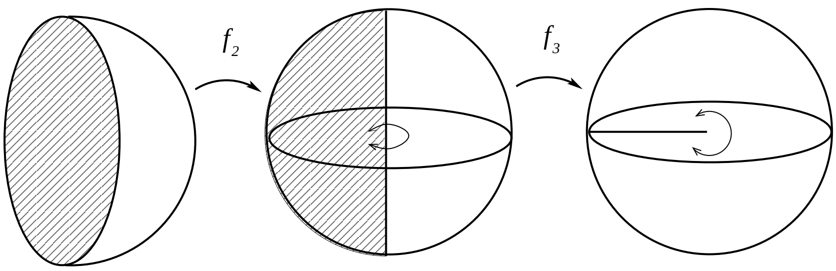
Example 5.1. *It is well-known that one may map the unit ball $\mathbb{B}^n$ quasiconformally onto the half-ball $\mathbb{B}^n_+ = \{x : |x| < 1$ and $x_1 > 0\}$. Denote by f_1 a quasiconformal mapping such that $f_1 \colon \mathbb{B}^n \to \mathbb{B}^n_+ = f_1(\mathbb{B}^n)$. Let f_2 be the winding mapping defined by*

$$f_2(x_1, \ldots, x_n) = \big(r_2 \cos(2\varphi_2), r_2 \sin(2\varphi_2), x_3, \ldots, x_n\big),$$

defined in the cylindrical coordinates such that $r_2 = \sqrt{x_1^2 + x_2^2}$ and $\varphi_2 = \arctan(x_2/x_1)$ for $x \in \mathbb{R}^n$. The mapping $f_2 \colon \mathbb{R}^n \to \mathbb{R}^n$ is a well-known example of a quasiregular mapping (see e.g. [15, 3.1]). In particular, we have $f_2(\mathbb{B}^n_+) = \mathbb{B}^n \setminus \{x : x_1 \le 0,\ x_2 = 0\}$. Similarly, let $f_3 \colon \mathbb{R}^n \to \mathbb{R}^n$ be the winding mapping defined by

$$f_3(x_1, \ldots, x_n) = \big(r_3 \cos(2\varphi_3), x_2, r_3 \sin(2\varphi_3), x_4, \ldots, x_n\big),$$

where $r_3 = \sqrt{x_1^2 + x_3^2}$ and $\varphi_3 = \arctan(x_3/x_1)$ for $x \in \mathbb{R}^n$. Then the quasiregular mapping $f = f_3 \circ f_2 \circ f_1$ maps the unit ball onto the domain $\mathbb{B}^n \setminus \{x \in \mathbb{R}^n : x_1 \le 0,\ x_2 = x_3 = 0\}$.



In particular, for $n = 3$ the image set is the unit ball with the negative x_1-axis removed. However, the cluster set of this mapping clearly consists of the unit sphere $\mathbb{S}^2$ and the two-dimensional disk

$$\mathbb{D} = \{x \in \mathbb{R}^3 : \sqrt{x_1^2 + x_3^2} \le 1 \text{ and } x_2 = 0\},$$

and thus the mapping f is not closed.

Our results in this section, Theorems 5.3 and 5.4, are generalizations of similar results for quasiconformal mappings (see [18]).

Boundary regularity conditions

Recall that a quasiconformal map of $\mathbb{B}^n$ onto $\mathbb{B}^n$ has a homeomorphic extension to $\overline{\mathbb{B}}^n$, see [16, Theorem 2]. The following definition is from [18, 17.5].

Definition 5.2. *Let G be a domain in $\overline{\mathbb{R}}^n$ and let $b \in \partial G$.*

(1) The domain G is locally connected at b if b has arbitrarily small neighborhoods U such that $U \cap G$ is connected.

(2) The domain G is finitely connected at b if b has arbitrarily small neighborhoods U such that $U \cap G$ has a finite number of components.

(3) The domain G has property P_1 at b if the following condition is satisfied: Whenever E and F are connected subsets of G such that $b \in \overline{E} \cap \overline{F}$ we have $\mathrm{M}(\Delta(E, F; G)) = \infty$.

(4) The domain G has property P_2 at b if: For each point $b_1 \in \partial G$, $b_1 \neq b$, there is a compact set $F \subset G$, and a constant $\delta > 0$, such that $\mathrm{M}(\Delta(E, F; G)) \geq \delta$ whenever E is a connected set in G such that $\overline{E}$ contains b and b_1.

(5) The domain G is locally quasiconformally collared at b if there is a neighborhood U of b and a homeomorphism g of $U \cap \overline{G}$ onto the set $\{x \in \overline{\mathbb{R}}^n : |x| < 1$ and $x_n \geq 0\}$ such that $g|U \cap G$ is quasiconformal.

(6) The domain G is said to have one of the above properties at the boundary if it has it at every boundary point.

Theorem 5.3. *Suppose that G and G' are domains in $\overline{\mathbb{R}}^n$, and let $f\colon \overline{G} \to \overline{G'}$ be a continuous function such that $fG = G'$, and the mapping $f_1 = f|G$ is quasiregular and closed. If G is a P_1 domain, and G' is locally connected on the boundary, then G' is P_1.*

Theorem 5.4. *Suppose that G and G' are domains in $\overline{\mathbb{R}}^n$, and let $f\colon \overline{G} \to \overline{G'}$ be a continuous function such that $fG = G'$ and the mapping $f_1 = f|G$ is quasiregular and closed. If G is a P_2 domain, then G' also is P_2.*

Recall the next result from [18, 17.7]:

Theorem 5.5. *The following conditions are equivalent:*

(1) G is finitely connected at b.

(2) Every neighborhood U of b contains a neighborhood V of b such that $V \cap G$ is contained in the union of a finite number of components of $U \cap G$.

(3) If U is a neighborhood of b and if (x_j) is a sequence of points such that $x_j \to b$ and $x_j \in G$, then there is a subsequence which is contained in a single component of $U \cap G$.

The next theorem, due to M. Vuorinen, is a generalization of [18, 17.13].

Theorem 5.6. *[19, Theorem 4.2] Suppose that $f\colon G \to G'$ is a closed quasiregular mapping and that G has the property P_1 at the point $b \in \partial G$. Then the set $C(f, b)$ contains at most one point at which G' is finitely connected.*

The combination of Theorems 5.5 and 5.6 easily implies the following result about the extension of closed quasiregular mappings.

Corollary 5.7. *Let $f\colon G \to G'$ be a closed quasiregular mapping, let G be a P_1 domain, and let G' be finitely connected on the boundary. Then f can be extended to a continuous mapping $\overline{f}\colon \overline{G} \to \overline{G'}$.*

Now we are ready to prove Theorems 5.3 and 5.4.

Proof. [Proof of Theorem 5.3] Let $b' \in \partial G'$. For a point b in ∂G we define a set $V(b, r)$ to be the b-component of the set

$$f^{-1}\Big(\overline{G'} \cap \mathbb{B}^n(f(b), r)\Big).$$

Because $C(f_1) = \partial G'$ by Corollary 3.3, we may find a point $b \in \partial G$ such that $f(b) = b'$.

Now, let E', F' be any continua in G' such that $b' \in \overline{E'} \cap \overline{F'}$. As G' locally connected on the boundary, each neighborhood U of b' is connected and intersects with E' and F'. Let V be the b-component of $f^{-1}U$. We choose E, F to be the b-components of $(f^{-1}E') \cap V \cap G$ and $(f^{-1}F') \cap V \cap G$, respectively. It follows that E, F are continua in G and $b \in \overline{E} \cap \overline{F}$. As G is a P_1 domain, $\mathrm{M}(\Delta(E, F; G)) = \infty$. By Theorem 3.2(5),

$$N(f, G) = p < \infty.$$

Let $\Gamma = \Delta(E, F; G)$. By Theorem 2.3

$$\mathrm{M}(\Gamma) \le N(f_1, G) K_O(f_1) \mathrm{M}(f_1 \Gamma),$$

and thus

$$\infty = \mathrm{M}(\Gamma) \le N(f, G) K_O(f_1) \mathrm{M}(f_1 \Gamma) \le p K_O(f_1) \mathrm{M}(\Delta(E', F'; G')).$$

So, we have concluded that $\mathrm{M}(\Delta(E', F'; G')) = \infty$, and the claim is proved. □

Proof. [Proof of Theorem 5.4] Let $b', b'_1 \in \partial G'$ such that $b'_1 \neq b'$ and $E' \subset G'$ be a connected set such that $b', b'_1 \in \overline{E'}$. By Lemma 3.4 we may choose

$$E \subset (f^{-1}E') \cap G$$

such that $fE = E'$ and E is connected. As by Lemma 3.4

$$f\partial E = \partial fE, \quad f\partial G = \partial fG,$$

and

$$b', b'_1 \in \partial E' \cap \partial G' = \partial fE \cap \partial fG.$$

Hence, we may conclude that $\partial G \cap \partial E$ contains at least two separate points, $b \in f^{-1}(b')$ and $b_1 \in f^{-1}(b'_1)$.

Now $b, b_1 \in \partial G$ are separate points and E is a continuum such that $b, b_1 \in \overline{E}$. It was assumed, that G is a P_2 domain, and so there exists a compact set F and a constant $\delta > 0$ such that $\mathrm{M}(\Delta(E, F; G)) \ge \delta$. As f_1 is a closed quasiregular mapping, by Theorem 3.2, $N(f_1, G) = p < \infty$. By Theorem 2.3

$$\delta \le \mathrm{M}\Big(\Delta(E, F; G)\Big) \le N(f_1, G) K_O(f_1) \mathrm{M}\Big(\Delta(E', f_1 F; G')\Big).$$

We may choose $F' = f_1 F$ and

$$\delta' = \frac{\delta}{p K_O(f_1)} > 0.$$

It follows that

$$\mathrm{M}(\Delta(E', F'; G')) \ge \delta' > 0.$$

As $f_1 F$ is a compact set, the set G' is a P_2 domain with the corresponding compact set F' and the constant δ', proving the claim. □

The following problem related to the branch set of a closed quasiregular mappings was given by M. Vuorinen in 1980's [22, p. 193], and it is still open.

Problem 5.8. *Let $f\colon \mathbb{B}^n \to f\mathbb{B}^n \subset \mathbb{B}^n$ be discrete, open and proper. Assume that $n \ge 3$ and B_f is compact. Is f one-to-one? The answer is yes, if $f\mathbb{B}^n = \mathbb{B}^n$.*

Remark 5.9. *A mapping $f\colon G \to \mathbb{R}^n$ is called harmonic if all its coordinate functions $u_j\colon G \to \mathbb{R}$ satisfy the Laplace equation $\Delta u_j = 0$. In particular, analytic functions are harmonic. Recall that each closed analytic function is a finite Blaschke product or a constant (see Remark 3.5). The class of harmonic mappings has been extensively studied [4], and certain topological properties of harmonic mappings have been considered in [10]. However, to our knowledge, the class of closed harmonic mappings has not been studied.*

6. Boundary behavior

In this section, we prove some boundary behavior results for closed quasiregular mappings.

Existence of arcwise limits.

A classical theorem by P. Koebe states that a conformal mapping of a simply connected domain G in the complex plane $\mathbb{C}$ has arcwise limits along all end-cuts of G. R. Näkki [14] proved a similar result for quasiconformal mappings in $\mathbb{R}^n$. We show that this result holds for closed quasiregular mappings as well.

Let G a domain $\mathbb{R}^n$. A point $b \in \partial G$ is called *accessible* from G if there is a closed Jordan arc γ contained in G except for one endpoint, b. Then γ is called an *end-cut* of G from b. Suppose that f is a mapping of G into $\overline{\mathbb{R}}^n$. The cluster set of f at b along an end-cut γ from b is denoted by $C_\gamma(f, b)$. If $C_\gamma(f, b) = \{b'\}$, then b' is called an *arcwise limit* of f at b.

Definition 6.1. *The spherical (chordal) metric q in $\overline{\mathbb{R}}^n$ is defined by*

$$\begin{cases} q(x,y) &= \frac{|x-y|}{\sqrt{1+|x|^2}\,\sqrt{1+|y|^2}}, \text{ for } x \neq \infty \neq y\,, \\ q(x,\infty) &= \frac{1}{\sqrt{1+|x|^2}}. \end{cases}$$

For a set E in $\overline{\mathbb{R}}^n$, we denote by $q(E)$ the diameter of E with respect to the metric $q(x, y)$.

Lemma 6.2. ([13]) *Let G be a locally quasiconformally collared domain and let E, F be nondegenerate continua in G. Then for each $r > 0$ there exists $\delta > 0$ such that $\mathrm{M}(\Delta(E, F; D)) \geq \delta$ whenever $q(E) \geq r$ and $q(F) \geq r$.*

Theorem 6.3. *Suppose that G is domain in $\mathbb{R}^n$, $f\colon G \to G' = fG$ is a closed quasiregular mapping, and G' is a locally quasiconformally collared domain. Then f has arcwise limits along all end-cuts of G.*

Proof. Let $b \in \partial G$, and suppose that γ is an end-cut from the point b. Fix a continuum $C \subset G$. We choose a sequence of neighborhoods U_k of b such that

$$\bigcap_{k=1}^{\infty} U_k = \{b\} \text{ and } \gamma_k = U_k \cap G \cap \gamma$$

is connected for $k = 1, 2, \ldots$. Write $C' = fC$. By Theorem 3.2, $f^{-1}C'$ is compact, and by Lemma 3.1 every path in $\Delta(C', |f(\gamma_k)|; G')$ has a lifting in G beginning at $|\gamma_k|$ and leading to $f^{-1}C'$. Denote by Γ_k the family of these liftings. Then

$$\lim_{k\to\infty} \mathrm{M}(\Gamma_k) = 0,$$

and $f(\Gamma_k) < \Delta(C', |f(\gamma_k)|; G')$. Hence, by Theorem 2.2, we have

$$\mathrm{M}(\Delta(C', |f(\gamma_k)|; G')) \leq \mathrm{M}(f(\Gamma_k)) \leq K_I(f)\mathrm{M}(\Gamma_k) \to 0$$

as $k \to \infty$. Then it follows by Lemma 6.2 that $\lim_{k\to\infty} q(|f(\gamma_k)|) = 0$ and hence f has a limit at b along γ. □

Relative size of preimages

By Theorem 3.2, a set D has at most $p < \infty$ preimages under a closed quasiregular mapping. Next we give an upper bound for the diameter of a preimage in terms of the diameter of another preimage, i.e., we will prove that only the images of the sets of roughly similar size can coincide in a closed quasiregular mapping. Our result reads as follows.

Theorem 6.4. *Let $f\colon G \to \mathbb{R}^n$ be a closed K-quasiregular mapping. Suppose that $0 < t < 1$, and $A_1, A_2 \subset \mathbb{B}^n(x, tr)$ are nondegenerate continua with $A_1 \cap A_2 = \emptyset$ such that $fA_1 = fA_2$ and $\overline{\mathbb{B}}^n(x, r) \subset G$. Then there is a homeomorphism $h\colon [0, \infty) \to [0, \infty)$ depending only on n, K, t and $N(f, \mathbb{B}^n(x, r))$ such that $d(A_1) \geq h(d(A_2))$.*

Before the proof of Theorem 6.4, we introduce two lemmas.

Lemma 6.5. *[9, Lemma 2.31.] Let* $0 < r_0 < 1$. *Then*

$$C(n, r_0)\mathrm{M}\Big(\Delta(\mathbb{B}^n(r), \mathbb{S}^{n-1})\Big) \le \gamma_n(1/r) \le \mathrm{M}\Big(\Delta(\mathbb{B}^n(r), \mathbb{S}^{n-1})\Big)$$

for $r_0 > r > 0$, *where*

$$C(n, r_0) = \Big(1 - \frac{\log \lambda_n}{\log r_0}\Big)^{1-n}.$$

Lemma 6.6. *[22, 1.43] Let* $0 < s < 1$. *Then for all* $a, x, y \in \overline{\mathbb{B}}^n(s)$

$$\frac{1-s^2}{(1+s^2)^2}|x-y| \le |T_a x - T_a y| \le \frac{1}{1-s^2}|x-y|.$$

Proof. [Proof of Theorem 6.4] Let $p = N(f, \mathbb{B}^n(x, r)) < \infty$. By replacing f with the mapping $f \circ g$, where $g\colon z \mapsto (z-x)/r$, if necessary, we may assume that $\mathbb{B}^n(x, r) = \mathbb{B}^n$. We choose the points $z_1, z_2 \in \overline{A_1}$ and $y_1, y_2 \in \overline{A_2}$ such that $d(A_1) \le 2|z_1 - z_2|$ and $d(A_2) \le 2|y_1 - y_2|$, respectively. Next we estimate the modulus of curve family $\Delta(A_1, \mathbb{S}^{n-1})$ with the capacity of spherical annulus (2), and then apply Theorem 2.2 to obtain the estimate:

$$\begin{aligned}\omega_{n-1}\Big[\log\Big(\frac{1}{2|T_{z_1}(z_2)|}\Big)\Big]^{1-n} &\ge \mathrm{M}\Big(\Delta(A_1, \mathbb{S}^{n-1})\Big) \ge \frac{\mathrm{M}\Big(f(\Delta(A_1, \mathbb{S}^{n-1}))\Big)}{K_I(f)} \\ &= \frac{\mathrm{M}\Big(\Delta(fA_1, f\mathbb{S}^{n-1})\Big)}{K_I(f)} = \frac{\mathrm{M}\Big(f(\Delta(A_2, \mathbb{S}^{n-1}))\Big)}{K_I(f)}.\end{aligned}$$

Now we apply the K_O-inequality, and then estimate the modulus in terms of the capacity of the Grötzsch ring domain

$$\frac{\mathrm{M}\Big(f(\Delta(A_2, \mathbb{S}^{n-1}))\Big)}{K_I(f)} \ge \frac{\mathrm{M}\Big(\Delta(A_2, \mathbb{S}^{n-1})\Big)}{pK_I(f)K_O(f)} \ge \frac{\gamma\Big(|T_{y_1}(y_2)|^{-1}\Big)}{pK_I(f)K_O(f)}.$$

By combining these estimates with Lemma 6.5 and (3) we obtain

$$\omega_{n-1}\Big[\log\Big(\frac{1}{2|T_{z_1}(z_2)|}\Big)\Big]^{1-n} \ge \frac{\gamma\Big(|T_{y_1}(y_2)|^{-1}\Big)}{pK_I(f)K_O(f)} \ge \frac{C(n,t)\omega_{n-1}}{pK_I(f)K_O(f)}\Big[\log\Big(\frac{\lambda_n}{|T_{y_1}(y_2)|}\Big)\Big]^{1-n}.$$

We have

$$\Big(2\lambda_n |T_{z_1}(z_2)|\Big)^{C(K,n,p,t)} \ge |T_{y_1}(y_2)|,$$

and by applying Lemma 6.6 we obtain

$$\Big[2\lambda_n \frac{1-t^2}{(1+t^2)^2}|z_1 - z_2|\Big]^{C(K,n,p,t)} \ge \frac{1}{1-t^2}|y_1 - y_2|,$$

proving the claim. □

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