Filomat 27:2 (2013), 411–427
DOI 10.2298/FIL1302411A

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

On q-Sumudu Transforms of Certain q-Polynomials

Durmuş Albayrak[a], Sunil Dutt Purohit[b], Faruk Uçar[a,*]

[a]*Department of Mathematics, Marmara University, TR-34722, Kadıköy, Istanbul, Turkey*
[b]*Department of Basic Science (Mathematics), College of Technology and Engineering, M.P. University of Agriculture and Technology Udaipur-313001 (Rajasthan), India*

Abstract. Although Sumudu transform is the theoritical dual of the Laplace transform, it has many applications in sciences and engineering for its special fundamental properties. In a previous paper [3], we studied q-analogues of the Sumudu transform and derived some fundamental properties. This paper follows the previous paper and aims to provide some applications of the q-Sumudu transform. The authors give q-Sumudu transforms of some q-polynomials and q-functions. Also, we evaluated the q-Sumudu transform of basic analogue of Fox's H-function.

1. Introduction and Preliminaries

In the classical analysis, differential equations play a major role in mathematics, physics and engineering. There are lots of different techniques for solving differential equations. Integral transforms were widely used and thus a lot of work has been done on the theory and applications of integral transforms. Most popular integral transforms are due to Laplace, Fourier, Mellin and Hankel. In 1993, the Sumudu transform was proposed originally by Watugala [17] and he applied it to the solution of ordinary differential equations in control engineering problems. The Sumudu transform plays a curious role in the solution of ordinary differential equations and other branches of Mathematics and Physics. Nevertheless, this new transform rivals the Laplace transform in problem solving. Its main advantage is the fact that it may be used to solve problems without resorting to a new frequency domain, because it preserves scale and unit properties [4]. For further detail, one may refer to the recent papers [9]-[10] on this subject.

The theory of q-analysis, the foundation 18^{th}century, in recent past have been applied in the many areas of mathematics and physics like ordinary fractional calculus, optimal control problems, q-transform analysis and in finding solutions of the q-difference and q-integral equations. In 1910, Jackson [7] presented a precise definition of so-called the q-Jackson integral and developed q-calculus in a systematic way. It is well known that, in the literature, there are two types of the q-Laplace transform and they are studied in detail by many authors. For example [1, 6, 14].

Recently, authors [3] have introduced and study q-analogues of the Sumudu transform and derived their fundamental properties. The aim of this paper is to give q-Sumudu transform of certain q-functions and

2010 *Mathematics Subject Classification*. 33D45, 33D60, 33D05, 33D15, 05A30
Keywords. q-Sumudu transforms, q-Hypergeometric functions, q-special polynomials, Fox's H-function.
Received: 13 June 2012; Accepted: 1 November 2012
Communicated by Predrag Stanimirovic
This paper was supported by the Marmara University, Scientific Research Commission (BAPKO) under Grant 2011 FEN-A-110411-0101.
Corresponding author
Email addresses: durmusalbayrak@marun.edu.tr (Durmuş Albayrak), sunil-a-purohit@yahoo.com (Sunil Dutt Purohit), sunil_a_purohit@yahoo.com (Faruk Uçar)

their special cases. For the convenience of the reader, we deem it proper to give here the basic definitions and facts from the q-calculus.

Throughout this paper, we will assume that q satisfies the condition $0 < \left|q\right| < 1$. The q-derivative $D_q f$ of an arbitrary function f is given by

$$(D_q f)(x) = \frac{f(x) - f(qx)}{(1-q)x},$$

where $x \neq 0$. Clearly, if f is differentiable, then

$$\lim_{q\to 1^-} (D_q f)(x) = \frac{df(x)}{dx}.$$

For any real number α,

$$[\alpha] := \frac{q^\alpha - 1}{q - 1}.$$

In particular, if $n \in \mathbb{Z}^+$, we denote

$$[n] = \frac{q^n - 1}{q-1} = q^{n-1} + \cdots + q + 1.$$

Following usual notation are very useful in the theory of q-calculus:

$$(a;q)_n = \prod_{k=0}^{n-1}\left(1 - aq^k\right), \qquad (a;q)_\infty = \prod_{k=0}^{\infty}\left(1 - aq^k\right),$$

$$(a;q)_t = \frac{(a;q)_\infty}{(aq^t;q)_\infty} \qquad (t \in \mathbb{R}).$$

The q-analogues of the classical exponential functions are defined by

$$e_q(t) = \sum_{n=0}^{\infty} \frac{t^n}{(q;q)_n} = \frac{1}{(t;q)_\infty} \qquad |t| < 1, \tag{1.1}$$

$$E_q(t) = \sum_{n=0}^{\infty} \frac{(-1)^n q^{n(n-1)/2} t^n}{(q;q)_n} = (t;q)_\infty \qquad (t \in \mathbb{C}). \tag{1.2}$$

By means of the (1.1) and (1.2), q-trigonometric functions defined as

$$\sin_q t = \frac{e_q(it) - e_q(-it)}{2i} = \sum_{n=0}^{\infty} (-1)^n \frac{t^{2n+1}}{(q;q)_{2n+1}}, \tag{1.3}$$

$$\mathrm{Sin}_q t = \frac{E_q(-it) - E_q(it)}{2i} = \sum_{n=0}^{\infty} (-1)^n q^{n(2n+1)} \frac{t^{2n+1}}{(q;q)_{2n+1}}, \tag{1.4}$$

$$\cos_q t = \frac{e_q(it) + e_q(-it)}{2} = \sum_{n=0}^{\infty} (-1)^n \frac{t^{2n}}{(q;q)_{2n}}, \tag{1.5}$$

$$\mathrm{Cos}_q t = \frac{E_q(it) + E_q(-it)}{2} = \sum_{n=0}^{\infty} (-1)^n q^{n(2n-1)} \frac{t^{2n}}{(q;q)_{2n}}. \tag{1.6}$$

Furthermore, q-hypergeometric functions are defined by

$$ {}_r\phi_s\left[\begin{matrix} a_1 & a_2 & \cdots & a_r \\ b_1 & b_2 & \cdots & b_s \end{matrix};q,z\right] = \sum_{n=0}^{\infty} \frac{(a_1,a_2,\ldots,a_r;q)_n}{(b_1,b_2,\ldots,b_s;q)_n} \frac{z^n}{(q;q)_n}, $$

$$ {}_r\psi_s\left[\begin{matrix} a_1 & a_2 & \cdots & a_r \\ b_1 & b_2 & \cdots & b_s \end{matrix};q,z\right] = \sum_{n=-\infty}^{\infty} \frac{(a_1,a_2,\ldots,a_r;q)_n}{(b_1,b_2,\ldots,b_s;q)_n} \left[(-1)^n q^{\binom{n}{2}}\right]^{s-r} z^n, $$

and

$$ {}_{m-k}\Phi_{m-1}\left[\begin{matrix} a_1 & a_2 & \cdots & a_{m-k} \\ b_1 & b_2 & \cdots & b_{m-1} \end{matrix};q,z\right] = \sum_{n=0}^{\infty} \frac{(a_1,a_2,\ldots,a_{m-k};q)_n}{(b_1,b_2,\ldots,b_{m-1};q)_n} \left[(-1)^n q^{\binom{n}{2}}\right]^{k} \frac{z^n}{(q;q)_n}. $$

where

$$ (a_1,a_2,\ldots,a_r;q)_n = \prod_{i=1}^{r} (a_i;q)_n . $$

For further detail and properties about q-hypergeometric functions see [5, 12, 13] and many others.

If a function $f(x)$ has a series expansion as follow

$$ f(x) = \sum_{n=-\infty}^{\infty} a_n x^n, $$

then the following function is well defined

$$ f[x-y] = \sum_{n=-\infty}^{\infty} a_n (x-y)_q^n . \tag{1.7} $$

The improper integral is defined by [7, 11]

$$ \int_0^x f(t)d_qt = x(1-q)\sum_{k=0}^{\infty} q^k f(xq^k), \tag{1.8} $$

$$ \int_0^{\infty/A} f(x)d_qx = (1-q)\sum_{k\in\mathbb{Z}} \frac{q^k}{A} f(\frac{q^k}{A}). \tag{1.9} $$

q-analogues of gamma and beta functions defined as follow [11]

$$ \Gamma_q(\alpha) = \int_0^{1/(1-q)} x^{\alpha-1} E_q(q(1-q)x)\, d_qx \quad (\alpha>0), \tag{1.10} $$

$$ \Gamma_q(\alpha) = K(A;\alpha)\int_0^{\infty/A(1-q)} x^{\alpha-1} e_q(-(1-q)x)\, d_qx \ \ (\alpha>0), \tag{1.11} $$

where

$$ K(A;t) = A^{t-1}\frac{(-q/A;q)_\infty}{(-q^t/A;q)_\infty}\frac{(-A;q)_\infty}{(-Aq^{1-t};q)_\infty} \quad (t\in\mathbb{R}). \tag{1.12} $$

The function $K(A;t)$ provides the following equation for the variable t (see [11]):

$$ K(x;t+1) = q^t K(x;t) \tag{1.13} $$

q-gamma function has the following property

$$\Gamma_q(x) = \frac{(q;q)_\infty}{(q^x;q)_\infty}(1-q)^{1-x} = \frac{(q;q)_{x-1}}{(1-q)^{x-1}}, \tag{1.14}$$

and we have

$$\lim_{q\to 1^-} \Gamma_q(\alpha) = \Gamma(\alpha).$$

q-Bessel functions were introduced by Jackson [8] in 1905 and are therefore referred to as Jackson's q-Bessel functions. Some q-analogues of the Bessel functions are given by

$$\begin{aligned} J_\nu^{(1)}(z;q) &= \frac{\left(q^{\nu+1};q\right)_\infty}{(q;q)_\infty}\left(\frac{z}{2}\right)^\nu {}_2\Phi_1\left[\begin{matrix} 0 & 0 \\ q^{\nu+1} & \end{matrix} ;q,-\frac{z^2}{4}\right], \qquad |z|<2 \\ &= \left(\frac{z}{2}\right)^\nu \sum_{n=0}^{\infty} \frac{\left(-z^2/4\right)^n}{(q;q)_{\nu+n}(q;q)_n} \end{aligned} \tag{1.15}$$

and

$$\begin{aligned} J_\nu^{(2)}(z;q) &= \frac{\left(q^{\nu+1};q\right)_\infty}{(q;q)_\infty}\left(\frac{z}{2}\right)^\nu {}_0\Phi_1\left[\begin{matrix} - \\ q^{\nu+1} \end{matrix} ;q,-\frac{q^{\nu+1}z^2}{4}\right] \\ &= \left(\frac{z}{2}\right)^\nu \sum_{n=0}^{\infty} \frac{q^{n(n+\nu)}\left(-z^2/4\right)^n}{(q;q)_{\nu+n}(q;q)_n}. \end{aligned} \tag{1.16}$$

q-Bessel functions are related by the following equality

$$J_\nu^{(2)}(z;q) = \left(-\frac{z^2}{4};q\right)_\infty J_\nu^{(1)}(z;q), \qquad |z|<2.$$

q-Bessel functions are q-extensions of the Bessel function of the first kind since

$$\lim_{q\to 1^-} J_\nu^{(k)}((1-q)z;q) = J_\nu(z), \qquad k=1,2.$$

The third kind q-analogue of the Bessel function is given by following formula

$$\begin{aligned} J_\nu^{(3)}(z;q) &= \frac{\left(q^{\nu+1};q\right)_\infty}{(q;q)_\infty} z^\nu {}_1\Phi_1\left[\begin{matrix} 0 \\ q^{\nu+1} \end{matrix} ;q,qz^2\right] \\ &= z^\nu \sum_{n=0}^{\infty} \frac{(-1)^n q^{n(n-1)/2}\left(qz^2\right)^n}{(q;q)_{\nu+n}(q;q)_n}. \end{aligned} \tag{1.17}$$

This third kind q-Bessel function is also known as the Hahn-Exton q-Bessel function. This is also q-extension of the Bessel function of the first kind since

$$\lim_{q\to 1^-} J_\nu^{(3)}((1-q)z;q) = J_\nu(2z).$$

Albayrak, Purohit and Uçar [3] defined the q-analogues of the Sumudu transform by means of the following q-integrals

$$S_q\{f(t);s\} = \frac{1}{(1-q)s}\int_0^s E_q\left(\frac{q}{s}t\right) f(t)\, d_qt, \qquad s\in(-\tau_1,\tau_2), \tag{1.18}$$

over the set of functions

$$A = \left\{ f(t) | \exists M, \tau_1, \tau_2 > 0,\ |f(t)| < ME_q\left(|t| / \tau_j\right),\ t \in (-1)^j \times [0, \infty) \right\}$$

and

$$\mathbb{S}_q\{f(t); s\} = \frac{1}{(1-q)s} \int_0^\infty e_q\left(-\frac{1}{s}t\right) f(t)\, d_q t, \qquad s \in (-\tau_1, \tau_2), \tag{1.19}$$

over the set of functions

$$B = \left\{ f(t) | \exists M, \tau_1, \tau_2 > 0, |f(t)| < Me_q\left(|t| / \tau_j\right), t \in (-1)^j \times [0, \infty) \right\}.$$

By virtue of (1.8) and (1.9), q-Sumudu transforms can be expressed as

$$S_q\{f(t); s\} = (q;q)_\infty \sum_{k=0}^\infty \frac{q^k f\left(sq^k\right)}{(q;q)_k}, \tag{1.20}$$

and

$$\mathbb{S}_q\{f(t); s\} = \frac{s^{-1}}{\left(-\frac{1}{s}; q\right)_\infty} \sum_{k \in \mathbb{Z}} q^k f(q^k) \left(-\frac{1}{s}; q\right)_k. \tag{1.21}$$

2. Main Theorems

In this section, we shall investigate certain theorems, which gives a number of image formulas involving q-hypergeometric functions, under the q-Sumudu transforms.

Theorem 2.1. *In correspondence to the bounded sequence A_n, let $f(x)$ is given by*

$$f(x) = \sum_{n=0}^\infty A_n x^n, \tag{2.1}$$

then for $\alpha > 0$ the following results hold:

$$S_q\left\{x^{\alpha-1} f(x); s\right\} = s^{\alpha-1}(1-q)^{\alpha-1} \sum_{n=0}^\infty A_n \Gamma_q(\alpha+n) \left[(1-q)s\right]^n, \tag{2.2}$$

$$\mathbb{S}_q\left\{x^{\alpha-1} f(x); s\right\} = s^{\alpha-1}(1-q)^{\alpha-1} \sum_{n=0}^\infty A_n \frac{\Gamma_q(\alpha+n)}{K(s;\alpha+n)} \left[(1-q)s\right]^n. \tag{2.3}$$

Proof. In view of (1.20) and (2.1), we have

$$\begin{aligned} S_q\left\{x^{\alpha-1} f(x); s\right\} &= (q;q)_\infty \sum_{k=0}^\infty \frac{q^k \left(sq^k\right)^{\alpha-1}}{(q;q)_k} f\left(sq^k\right) \\ &= (q;q)_\infty \sum_{k=0}^\infty \frac{q^k \left(sq^k\right)^{\alpha-1}}{(q;q)_k} \sum_{n=0}^\infty A_n \left(sq^k\right)^n \\ &= s^{\alpha-1}(q;q)_\infty \sum_{n=0}^\infty A_n s^n \sum_{k=0}^\infty \frac{(q^{\alpha+n})^k}{(q;q)_k}. \end{aligned} \tag{2.4}$$

Substituting (1.1) into (2.4) and from (1.14) we deduce that

$$\begin{aligned} S_q\left\{x^{\alpha-1}f(x);s\right\} &= s^{\alpha-1}(q;q)_\infty \sum_{n=0}^{\infty} A_n s^n e_q\left(q^{\alpha+n}\right) \\ &= s^{\alpha-1}\sum_{n=0}^{\infty} A_n s^n \frac{(q;q)_\infty}{(q^{\alpha+n};q)_\infty} \\ &= s^{\alpha-1}(1-q)^{\alpha-1}\sum_{n=0}^{\infty} A_n \Gamma_q(\alpha+n)\left[(1-q)s\right]^n. \end{aligned}$$

The proof for $\mathbb{S}_q$-transform is similar. Using the definition (1.21) of $\mathbb{S}_q$-transform, we have

$$\begin{aligned} \mathbb{S}_q\left\{x^{\alpha-1}f(x);s\right\} &= \frac{s^{-1}}{(-1/s;q)_\infty}\sum_{k\in\mathbb{Z}} q^k(-1/s;q)_k\left(q^k\right)^{\alpha-1} f(q^k) \\ &= \frac{s^{-1}}{(-1/s;q)_\infty}\sum_{k\in\mathbb{Z}} q^k(-1/s;q)_k\left(q^{\alpha-1}\right)^k \sum_{n=0}^{\infty} A_n\left(q^k\right)^n \\ &= \frac{s^{-1}}{(-1/s;q)_\infty}\sum_{n=0}^{\infty} A_n \sum_{k\in\mathbb{Z}} \frac{(-1/s;q)_k}{(0;q)_k}\left(q^{\alpha+n}\right)^k. \end{aligned}$$

An easy computation shows that

$$\begin{aligned} \mathbb{S}_q\left\{x^{\alpha-1}f(x);s\right\} &= \frac{s^{-1}}{(-1/s;q)_\infty}\sum_{n=0}^{\infty} A_n\, {}_1\psi_1\left(-1/s;0;q,q^{\alpha+n}\right) \\ &= \frac{s^{-1}}{(-1/s;q)_\infty}\sum_{n=0}^{\infty} A_n \frac{\left(q,0,-q^{\alpha+n}/s,q^{1-(\alpha+n)}s;q\right)_\infty}{(0,-qs,q^{\alpha+n},0;q)_\infty} \\ &= s^{-1}\sum_{n=0}^{\infty} A_n \frac{(q;q)_\infty}{(q^{\alpha+n};q)_\infty}\frac{\left(-q^{\alpha+n}/s,q^{1-(\alpha+n)}s;q\right)_\infty}{(-1/s,-qs;q)_\infty}. \end{aligned}$$

Using (1.12) and (1.14) , we conclude that

$$\begin{aligned} \mathbb{S}_q\left\{x^{\alpha-1}f(x);s\right\} &= s^{-1}\sum_{n=0}^{\infty} A_n\ (1-q)^{\alpha+n-1}\Gamma_q(\alpha+n)\frac{s^\alpha}{K(s;\alpha+n)} \\ &= s^{\alpha-1}(1-q)^{\alpha-1}\sum_{n=0}^{\infty} A_n\frac{\Gamma_q(\alpha+n)}{K(s;\alpha+n)}\left[(1-q)s\right]^n. \end{aligned}$$

We obtain the following useful identities for q-Sumudu transforms:

Corollary 2.1. *The following identities are valid:*

$$S_q\left\{x^{\alpha-1};s\right\} = s^{\alpha-1}(1-q)^{\alpha-1}\Gamma_q(\alpha), \tag{2.5}$$

$$\mathbb{S}_q\left\{x^{\alpha-1};s\right\} = s^{\alpha-1}(1-q)^{\alpha-1}\frac{\Gamma_q(\alpha)}{K(s;\alpha)}. \tag{2.6}$$

Proof. If we set $A_0 = 1$, $A_n = 0$ $(n \geq 1)$ in (2.1), then we have $f(x) = 1$ and making use of the above theorem one can easily obtain the desired results.

Corollary 2.2. *The q-Sumudu transform of first kind Bessel function is given by*

$$S_q\left\{x^{\alpha-1}J_{2v}^{(1)}\left(2\sqrt{ax};q\right);s\right\} = \frac{a^v}{\Gamma_q(2v+1)}s^{\alpha+v-1}\frac{(q;q)_{a+\nu-1}}{(1-q)^{2v}}\,{}_2\Phi_1\left[\begin{array}{cc} q^{\alpha+v} & 0 \\ q^{2\nu+1} & \end{array};q,-as\right]. \tag{2.7}$$

Proof. If we set $\alpha = \alpha + v > 0$ and

$$A_n = \frac{(-1)^n a^{n+v}}{(q^{2v+1};q)_n (q;q)_n (1-q)^{2v}\Gamma_q(2v+1)}$$

in (2.1), we have

$$f(x) = x^{-v}J_{2v}^{(1)}\left(2\sqrt{ax};q\right). \tag{2.8}$$

The assertion of the corollary follows easily from the theorem.

Further, on setting $\alpha = \nu/2 + 1$ and $\nu = \nu/2$ in the assertion (2.7) respectively, we obtain that

$$S_q\left\{x^{\nu/2}J_v^{(1)}\left(2\sqrt{ax};q\right);s\right\} = a^{\nu/2}s^{\nu}e_q(-as). \tag{2.9}$$

Next, we put $\nu = 0$ in (2.9), we get

$$S_q\left\{J_0^{(1)}\left(2\sqrt{ax};q\right);s\right\} = e_q(-as).$$

If we put first $\nu = 0$ in (2.7) and next choose $a = 0$, we get

$$S_q\left\{x^{\alpha-1};s\right\} = s^{\alpha-1}(1-q)^{\alpha-1}\Gamma_q(\alpha)$$

which is given in (2.5).

Theorem 2.2. *q-Sumudu transforms of the function* $f(x) = {}_{m-k}\Phi_{m-1}\left[\begin{array}{cccc} a_1 & a_2 & \cdots & a_{m-k} \\ b_1 & b_2 & \cdots & b_{m-1}\end{array};q,ax\right]$ *are given by*

$$S_q\left\{x^{\alpha-1}f(x);s\right\} = s^{\alpha-1}(q;q)_{\alpha-1}\,{}_{m-k+1}\Phi_m\left[\begin{array}{ccccc} a_1 & a_2 & \cdots & a_{m-k} & q^{\alpha} \\ b_1 & b_2 & \cdots & b_{m-1} & 0\end{array};q,as\right], \tag{2.10}$$

and

$$\mathbb{S}_q\left\{x^{\alpha-1}f(x);s\right\} = \frac{s^{\alpha-1}(q;q)_{\alpha-1}}{K(s;\alpha)}\,{}_{m-k+1}\Phi_{m-1}\left[\begin{array}{ccccc} a_1 & a_2 & \cdots & a_{m-k} & q^{\alpha} \\ b_1 & b_2 & \cdots & b_{m-1} & -\end{array};q,-\frac{as}{q^{\alpha}}\right]. \tag{2.11}$$

Proof. To prove the above results, we set

$$A_n = \frac{(a_1;q)_n(a_2;q)_n\cdots(a_{m-k};q)_n}{(b_1;q)_n(b_2;q)_n\cdots(b_{m-1};q)_n}\left[(-1)^n q^{\binom{n}{2}}\right]^k\frac{a^n}{(q;q)_n}$$

in (2.1), then we have

$$f(x) = {}_{m-k}\Phi_{m-1}\left[\begin{array}{cccc} a_1 & a_2 & \cdots & a_{m-k} \\ b_1 & b_2 & \cdots & b_{m-1}\end{array};q,ax\right].$$

Thus we find that

$$\begin{aligned} S_q\left\{x^{\alpha-1}f(x);s\right\} &= s^{\alpha-1}(q;q)_{\alpha-1}\sum_{n=0}^{\infty}A_n(q^{\alpha};q)_n s^n \\ &= s^{\alpha-1}(q;q)_{\alpha-1}\,{}_{m-k+1}\Phi_m\left[\begin{array}{ccccc} a_1 & a_2 & \cdots & a_{m-k} & q^{\alpha} \\ b_1 & b_2 & \cdots & b_{m-1} & 0\end{array};q,as\right]. \end{aligned}$$

Similarly, we obtain

$$\mathbb{S}_q\left\{x^{\alpha-1}f(x);s\right\} = s^{\alpha-1}(1-q)^{\alpha-1}\sum_{n=0}^{\infty}A_n\frac{\Gamma_q(\alpha+n)}{K(s;\alpha+n)}\left[(1-q)s\right]^n$$
$$= s^{\alpha-1}(q;q)_{\alpha-1}\sum_{n=0}^{\infty}A_n\left[(-1)^n q^{\binom{n}{2}}\right]^{-1}\frac{(q^\alpha;q)_n}{K(s;\alpha)}\left(\frac{-1}{q^\alpha}\right)^n s^n$$
$$= \frac{s^{\alpha-1}(q;q)_{\alpha-1}}{K(s;\alpha)}\ {}_{m-k+1}\Phi_{m-1}\left[\begin{matrix} a_1 & a_2 & \cdots & a_{m-k} & q^\alpha \\ b_1 & b_2 & \cdots & b_{m-1} & - \end{matrix};q,-\frac{as}{q^\alpha}\right].$$

Now, if we choose $\alpha = m = k = 1$ in the previous theorem, we obtain the following result:

Corollary 2.3. *The q-Sumudu transforms of q-exponential functions are*

$$S_q\left\{E_q(ax);s\right\} = {}_1\Phi_1(q;0;q,as), \tag{2.12}$$
$$\mathbb{S}_q\left\{E_q(ax);s\right\} = \frac{q}{q+as},\quad |as| < |q|. \tag{2.13}$$

Theorem 2.3. *q-Sumudu transforms of the function* $f(x) = {}_r\phi_p\left[\begin{matrix} a_1 & a_2 & \cdots & a_r \\ b_1 & b_2 & \cdots & b_p \end{matrix};q,ax\right]$ *are given by*

$$S_q\left\{x^{\alpha-1}f(x);s\right\} = s^{\alpha-1}(q;q)_{\alpha-1}\ {}_{r+1}\phi_p\left[\begin{matrix} a_1 & a_2 & \cdots & a_r & q^\alpha \\ b_1 & b_2 & \cdots & b_p & - \end{matrix};q,as\right] \tag{2.14}$$

$$\mathbb{S}_q\left\{x^{\alpha-1}f(x);s\right\} = \frac{s^{\alpha-1}(q;q)_{\alpha-1}}{K(s;\alpha)}\ {}_{r+1}\Phi_{r-1}\left[\begin{matrix} a_1 & a_2 & \cdots & a_r & q^\alpha \\ b_1 & b_2 & \cdots & b_p & \underbrace{0...0}_{(r-p-1)\ times} \end{matrix};q,-as/q^\alpha\right] \tag{2.15}$$

Proof. Setting

$$A_n = \frac{(a_1;q)_n(a_2;q)_n\cdots(a_r;q)_n}{(b_1;q)_n(b_2;q)_n\cdots(b_p;q)_n}\frac{a^n}{(q;q)_n}$$

in (2.1), we have

$$f(x) = {}_r\phi_p\left[\begin{matrix} a_1 & a_2 & \cdots & a_r \\ b_1 & b_2 & \cdots & b_p \end{matrix};q,ax\right].$$

Therefore

$$S_q\left\{x^{\alpha-1}f(x);s\right\} = s^{\alpha-1}(q;q)_{\alpha-1}\sum_{n=0}^{\infty}A_n(q^\alpha;q)_n s^n$$
$$= s^{\alpha-1}(q;q)_{\alpha-1}\ {}_{r+1}\phi_p\left[\begin{matrix} a_1 & a_2 & \cdots & a_r & q^\alpha \\ b_1 & b_2 & \cdots & b_p & - \end{matrix};q,as\right]$$

Similarly, we have

$$\mathbb{S}_q\left\{x^{\alpha-1}f(x);s\right\} = s^{\alpha-1}(1-q)^{\alpha-1}\sum_{n=0}^{\infty}A_n\frac{\Gamma_q(\alpha+n)}{K(s;\alpha+n)}\left[(1-q)s\right]^n$$
$$= s^{\alpha-1}(q;q)_{\alpha-1}\sum_{n=0}^{\infty}A_n\left[(-1)^n q^{\binom{n}{2}}\right]^{-1}\frac{(q^\alpha;q)_n}{K(s;\alpha)}\left(-\frac{1}{q^\alpha}\right)^n s^n$$
$$= \frac{s^{\alpha-1}(q;q)_{\alpha-1}}{K(s;\alpha)}\,{}_{r+1}\Phi_{r-1}\left[\begin{array}{cccccc} a_1 & a_2 & \cdots & a_r & q^\alpha & ;q,\dfrac{-as}{q^\alpha} \\ b_1 & b_2 & \cdots & b_p & \underbrace{0...0}_{(r-p-1)\text{ times}} & \end{array}\right].$$

Again, if we choose $\alpha = r = 1$ and $p = 0$ in the previous theorem, we obtain the following corollary:

Corollary 2.4. *The q-Sumudu transform of q-exponential functions are given by*

$$S_q\left\{e_q(ax);s\right\} = {}_2\phi_1(q,0;-;q,as)$$
$$= \sum_{n=0}^{\infty}(as)^n = \frac{1}{1-as}, \quad |as| < 1.$$

Now, on using the definitions (1.3)-(1.6) of q-trigonometric functions, results (2.12) and (2.13), and by the linearity of q-Sumudu transforms (see [3]), we get the following result:

Corollary 2.5. *q-Sumudu transforms of q-trigonometric functions given as*

$$S_q\left\{\operatorname{Sin}_q(ax);s\right\} = as\,{}_1\Phi_1\left(q^4;0;q^4,a^2s^2q^3\right),$$
$$S_q\left\{\operatorname{Cos}_q(ax);s\right\} = {}_1\Phi_1\left(q^4;0;q^4,a^2s^2q\right),$$
$$\mathbb{S}_q\left\{\operatorname{Sin}_q(ax);s\right\} = \frac{qas}{q^2+a^2s^2},$$
$$\mathbb{S}_q\left\{\operatorname{Cos}_q(ax);s\right\} = \frac{q^2}{q^2+a^2s^2},$$
$$S_q\left\{\sin_q(ax);s\right\} = \frac{as}{1+a^2s^2}, \quad |as| < 1,$$
$$S_q\left\{\cos_q(ax);s\right\} = \frac{1}{1+a^2s^2}, \quad |as| < 1.$$

Here, for the sake of brevity we omit the detailed proof of the above results.

Corollary 2.6. *We show that*

$$\mathbb{S}_q\left\{x^{\alpha-1}J_{2\nu}^{(3)}\left(\sqrt{ax};q\right);s\right\} = \frac{\Gamma_q(\alpha+\nu)a^\nu}{\Gamma_q(2\nu+1)}\frac{(1-q)^{\alpha-1-\nu}.s^{\alpha-1+\nu}}{K(s;\alpha+\nu)}\,{}_2\Phi_1\left[\begin{array}{cc} 0 & q^{\alpha+\nu} \\ q^{2\nu+1} & \end{array};q,\frac{-as}{q^{\alpha+\nu-1}}\right]. \tag{2.16}$$

Proof. Using the definition of Hahn-Exton q-Bessel function, we obtain

$$f(x) = x^{\alpha-1}J_{2\nu}^{(3)}\left(\sqrt{ax};q\right)$$
$$= \frac{a^\nu}{(q;q)_{2\nu}}x^{\alpha-1+\nu}\,{}_1\Phi_1\left[\begin{array}{c} 0 \\ q^{2\nu+1} \end{array};q,qax\right]. \tag{2.17}$$

Substituting (2.17) into the definition (1.21) of $\mathbb{S}_q$-transform, we have

$$\mathbb{S}_q\left\{x^{\alpha-1}J_{2\nu}^{(3)}\left(\sqrt{ax};q\right);s\right\}=\frac{a^{\nu}}{(q;q)_{2\nu}}\mathbb{S}_q\left\{x^{\alpha-1+\nu}\,{}_1\Phi_1\left[\begin{matrix}0\\q^{2\nu+1}\end{matrix};q,qax\right];s\right\},$$

which in view of result (2.11) of Theorem 2.2, we obtain

$$\mathbb{S}_q\left\{x^{\alpha-1}J_{2\nu}^{(3)}\left(\sqrt{ax};q\right);s\right\}=\frac{a^{\nu}}{(q;q)_{2\nu}}\frac{s^{\alpha-1+\nu}(q;q)_{\alpha-1+\nu}}{K(s;\alpha+\nu)}\,{}_2\Phi_1\left[\begin{matrix}0 & q^{\alpha+\nu}\\q^{2\nu+1} & \end{matrix};q,\frac{-as}{q^{\alpha+\nu-1}}\right].$$

Making use of the formula (1.14), we can easily arrive at (2.16).
Now, if we write $\alpha=\nu/2+1$ and $\nu=\nu/2$ in the assertion (2.16), we get

$$\mathbb{S}_q\left\{x^{\nu/2}J_{\nu}^{(3)}\left(\sqrt{ax};q\right);s\right\}=\frac{a^{\nu/2}.s^{\nu}}{K(s;\nu+1)}e_q\left(-\frac{as}{q^{\nu}}\right).$$

Setting $\nu=0$, we find

$$\mathbb{S}_q\left\{J_0^{(3)}\left(\sqrt{ax};q\right);s\right\}=e_q(-as).$$

If we write $\nu=0$ and then choose $a=0$ in the assertion (2.16), we find

$$\mathbb{S}_q\left\{x^{\alpha-1};s\right\}=s^{\alpha-1}(1-q)^{\alpha-1}\frac{\Gamma_q(\alpha)}{K(s;\alpha)}$$

which is the previously obtained (2.6) in Corollary 2.1.

3. q-Sumudu Transforms of Certain q-Polynomials

In this section, we derive a theorem, which give rise to q-Sumudu transforms of a general class of q-polynomials.

Theorem 3.1. *Let* $\left\{S_{j,q}\right\}_{j=0}^{\infty}$ *be a bounded complex sequence and define a family of q-polynomials* $\{f_{n,N}(x;q)\}_{n=0}^{\infty}$ *by*

$$f_{n,N}(x;q)=\sum_{j=0}^{[n/N]}\begin{bmatrix}n\\Nj\end{bmatrix}_q S_{j,q}x^{j},\quad (n=0,1,2,...) \tag{3.1}$$

where N is a positive integer. Then we have

$$S_q\{x^{\alpha}f_{n,N}(x;q);s\}=s^{\alpha}(1-q)^{\alpha}\Gamma_q(\alpha\mu+1)\sum_{j=0}^{[n/N]}\begin{bmatrix}n\\Nj\end{bmatrix}_q S_{j,q}s^{j}\left(q^{\alpha+1};q\right)_j,$$

$$\mathbb{S}_q\{x^{\alpha}f_{n,N}(x;q);s\}=s^{\alpha}(1-q)^{\alpha}\frac{\Gamma_q(\alpha+1)}{K(s;\alpha+1)}\sum_{j=0}^{[n/N]}\begin{bmatrix}n\\Nj\end{bmatrix}_q S_{j,q}s^{j}\frac{\left(q^{\alpha+1};q\right)_j}{(q^{\alpha})^{j}q^{j(j+1)/2}}.$$

Proof. By the definition (1.18) of S_q-transform, we have

$$S_q\{x^{\alpha}f_{n,N}(x;q);s\}=\sum_{j=0}^{[n/N]}\begin{bmatrix}n\\Nj\end{bmatrix}_q S_{j,q}\,S_q\left\{x^{\alpha+j};s\right\}.$$

Making use of (2.5), we get

$$S_q\{x^\alpha f_{n,N}(x;q);s\} = \sum_{j=0}^{[n/N]} \begin{bmatrix} n \\ Nj \end{bmatrix}_q S_{j,q} s^{\alpha+j}(1-q)^{\alpha+j}\Gamma_q(\alpha+j+1)$$

$$= s^\alpha (1-q)^\alpha \Gamma_q(\alpha+1) \sum_{j=0}^{[n/N]} \begin{bmatrix} n \\ Nj \end{bmatrix}_q S_{j,q} s^j \left(q^{\alpha+1};q\right)_j.$$

Similarly, by the definition (1.19) of $\mathbb{S}_q$-transform, we have

$$\mathbb{S}_q\{x^\alpha f_{n,N}(x;q);s\} = \sum_{j=0}^{[n/N]} \begin{bmatrix} n \\ Nj \end{bmatrix}_q S_{j,q} \mathbb{S}_q\left\{x^{\alpha+j};s\right\}.$$

Making use of (2.6), we get the desired result.

$$\mathbb{S}_q\{x^\alpha f_{n,N}(x;q);s\} = \sum_{j=0}^{[n/N]} \begin{bmatrix} n \\ Nj \end{bmatrix}_q S_{j,q} s^{\alpha+j}(1-q)^{\alpha+j} \frac{\Gamma_q(\alpha+j+1)}{K(s;\alpha+j+1)}$$

$$= s^\alpha (1-q)^\alpha \frac{\Gamma_q(\alpha+1)}{K(s;\alpha+1)} \sum_{j=0}^{[n/N]} \begin{bmatrix} n \\ Nj \end{bmatrix}_q S_{j,q} s^j \frac{\left(q^{\alpha+1};q\right)_j}{(q^\alpha)^j q^{j(j+1)/2}}.$$

Now, we consider some consequences and applications of the above results. On suitably specializing bounded sequence A_n or the coefficient $S_{j,q}$ in $f(x)$, given by (2.1), or the q-polynomials family $f_{n,N}(x;q)$ yields a number of known q-polynomials as its special cases. These include, among others, the q-Stieltjes-Wigert polynomials, the q-Laguerre polynomials, the q-Jacobi polynomials, the q-Charlier polynomials, the q-Konhauser polynomials and several others. Therefore, by assigning suitable special values to the arbitrary sequence our main results of Theorems 2.1 and 3.1 can be applied to derive q-Sumudu transforms of q-hypergeometric polynomials. To illustrate that we consider the following examples.

Example 1. We show that

$$S_q\left\{x^{\alpha-1}S_k(x;q)\right\} = s^{\alpha-1}(q;q)_{\alpha-1}\ {}_2\Phi_2\left(q^{-k},q^\alpha;0,0;q,-q^{k+1}s\right), \tag{3.2}$$

$$\mathbb{S}_q\left\{x^{\alpha-1}S_k(x;q)\right\} = \frac{s^{\alpha-1}(q;q)_{\alpha-1}}{K(s;\alpha)}\ {}_2\Phi_1\left(q^{-k},q^\alpha;0;q,q^{k+1-\alpha}s\right). \tag{3.3}$$

If we set

$$A_n = \frac{\left(q^{-k};q\right)_n}{(q;q)_n} q^{n(2k+n+1)/2},$$

in (2.1), then we have

$$f(x) = S_k(x;q) = {}_1\Phi_1\left(q^{-k};0;q,-q^{k+1}x\right),$$

where $S_k(x;q)$ is the Stieltjes-Wigert polynomial [12, p. 61]. Thus, assertions of the example follows from Theorem 2.1.

Example 2. We show that

$$S_q\left\{x^{\alpha-1}L_k^{(\beta)}(x;q)\right\} = s^{\alpha-1}(q;q)_{\alpha-1}\frac{\left(q^\beta;q\right)_k}{(q;q)_k}\ {}_2\Phi_2\left(q^{-k},q^\alpha;q^{\beta+1},0;q,-q^{k+\beta+1}s\right), \tag{3.4}$$

$$\mathbb{S}_q\left\{x^{\alpha-1}L_k^{(\beta)}(x;q)\right\} = s^{\alpha-1}\frac{(q;q)_{\alpha-1}}{K(s;\alpha)}\frac{\left(q^\beta;q\right)_k}{(q;q)_k}\ {}_2\Phi_1\left(q^{-k},q^\alpha;q^{\beta+1};q,q^{k+\beta-\alpha+1}s\right). \tag{3.5}$$

If we set

$$A_n = \frac{\left(q^{\beta};q\right)_k}{(q;q)_k}\frac{\left(q^{-k};q\right)_n}{\left(q^{\beta+1};q\right)_n}q^{n\left[(n+1)/2+\left(k+\beta\right)\right]},$$

in (2.1), then we have

$$f(x) = L_k^{(\beta)}(x;q) = \frac{\left(q^{\beta};q\right)_k}{(q;q)_k}\,{}_1\Phi_1\left(q^{-k};q^{\beta+1};q,-q^{k+1}s\right),$$

where $L_k^{(\beta)}(x;q)$ denotes the q-Laguerre polynomials [12, p.108]. Thus, from Theorem 2.1 we get the desired result.

Example 3. We show that

$$S_q\left\{x^{\alpha-1}K_m(x;a;q)\right\} = s^{\alpha-1}(q;q)_{\alpha-1}\;{}_3\Phi_2\left(q^{-m},-aq^m,q^{\alpha};0,0;q,qs\right), \tag{3.6}$$

$$\mathbb{S}_q\left\{x^{\alpha-1}K_m(x;a;q)\right\} = s^{\alpha-1}\frac{(q;q)_{\alpha-1}}{K(s;\alpha)}\;{}_3\Phi_1\left(q^{-m},-aq^m,q^{\alpha};0;q,-q^{-\alpha+1}s\right). \tag{3.7}$$

Similarly, if we set

$$A_n = \frac{\left(q^{-m};q\right)_n\left(-aq^m;q\right)_n}{(q;q)_n}q^n,$$

in (2.1), we have

$$f(x) = K_m(x;a;q),$$

where $K_n(x;a;q)$ are q-Charlier polynomials [12, p.110]. Hence, making use of Theorem 2.1, we obtain the assertions of example.

Now, we explore the similar type of applications for the results of Theorem 3.1.

Example 4. If we set $N = 1$, $\rho \geq 1$ and

$$S_{j,q} = \frac{\Gamma_q(\rho n+\alpha+1)(-1)^j q^{j(j-1)}}{(q;q)_n\Gamma_q(\rho j+\alpha+1)},$$

and replace x by x^{ρ} in equation (3.1) to obtain

$$Z_n^{\alpha}(x;q,\rho) = \frac{\Gamma_q(\rho n+\alpha+1)}{(q;q)_n}\sum_{j=0}^{n}\begin{bmatrix}n\\j\end{bmatrix}_q\frac{(-1)^j q^{j(j-1)}}{\Gamma_q(\rho j+\alpha+1)}x^{\rho j}, \tag{3.8}$$

where $Z_n^{\alpha}(x;q,\rho)$ denotes the q-Konhouser biorthogonal polynomials due to Yadav and Singh [19, p,185, eqn.(2.1)]. Thus for $\rho = 1$, the above polynomial in view of Theorem 3.1 (with α replace $\mu+1$ therein), yields the following formulas:

$$\begin{aligned} S_q\left\{x^{\mu}Z_n^{\alpha}(x;q,1);s\right\} &= s^{\mu}(1-q)^{\mu+\alpha}\frac{\Gamma_q(\mu+1)\Gamma_q(n+\alpha+1)}{(q;q)_n(q;q)_{\alpha}} \\ &\times\;{}_2\Phi_2\left(q^{-n},q^{\mu+1};q^{\alpha+1},0;q,-(1-q)q^n s\right), \end{aligned}$$

$$\begin{aligned} \mathbb{S}_q\left\{x^{\mu}Z_n^{\alpha}(x;q,1);s\right\} &= s^{\mu}(1-q)^{\mu+\alpha}\frac{\Gamma_q(\mu+1)\Gamma_q(n+\alpha+1)}{K(s;\mu+1)(q;q)_n(q;q)_{\alpha}} \\ &\times\;{}_2\Phi_1\left(q^{-n},q^{\mu+1};q^{\alpha+1};q,(1-q)q^{n-\mu+1}s\right). \end{aligned}$$

Example 5. Finally, on setting $N = 1$ and

$$S_{j,q} = \frac{\Gamma_q(\alpha+1+n)(1-q)^n\,\Gamma_q(\alpha+\beta+n+1+j)(-1)^j q^{j(j+1)/2-nj}}{(q;q)_n\,\Gamma_q(\alpha+\beta+n+1)\,\Gamma_q(\alpha+1+j)}$$

in equation (3.1) to obtain

$$P_n^{(\alpha,\beta)}(x;q) = \frac{\left(q^{\alpha+1};q\right)_n}{(q;q)_n}\,{}_2\Phi_1\left(q^{-n}, q^{\alpha+\beta+n+1}; q^{\alpha+1}; q, xq\right),$$

where $P_n^{(\alpha,\beta)}(x;q)$ denotes the q-Jacobi polynomials (cf. [19, p, 185, eqn. (2.1)]). Thus, by Theorem 3.1 we find that

$$\begin{aligned} S_q\left\{x^\mu P_n^{(\alpha,\beta)}(x;q);s\right\} =& s^\mu (1-q)^{\mu+\alpha+n}\,\frac{\Gamma_q(\mu+1)\,\Gamma_q(n+\alpha+1)}{(q;q)_n\,(q;q)_\alpha} \\ &\times\,{}_3\Phi_2\left(q^{-n}, q^{\alpha+\beta+n+1}, q^{\mu+1}; q^{\alpha+1}, 0; q, qs\right), \end{aligned}$$

$$\begin{aligned} \mathbb{S}_q\left\{x^\mu P_n^{(\alpha,\beta)}(x;q);s\right\} =& s^\mu (1-q)^{\mu+\alpha+n}\,\frac{\Gamma_q(\mu+1)}{K(s;\mu+1)}\,\frac{\Gamma_q(n+\alpha+1)}{(q;q)_n\,(q;q)_\alpha} \\ &\times\,{}_3\Phi_1\left(q^{-n}, q^{\alpha+\beta+n+1}, q^{\mu+1}; q^{\alpha+1}; q, -q^{n-\mu}s\right). \end{aligned}$$

Though several similar results can be obtained from our theorems, we omit further details. Now we give some results on q-Sumudu transform of basic analogue of Fox's H-function.

4. Some Results on *q*-Sumudu Transform of Basic Analogue of Fox's *H*-Function

The basic analogue of the Fox's H-function of one variable due to Saxena, Modi and Kalla [16] is given by

$$\begin{aligned} &H_{A,B}^{m_1,n_1}\left[\,x;q\,\left|\begin{matrix}(a_1,\alpha_1),\ (a_2,\alpha_2),\ \cdots,\ (a_A,\alpha_A)\\ (b_1,\beta_1),\ (b_2,\beta_2),\ \cdots,\ (b_B,\beta_B)\end{matrix}\right.\right] \\ &= \frac{1}{2\pi i}\int_C \frac{\prod_{j=1}^{m_1} G(q^{b_j-\beta_j s})\prod_{j=1}^{n_1} G(q^{1-a_j+\alpha_j s})\pi x^s}{\prod_{j=m_1+1}^{B} G(q^{1-b_j+\beta_j s})\prod_{j=n_1+1}^{A} G(q^{a_j-\alpha_j s})G(q^{1-s})\sin\pi s}\,d_q s, \end{aligned} \tag{4.1}$$

where

$$G(q^\alpha) = \prod_{n=0}^{\infty}\left\{(1-q^{\alpha+n})\right\}^{-1} = \frac{1}{(q^\alpha;q)_\infty}, \tag{4.2}$$

and $0 \le m_1 \le B; 0 \le n_1 \le A; \alpha_j$ and β_j are all positive integers. The contour C is a parallel to $\mathrm{Re}(\omega s) = 0$, with indentations, if necessary, in such a manner that all the poles of $G(q^{b_j-\beta_j s})$ $(1 \le j \le m_1)$ are its right, and those of $G(q^{1-a_j+\alpha_j s})$ $(1 \le j \le n_1)$ are to the left of C. The basic integral converges if $\mathrm{Re}\left[s\log(x) - \log\sin\pi s\right] < 0$, for large values of $|s|$ on the contour C, that is if $\left|\left\{\arg(x) - \omega_2\omega_1^{-1}\log|x|\right\}\right| < \pi$, where $\left|q\right| < 1$, $\log q = -\omega = -(\omega_1 + i\omega_2)$, ω_1 and ω_2 being real.

Further, if we set $\alpha_i = \beta_j = 1$, for all i and j in definition (4.1), we obtain the following basic analogue of Meijer's G-function:

$$G_{A,B}^{m_1,n_1}\left[\ x;q\ \middle|\begin{matrix} a_1,,\ a_2,\ \cdots,\ a_A \\ \\ b_1,\ b_2,\ \cdots,\ b_B \end{matrix}\right]$$

$$= \frac{1}{2\pi i}\int_C \frac{\prod_{j=1}^{m_1} G(q^{b_j-s})\prod_{j=1}^{n_1} G(q^{1-a_j+s})\pi x^s}{\prod_{j=m_1+1}^{B} G(q^{1-b_j+s})\prod_{j=n_1+1}^{A} G(q^{a_j-s})G(q^{1-s})\sin\pi s}\, d_q s, \tag{4.3}$$

where $0 \le m_1 \le B; 0 \le n_1 \le A$ and $\mathrm{Re}\left[s\log(x) - \log\sin\pi s\right] < 0$.

Further, if we set $n_1 = 0$ and $m_1 = B$ in the equation (4.3), we get the basic analogue of MacRobert's E-function due to Agarwal [2], namely

$$G_{A,B}^{B,0}\left[\ x;q\ \middle|\begin{matrix} a_1,,\ a_2,\ \cdots,\ a_A \\ \\ b_1,\ b_2,\ \cdots,\ b_B \end{matrix}\right] \equiv E_q\left[B;b_j : A;a_j : x\right]$$

$$= \frac{1}{2\pi i}\int_C \frac{\prod_{j=1}^{B} G(q^{b_j-s})\pi x^s}{\prod_{j=1}^{A} G(q^{a_j-s})G(q^{1-s})\sin\pi s}\, d_q s, \tag{4.4}$$

where $\mathrm{Re}\left[s\log(x) - \log\sin\pi s\right] < 0$.

Saxena and Kumar [15], introduced the basic analogues of $J_\nu(x),\ Y_\nu(x),\ K_\nu(x),\ \mathbf{H}_\nu(x)$ in terms of $H_q(\cdot)$ function as under:

$$J_\nu(x;q) = \{G(q)\}^2 H_{0,3}^{1,0}\left[\ \frac{x^2(1-q)^2}{4};q\ \middle|\begin{matrix} \\ \\ (\frac{\nu}{2},1),\ (\frac{-\nu}{2},1),\ (1,1) \end{matrix}\right] \tag{4.5}$$

where $J_\nu(x;q)$ denotes the q-analogue of Bessel function of first kind $J_\nu(x)$.

$$Y_\nu(x;q) = \{G(q)\}^2 H_{1,4}^{2,0}\left[\ \frac{x^2(1-q)^2}{4};q\ \middle|\begin{matrix} (\frac{-\nu-1}{2},1) \\ \\ (\frac{\nu}{2},1),\ (\frac{-\nu}{2},1),\ (\frac{-\nu-1}{2},1),\ (1,1) \end{matrix}\right] \tag{4.6}$$

where $Y_\nu(x;q)$ denotes the q-analogue of the Bessel function $Y_\nu(x)$.

$$K_\nu(x;q) = (1-q) H_{0,3}^{2,0}\left[\ \frac{x^2(1-q)^2}{4};q\ \middle|\begin{matrix} \\ \\ (\frac{\nu}{2},1),\ (\frac{-\nu}{2},1),\ (1,1) \end{matrix}\right] \tag{4.7}$$

where $K_\nu(x;q)$ denotes the basic analogue of the Bessel function of the third kind $K_\nu(x)$.

$$\mathbf{H}_\nu(x;q) = \left(\frac{1-q}{2}\right)^{1-\alpha} H_{1,4}^{3,1}\left[\ \frac{x^2(1-q)^2}{4};q\ \middle|\begin{matrix} (\frac{1+\alpha}{2},1) \\ \\ (\frac{\nu}{2},1),\ (\frac{-\nu}{2},1),\ (\frac{1+\alpha}{2},1),\ (1,1) \end{matrix}\right] \tag{4.8}$$

where $\mathbf{H}_\nu(x;q)$ is the basic analogue of Struve's function $\mathbf{H}_\nu(x)$.

Following Yadav, Purohit and Kalla [18], we have the following q-extensions of some elementary functions in terms of a basic analogue of the Fox's H-function as:

$$e_q(-x) = G(q)H^{1,0}_{0,2}\left[\begin{array}{c|c} x(1-q);q & \\ & (0,1),(1,1) \end{array}\right] \tag{4.9}$$

$$\sin_q(x) = \sqrt{\pi}(1-q)^{-1/2}\{G(q)\}^2 H^{1,0}_{0,3}\left[\begin{array}{c|c} \frac{x^2(1-q)^2}{4};q & - \\ & (\frac{1}{2},1),\ (0,1),\ (1,1) \end{array}\right] \tag{4.10}$$

$$\cos_q(x) = \sqrt{\pi}(1-q)^{-1/2}\{G(q)\}^2 H^{1,0}_{0,3}\left[\begin{array}{c|c} \frac{x^2(1-q)^2}{4};q & - \\ & (0,1),\ (\frac{1}{2},1),\ (1,1) \end{array}\right] \tag{4.11}$$

$$\sinh_q(x) = \frac{\sqrt{\pi}}{i}(1-q)^{-1/2}\{G(q)\}^2 H^{1,0}_{0,3}\left[\begin{array}{c|c} -\frac{x^2(1-q)^2}{4};q & - \\ & (\frac{1}{2},1),\ (0,1),\ (1,1) \end{array}\right] \tag{4.12}$$

$$\cosh_q(x) = \sqrt{\pi}(1-q)^{-1/2}\{G(q)\}^2 H^{1,0}_{0,3}\left[\begin{array}{c|c} -\frac{x^2(1-q)^2}{4};q & - \\ & (0,1)\ (\frac{1}{2},1),\ (1,1) \end{array}\right] \tag{4.13}$$

Theorem 4.1. *Let $k \in I$ and λ be any complex number, then the q-Sumudu transform of $H_q(.)$ function is given by*

$$S_q\left\{x^\lambda H^{m_1,n_1}_{A,B}\left[\lambda x^k;q \left| \begin{array}{l} (a_1,\alpha_1),\cdots,(a_A,\alpha_A) \\ (b_1,\beta_1),\cdots,(b_B,\beta_B) \end{array}\right.\right];s\right\}$$

$$= \frac{s^\lambda}{G(q)} H^{m_1,n_1+1}_{A+1,B}\left[\lambda s^k;q \left| \begin{array}{l} (-\lambda,k),(a_1,\alpha_1),\cdots,(a_A,\alpha_A) \\ (b_1,\beta_1),\cdots,(b_B,\beta_B) \end{array}\right.\right], \quad k>0, \tag{4.14}$$

$$= \frac{s^\lambda}{G(q)} H^{m_1+1,n_1}_{A,B+1}\left[\lambda s^k;q \left| \begin{array}{l} (a_1,\alpha_1),\cdots,(a_A,\alpha_A) \\ (1+\lambda,-k),(b_1,\beta_1),\cdots,(b_B,\beta_B) \end{array}\right.\right], \quad k<0, \tag{4.15}$$

where $0 \le m_1 \le B$, $0 \le n_1 \le A$ and λ is arbitrary quantity.

Proof. To prove the result (4.14), we consider the left hand side of the Theorem 4.1 (say L) and make use of the definition (4.1) to obtain

$$L = S_q\left\{\frac{x^\lambda}{2\pi i}\int_C \frac{\prod_{j=1}^{m_1} G(q^{b_j-\beta_j z}) \prod_{j=1}^{n_1} G(q^{1-a_j+\alpha_j z})\pi(\lambda x^k)^z}{\prod_{j=m_1+1}^{B} G(q^{1-b_j+\beta_j z}) \prod_{j=n_1+1}^{A} G(q^{a_j-\alpha_j z})G(q^{1-z}) sin\pi z} d_q z;s\right\}.$$

On interchanging the order of summation and q-transform, which is valid under the conditions given with the equation (4.1), the above expression reduces to

$$L = \frac{1}{2\pi i}\int_C \frac{\prod_{j=1}^{m_1} G(q^{b_j-\beta_j z}) \prod_{j=1}^{n_1} G(q^{1-a_j+\alpha_j z})\pi(\lambda)^z}{\prod_{j=m_1+1}^{B} G(q^{1-b_j+\beta_j z}) \prod_{j=n_1+1}^{A} G(q^{a_j-\alpha_j z})G(q^{1-z}) sin\pi z} S_q\left\{x^{\lambda+kz};s\right\} d_q z.$$

Now, on using the formula [3, Theorem 8, (39)]

$$S_q\left\{x^{\alpha-1};s\right\} = s^{\alpha-1}\left(1-q\right)^{\alpha-1}\Gamma_q\left(\alpha\right),$$

we get

$$L = \frac{s^\lambda}{2\pi i}\int_C \frac{\prod_{j=1}^{m_1} G(q^{b_j-\beta_j z})\prod_{j=1}^{n_1} G(q^{1-a_j+\alpha_j z})\pi(\lambda s^k)^z}{\prod_{j=m_1+1}^{B} G(q^{1-b_j+\beta_j z})\prod_{j=n_1+1}^{A} G(q^{a_j-\alpha_j z})G(q^{1-z})sin\pi z}\left(1-q\right)^{\lambda+kz}\Gamma_q\left(\lambda+kz+1\right)d_q z.$$

Following the definition (4.2) and q-gamma function, we have

$$\left(1-q\right)^{\lambda+kz}\Gamma_q\left(\lambda+kz+1\right) = \frac{G(q^{\lambda+kz})}{G(q)},$$

therefore, we obtain

$$L = \frac{s^\lambda}{2\pi i G(q)}\int_C \frac{\prod_{j=1}^{m_1} G(q^{b_j-\beta_j z})\prod_{j=1}^{n_1} G(q^{1-a_j+\alpha_j z})G(q^{\lambda+kz})\pi(\lambda s^k)^z}{\prod_{j=m_1+1}^{B} G(q^{1-b_j+\beta_j z})\prod_{j=n_1+1}^{A} G(q^{a_j-\alpha_j z})G(q^{1-z})sin\pi z}d_q z$$

which on interpretation in view of the definition (4.1), leads us to the right hand side of the result (4.14). The second part of theorem, i.e. result (4.15) follows similarly.

Now, as an application of results (4.14) or (4.15), we can derive a number of results, by taking equations (4.3) to (4.13) into account. For example,

$$S_q\left\{x^\lambda e_q(-x);s\right\} = s^\lambda H^{1,1}_{1,2}\left[\ s(1-q);q\ \middle|\ \begin{matrix}(-\lambda,1)\\ \\(0,1),(1,1)\end{matrix}\ \right]. \quad (4.16)$$

References

[1] W.H. Abdi, On q-Laplace transforms. Proc. Nat. Acad. Sci.India Sect. A, **29** (1961) 389–408.

[2] R.P. Agarwal, A q-analogue of MacRobert's generalized E-function, Ganita, **11** (1960) 49–63.

[3] D. Albayrak, S.D. Purohit, F. Uçar, On q-Analogues of Sumudu Transforms, An. St. Univ. Ovidius Constanta, Ser. Mat vol. XX, fasc. 3, (2012), to appear.

[4] F.B.M. Belgacem, A.A. Karaball and S.L. Kalla, Analytical investigations of the Sumudu transform and applications to integral production equations, Mathematical Problems in Engineering, 3 (2003), 103–118

[5] G. Gasper and M. Rahman, Basic Hypergeometric Series, Encyclopedia of Mathematics and Its Applications, 35 (1990), Cambridge University Press, Cambridge and New York. ISBN: 0-521-35049-2.

[6] W. Hahn, Beitrage Zur Theorie der Heineschen Reihen. Die 24 Integrale der Hypergeometrischen q-Diferenzengleichung. Das q-Analogon der Laplace-Transformation, Math. Nachr., **2** (1949) 340–379.

[7] F. H. Jackson, On q-definite Integrals, Quarterly J. Pure and Appl. Mathematics, **41** (1910) 193–203.

[8] F.H. Jackson, The application of basic numbers to Bessel's and Legendre's functions, Proc. London Math. Soc. **2** (2) (1905) 192–220.

[9] F. Jarad, K. Bayram, T. Abdeljawad and D. Baleanu, On the discrete Sumudu transform, Romanian Reports in Physics, Vol. 64, No.2 (2012) 347–356.

[10] F. Jarad and K. Taş, On Sumudu transform method in discrete fractional Calculus, Hindawi Abstract and Applied Analysis,Vol. 2012, Article ID 270106, 16 pages, doi:10.1155/2012/270106.

[11] A. De Sole, V.G. Kac, On Integral representations of q-gamma and q-beta functions (English, Italian summary) , Atti Accad. Naz. Lincei Cl. Sci. Fis. Mat. Natur. Rend. Lincei (9) Mat. Appl. **16** (2005) no.1, 11–29.

[12] R. Koekoek, R. F. Swarttouw, The Askey-Scheme of hypergeometric orthogonal polynomials and its q-analogue, Report 94 − 05 (1994), Technical University Delft, available from http://fa.its.tudelft.nl/˜koekoek/documents/.

[13] E. Koelink, Quantum groups and q-special functions, Report 96 − 10 (1996), Universiteit van Amsterdam.

[14] S.D. Purohit, S.L. Kalla, On q-Laplace transforms of the q-Bessel functions, Fract. Calc. Appl. Anal., **10** (2), (2007) 189–196.

[15] R.K. Saxena and R. Kumar, Recurrence relations for the basic analogue of the H-function, J. Nat. Acad. Math., **8** (1990) 48–54.

[16] R.K. Saxena, G.C. Modi and S.L. Kalla, A basic analogue of Fox's H-function, Rev. Tec. Ing. Univ. Zulia, **6** (1983) 139–143.
[17] G. K. Watugala, Sumudu transform: a new integral transform to solve differential equations and control engineering problems. Internat. J. Math. Ed. Sci. Tech. **24** (1993), no. 1, 35–43.
[18] R. K. Yadav, S.D. Purohit and S.L. Kalla, On generalized Weyl fractional q-integral operator involving generalized basic hypergeometric functions, Fract. Calc. Appl. Anal., **11(2)** (2008) 129–142.
[19] R. K. Yadav, B. Singh, On a set of basic polynomials $Z_n^\alpha(x;k,q)$ suggested by basic Laguerre polynomials $L_n^\alpha(x,q)$, Math. Student **73** (2004) no. 1-4, 183–189.