On defining the q-beta function for negative integers

Inci Ege[a]

[a]*Department of Mathematics, Faculty of Arts and Sciences, Adnan Menderes University 09010, Aydın-Turkey*

Abstract. The q-beta function $B_q(t, s)$ is defined for $s, t > 0$ and $0 < q < 1$. Its definition can be extended, by regularization, to negative non-integer values of t and s. In this paper we define the q-beta function $B_q(t, s)$ for negative integer values of t and s.

1. Introduction

Q-calculus was introduced and developed systematically by F.H. Jackson at the beginning of the twentieth century. After his advent, the new deformed calculus has found a lot of applications in mathematics and physics.

Since special functions play important roles in mathematical physics, it is motivated to think that some deformation of the ordinary special functions based on the q-calculus can also play similar roles in this area. New inequalities for some q-special functions are established by using their q-integral representations, [2, 3, 14].

In this paper, we aim to extend the definition of the q-analogue of classical Euler's beta function for negative integers via the theory of neutrices.

A neutrix is an additive group of negligible functions which satisfies the condition that does not contain any constant except zero. Taking the neutrix limit of a function is equivalent to picking out the Hadamard finite part from the function and taking the usual limit of that. Fisher at al. used neutrices to define the gamma and beta function for all real values [4, 5]. Özçağ et al. applied the neutrix limit to extend the definition of the incomplete beta function and its partial derivatives to negative integers, [11, 12]. It was shown that the neutrix limit of the q-gamma function and its partial derivatives exist for all values, [13]. Also Y. J. Ng and H. van Dam applied the neutrix calculus to quantum field theory to obtain finite results for the coefficients in the perturbation series, [9, 10].

2. Notations and preliminaries

In this section, we will give a summary of the mathematical notations and definitions required in this paper for the convenience of the reader.

Throughout of this work, we will fix $q \in (0, 1)$ and all of these definitions for the q-calculus can be found in [6, 8].

For any $n \in \mathbb{N}$ and $a, \alpha \in \mathbb{C}$, define

$$[\alpha]_q = \frac{1 - q^\alpha}{1 - q},$$

$$[n]_q! = [1]_q[2]_q \dots [n]_q,$$

$$(1 + x)_q^\alpha = \frac{(1 + x)_q^\infty}{(1 + q^\alpha x)_q^\infty}, \tag{1}$$

$$(x - a)_q^n = \begin{cases} 1, & n = 0 \\ (x - a)(x - qa)\dots(x - q^{n-1}a), & n \geq 1, \end{cases} \tag{2}$$

$$(a - x)_q^n = (-1)^n q^{n(n-1)/2}(x - q^{-n+1}a)_q^n. \tag{3}$$

The q-Jackson derivative $D_q f$ of a function f is given by

$$(D_q f)(x) = \frac{f(qx) - f(x)}{(q - 1)x} \quad \text{if} \ x \neq 0$$

and

$$(D_q f)(0) = f'(0) \quad \text{provided } f'(0) \text{ exists.}$$

Note that when $q \to 1$ it reduces to the standard derivative;

$$(D_q f)(x) = f'(x) \quad \text{if } f(x) \text{ is differentiable at } x.$$

The q-Jackson integral from 0 to b and from 0 to ∞ are defined respectively by

$$\int_0^b f(x)d_q x = (1 - q)b \sum_{n=0}^\infty f(bq^n)q^n,$$

$$\int_0^\infty f(x)d_q x = (1 - q) \sum_{n=-\infty}^\infty f(q^n)q^n$$

provided the sums converge absolutely.
The q-Jackson integral in a generic interval $[a, b]$ is given by

$$\int_a^b f(x)d_q x = \int_0^b f(x)d_q x - \int_0^a f(x)d_q x. \tag{4}$$

The formula of q-integration by parts is given for suitable functions f and g by

$$\int_a^b f(x)d_q g(x) = f(b)g(b) - f(a)g(a) - \int_a^b g(qx)d_q f(x). \tag{5}$$

A neutrix $\mathcal{N}$ is defined as a commutative additive group of functions $f(\epsilon)$ defined on a domain N' with values in an additive group N'', where further if for some f in $\mathcal{N}$, $f(\epsilon) = c$ (constant) for all $\epsilon \in N'$, then $c = 0$. The functions in $\mathcal{N}$ are called "negligible functions".

In this work, we let $\mathcal{N}$ be the neutrix having domain $N' = \{\epsilon : 0 < \epsilon < q^{-s}/2, 0 < q < 1\}$ and range $N'' = \mathbb{R}$, with the negligible functions being finite linear sums of the functions

$$\epsilon^\lambda \ln^{r-1} \epsilon, \ln^r \epsilon : \qquad \lambda < 0, r = 1, 2, \dots$$

and all functions $O(\epsilon)$ which converge to zero in the usual sense as ϵ tends to zero.

If $f(\epsilon)$ is a real (or complex) valued function defined on N' and if it is possible to find a constant c such that $f(\epsilon) - c$ belongs to $\mathcal{N}$, then c is called the neutrix limit of $f(\epsilon)$ as ϵ tends to a and we write N$-\lim_{\epsilon \to a} f(\epsilon) = c$.

The neutrix limit of a function is unique if it exists and if the usual limit of a function exists, it exists as a neutrix limit and the two limits are equal. The reader may find the general definition of the neutrix limit with some examples in [1].

3. Definitions of beta and q-beta functions

The beta function, introduced by Euler, is defined by the following definite integral

$$B(t,s) = \int_0^1 x^{t-1}(1-x)^{s-1}\,dx \qquad t,s > 0. \tag{6}$$

Since the beta function satisfies the property that

$$B(t,s) = \frac{\Gamma(t)\Gamma(s)}{\Gamma(s+t)},$$

it is natural to define the q-beta function by

$$B_q(t,s) = \frac{\Gamma_q(t)\Gamma_q(s)}{\Gamma_q(s+t)}$$

and it has the q-integral representation

$$B_q(t,s) = \int_0^1 x^{t-1}(1-qx)_q^{s-1}\,d_qx \qquad t,s > 0 \tag{7}$$

which is a q-analogue of Euler's formula, [8, 14].

The beta function is defined by using regularization technique for negative non-integer values of t and s by

$$
\begin{aligned}
B(t,s) = \int_0^{1/2} x^{t-1}&\Big[(1-x)^{s-1} - \sum_{i=0}^{r-1} \frac{(-1)^i\Gamma(s)}{i!\,\Gamma(s-i)}\,x^i\Big]dx + \\
&+ \sum_{i=0}^{r-1} \frac{(-1)^i\Gamma(s)}{2^{t+i}\,i!\,\Gamma(s-i)(t+i)} + \\
&+ \int_{1/2}^1 (1-x)^{s-1}\Big[x^{t-1} - \sum_{i=0}^{k-1} \frac{(-1)^i\Gamma(t)}{i!\,\Gamma(t-i)}(1-x)^i\Big]dx + \\
&+ \sum_{i=0}^{k-1} \frac{(-1)^i\Gamma(t)}{2^{s+i}\,i!\,\Gamma(t-i)(s+i)}
\end{aligned}
\tag{8}
$$

for $t > -r, s > -k$, and $t \neq 0, -1, \ldots, -r+1, s \neq 0, -1, \ldots, -k+1$, see [7].

Now, let N be a neutrix having domain the open interval $\{\epsilon : 0 < \epsilon < \frac{1}{2}\}$ and range N'' the real numbers of the functions

$$\epsilon^\lambda \ln^{r-1}\epsilon, \ \ln^r\epsilon : \qquad \lambda < 0, \ r = 1, 2, \ldots$$

and all functions $O(\epsilon)$ which converge to zero in the usual sense as ϵ tends to zero. It was shown that

$$B(t,s) = \mathrm{N}\!-\!\lim_{\epsilon\to 0} \int_\epsilon^{1-\epsilon} x^{t-1}(1-x)^{s-1}\,dx \tag{9}$$

and in general

$$\frac{\partial^{m+n}}{\partial t^m \partial s^n}B(t,s) = \mathrm{N}\!-\!\lim_{\epsilon\to 0} \int_\epsilon^{1-\epsilon} x^{t-1}\ln^m x(1-x)^{s-1}\ln^n(1-x)\,dx \tag{10}$$

for $t,s \neq 0, -1, -2, \ldots$ and $m,n = 0, 1, 2, \ldots$, see [4].

4. An extension of the q-beta function

Since $B_q(t,s)$ is a q-analogue of the classical beta function, we will use the regularization technique, used in the equation (8), and the concepts of the neutrix and neutrix limit to give a generalization of the equation (7) for $t > -r, r \neq 0, -1, -2, \ldots, -r+1, -n-1 < s < -n, n = 0, 1, 2, \ldots$.

Definition 4.1. The function $B_q(t,s)$ is defined by

$$\mathrm{N}-\lim_{\epsilon \to 0} \int_\epsilon^1 x^{t-1}(1-qx)_q^{s-1}\, d_q x \tag{11}$$

for $t > -r, t \neq 0, -1, \ldots, -r+1, -n-1 < s < -n, n = 0, 1, 2, \ldots$.

It is not immediately obvious that the neutrix limit in the equation (11) exists. In the following, we prove that this neutrix limit exits so that $B_q(t,s)$ is well-defined for the given values of t and s in the definition 4.1.

Theorem 4.2. *The neutrix limit as ϵ tends to zero of the q-integral*

$$\int_\epsilon^1 x^{t-1}(1-qx)_q^{s-1}\, d_q x$$

exists for $t > -r, t \neq 0, -1, \ldots, -r+1, -n-1 < s < -n, n = 0, 1, 2, \ldots$.

Proof. For proving our theorem, firstly we need the Taylor expansion of the function $(1-qx)_q^{s-1}$ about $x = 0$. Using the equation, given in [14],

$$D_q(1+bx)_q^t = b[t]_q(1+bqx)_q^{t-1}$$

for $t, b \in \mathbb{R}$, we have

$$D_q^j(1-qx)_q^{s-1}|_{x=0} = q^{j(j+1)/2}(-1)^j[s-1]_q[s-2]_q \ldots [s-j]_q.$$

Therefore we have

$$(1-qx)_q^{s-1} = \sum_{j=0}^\infty (-1)^j \begin{bmatrix} s-1 \\ j \end{bmatrix}_q q^{j(j+1)/2} x^j$$

by using the q-binomial coefficient

$$\begin{bmatrix} s \\ j \end{bmatrix}_q = \frac{[s]_q[s-1]_q \ldots [s-j+1]_q}{[j]_q!}.$$

Then the q-integral is obtained as

$$\int_\epsilon^1 x^{t-1}(1-qx)_q^{s-1}\, d_q x =$$

$$= \int_\epsilon^{q^{-s}/2} x^{t-1}\left[(1-qx)_q^{s-1} - \sum_{j=0}^r (-1)^j \begin{bmatrix} s-1 \\ j \end{bmatrix}_q q^{\frac{j(j+1)}{2}} x^j\right] d_q x +$$

$$+ \sum_{j=0}^r (-1)^j \begin{bmatrix} s-1 \\ j \end{bmatrix}_q q^{\frac{j(j+1)}{2}} \frac{1}{[t+j]_q}\left[(q^{-s}/2)^{t+j} - \epsilon^{t+j}\right] + \int_{q^{-s}/2}^1 x^{t-1}(1-qx)_q^{s-1}\, d_q x.$$

Since we have

$$(1-qx)_q^{s-1} = \frac{(1-qx)_q^\infty}{(1-q^s x)_q^\infty} = \frac{\prod_{j=0}^\infty (1-q^j x)}{\prod_{j=0}^\infty (1-q^{j+s} x)}$$

then $(1 - qx)_q^{s-1}$ have finite number of algebraic singularities at the points q^{-s-n} for $-n - 1 < s < -n$, $n = 0, 1, 2, \ldots$ on the interval $[q^{-s}/2, 1]$.

Then using the q-equation by parts, the q-integral

$$I = \int_{q^{-s}/2}^{1} x^{t-1}(1 - qx)_q^{s-1} \, d_q x$$

is obtained as

$$I = \int_{q^{-s}/2}^{q^{-s}+\epsilon} (1 - qx)_q^{s-1}\left[x^{t-1} - (q^{-s})^{t-1}\right] d_q x - \frac{1}{[s]_q}(q^{-s})^{t-1}\left[(1 - q^{-s} - \epsilon)_q^{s} - (1 - q^{-s}/2)_q^{s}\right] +$$

$$+ \sum_{k=0}^{n-1} \int_{q^{-s-k}+\epsilon}^{(q^{-s-k}+q^{-s-k-1})/2} (1 - qx)_q^{s-1}\left[x^{t-1} - (q^{-s-k})^{t-1}\right] d_q x +$$

$$- \frac{1}{[s]_q} \sum_{k=0}^{n-1}(q^{-s-k})^{t-1}\left[\left(1 - (q^{-s-k} + q^{-s-k-1})/2\right)_q^{s} - (1 - q^{-s-k} - \epsilon)_q^{s}\right] +$$

$$+ \sum_{k=0}^{n-1} \int_{(q^{-s-k}+q^{-s-k-1})/2}^{q^{-s-k-1}+\epsilon} (1 - qx)_q^{s-1}\left[x^{t-1} - (q^{-s-k-1})^{t-1}\right] d_q x +$$

$$- \frac{1}{[s]_q} \sum_{k=0}^{n-1}(q^{-s-k-1})^{t-1}\left[(1 - q^{-s-k-1} - \epsilon)_q^{s} - \left(1 - (q^{-s-k} + q^{-s-k-1})/2\right)_q^{s}\right] +$$

$$+ \int_{q^{-s-n}+\epsilon}^{1} (1 - qx)_q^{s-1}\left[x^{t-1} - (q^{-s-n})^{t-1}\right] d_q x + \frac{1}{[s]_q}(q^{-s-n})^{t-1}(1 - q^{-s-n} - \epsilon)_q^{s}$$

where the second, third, fourth and fifth sums are being empty in the right hand side of the last equation when $n = 0$.

We have from the generalized Gauss binomial formula that

$$\begin{aligned}
(1 - q^{-s-k} - \epsilon)_q^{s} &= \sum_{j=0}^{\infty} \begin{bmatrix} s \\ j \end{bmatrix}_q q^{\frac{j(j-1)}{2}}(-1)^j(q^{-s-k} + \epsilon)^j = \\
&= \sum_{j=0}^{\infty} \begin{bmatrix} s \\ j \end{bmatrix}_q q^{\frac{j(j-1)}{2}}(-1)^j(q^{-s-k})^j + O(\epsilon) = \\
&= (1 - q^{-s-k})_q^{s} + O(\epsilon).
\end{aligned}$$

Then

$$\mathrm{N}\!-\!\lim_{\epsilon \to 0}(1 - q^{-s-k} - \epsilon)_q^{s} = (1 - q^{-s-k})_q^{s} \tag{12}$$

and similarly

$$\mathrm{N}\!-\!\lim_{\epsilon \to 0}(1 - q^{-s-k-1} - \epsilon)_q^{s} = (1 - q^{-s-k-1})_q^{s} \tag{13}$$

for $k = 0, 1, \ldots, n-1$.

Now, recalling that the neutrix is given as in the section 2 and taking into account the neutrix limit of both sides of the equation we get

$$\underset{\epsilon \to 0}{\text{N}-\lim} \int_\epsilon^1 x^{t-1}(1-qx)_q^{s-1}\, d_q x =$$

$$= \int_0^{q^{-s}/2} x^{t-1}\left[(1-qx)_q^{s-1} - \sum_{j=0}^r (-1)^j \begin{bmatrix} s-1 \\ j \end{bmatrix}_q q^{\frac{j(j+1)}{2}} x^j\right] d_q x +$$

$$+ \sum_{j=0}^r (-1)^j \begin{bmatrix} s-1 \\ j \end{bmatrix}_q q^{\frac{j(j+1)}{2}} \frac{1}{[t+j]_q}(q^{-s}/2)^{t+j} +$$

$$+ \int_{q^{-s}/2}^{q^{-s}} (1-qx)_q^{s-1}\left[x^{t-1} - (q^{-s})^{t-1}\right] d_q x - \frac{1}{[s]_q}(q^{-s})^{t-1}\left[(1-q^{-s})_q^s - (1-q^{-s}/2)_q^s\right] +$$

$$+ \sum_{k=0}^{n-1} \int_{q^{-s-k}}^{(q^{-s-k}+q^{-s-k-1})/2} (1-qx)_q^{s-1}\left[x^{t-1} - (q^{-s-k})^{t-1}\right] d_q x +$$

$$- \frac{1}{[s]_q} \sum_{k=0}^{n-1} (q^{-s-k})^{t-1}\left[\left(1-(q^{-s-k}+q^{-s-k-1})/2\right)_q^s - (1-q^{-s-k})_q^s\right] +$$

$$+ \sum_{k=0}^{n-1} \int_{(q^{-s-k}+q^{-s-k-1})/2}^{q^{-s-k-1}} (1-qx)_q^{s-1}\left[x^{t-1} - (q^{-s-k-1})^{t-1}\right] d_q x +$$

$$- \frac{1}{[s]_q} \sum_{k=0}^{n-1} (q^{-s-k-1})^{t-1}\left[(1-q^{-s-k-1})_q^s - \left(1-(q^{-s-k}+q^{-s-k-1})/2\right)_q^s\right] +$$

$$+ \int_{q^{-s-n}}^1 (1-qx)_q^{s-1}\left[x^{t-1} - (q^{-s-n})^{t-1}\right] d_q x + \frac{1}{[s]_q}(q^{-s-n})^{t-1}(1-q^{-s-n})_q^s$$

for $t > -r, t \neq 0, -1, \ldots, -r+1$ and $-n-1 < s < -n, n = 0, 1, 2, \ldots$. Hence the proof is complete by using the equations 12 and 13 and also the property that the neutrix limit if it exist, is unique and it is precisely the same as the ordinary limit, if it exists. □

Lemma 4.3.

$$(1+q^\alpha x)_q^\beta = \frac{(1+x)_q^{\alpha+\beta}}{(1+x)_q^\alpha}$$

for $\alpha, \beta \in \mathbb{R}$.

Proof. The proof follows immediately by using the equation (1). □

Theorem 4.4. *The neutrix limit as ϵ tends to zero of the q-integral*

$$\int_\epsilon^{1-\epsilon} x^{-1}(1-qx)_q^{-1}\, d_q x$$

exists and

$$\underset{\epsilon \to 0}{\text{N}-\lim} \int_\epsilon^{1-\epsilon} x^{-1}(1-qx)_q^{-1}\, d_q x = \sum_{j=0}^\infty \frac{1}{[j+1]_q}.$$

Proof. Since from lemma 4.3 and equations (2) and (3) we can write

$$(1 - qx)_q^{-1} = \frac{(1 - x)_q^0}{(1 - x)_q} = \frac{1}{-(x - 1)_q} = \frac{1}{1 - x},$$

then we have

$$\int_\epsilon^{1-\epsilon} x^{-1}(1 - qx)_q^{-1} \, d_q x = \int_\epsilon^{1-\epsilon} x^{-1}(1 - x)^{-1} \, d_q x =$$

$$= \int_\epsilon^{1-\epsilon} \frac{1}{x} \, d_q x + \int_\epsilon^{1-\epsilon} \frac{1}{1 - x} \, d_q x.$$

We note that

$$\frac{1}{1 - x} = \sum_{j=0}^{\infty} x^j$$

is a formal power series and

$$\int_\epsilon^{1-\epsilon} \frac{1}{1 - x} \, d_q x = \sum_{j=0}^{\infty} \frac{x^{j+1}}{[j + 1]_q} \Big|_\epsilon^{1-\epsilon}.$$

Then

$$\int_\epsilon^{1-\epsilon} x^{-1}(1 - qx)_q^{-1} \, d_q x = \frac{q - 1}{\ln q} \left[\ln(1 - \epsilon) - \ln \epsilon \right] +$$

$$+ \sum_{j=0}^{\infty} \frac{(1 - \epsilon)^{j+1} - \epsilon^{j+1}}{[j + 1]_q}$$

and it follows that

$$\text{N--}\lim_{\epsilon \to 0} \int_\epsilon^{1-\epsilon} x^{-1}(1 - qx)_q^{-1} \, d_q x = \sum_{j=0}^{\infty} \frac{1}{[j + 1]_q}.$$

This leads us to the desired result. □

Definition 4.5. The q-analogue of the beta function at $t = s = 0$ is defined by

$$B_q(0, 0) = \text{N--}\lim_{\epsilon \to 0} \int_\epsilon^{1-\epsilon} x^{-1}(1 - qx)_q^{-1} \, d_q x.$$

Since we have

$$\text{N--}\lim_{\epsilon \to 0} \int_\epsilon^{1-\epsilon} x^{t-1}(1 - qx)_q^{s-1} \, d_q x = \text{N--}\lim_{\epsilon \to 0} \int_\epsilon^{1} x^{t-1}(1 - qx)_q^{s-1} \, d_q x =$$

$$= \text{N--}\lim_{\epsilon \to 0} \int_0^{1-\epsilon} x^{t-1}(1 - qx)_q^{s-1} \, d_q x = \int_0^{1} x^{t-1}(1 - qx)_q^{s-1} \, d_q x$$

for $s, t > 0$, this suggests us to give the following definition.

Definition 4.6.

$$B_q(0, s) = \text{N--}\lim_{\epsilon \to 0} \int_\epsilon^{1} x^{-1}(1 - qx)_q^{s-1} \, d_q x$$

and

$$B_q(t, 0) = \text{N--}\lim_{\epsilon \to 0} \int_0^{1-\epsilon} x^{t-1}(1 - qx)_q^{-1} \, d_q x.$$

In particular, since we have for $n > 1$

$$\int_\epsilon^1 x^{-1}(1-qx)_q^{n-1}\,d_qx = \int_\epsilon^1 x^{-1}(1-qx)_q^{n-2}(1-q^{n-1}x)\,d_qx =$$

$$= \int_\epsilon^1 x^{-1}(1-qx)_q^{n-2}\,d_qx - q^{n-1}\int_\epsilon^1 d_qx$$

then, if we continue this process we get

$$\int_\epsilon^1 x^{-1}(1-qx)_q^{n-1}\,d_qx = \int_\epsilon^1 x^{-1}\,d_qx - \sum_{i=1}^{n-1} q^i \int_\epsilon^1 d_qx.$$

Then

$$\mathrm{N}-\lim_{\epsilon\to 0}\int_\epsilon^1 x^{-1}(1-qx)_q^{n-1}\,d_qx = -\sum_{i=1}^{n-1} q^i.$$

Thus

$$B_q(0,n) = -\sum_{i=1}^{n-1} q^i$$

for $n > 1$.
Since

$$\int_0^{1-\epsilon}(1-qx)_q^{-1}\,d_qx = \int_0^{1-\epsilon}\frac{1}{1-x}\,d_qx = \sum_{j=0}^\infty \frac{x^{j+1}}{[j+1]_q}\bigg|_0^{1-\epsilon}$$

then we have

$$\mathrm{N}-\lim_{\epsilon\to 0}\int_0^{1-\epsilon}(1-qx)_q^{-1}\,d_qx = \sum_{i=1}^\infty \frac{1}{[i]_q}.$$

Also we have

$$\int_0^{1-\epsilon} x^{n-1}(1-qx)_q^{-1}\,d_qx =$$

$$= \int_0^{1-\epsilon}(1-x)^{-1}\,d_qx - \int_0^{1-\epsilon}\sum_{j=0}^{n-2} x^j\,d_qx = \sum_{j=0}^\infty \frac{x^{j+1}}{[j+1]_q}\bigg|_0^{1-\epsilon} - \sum_{j=0}^{n-2} \frac{x^{j+1}}{[j+1]_q}\bigg|_0^{1-\epsilon}$$

Then

$$\mathrm{N}-\lim_{\epsilon\to 0}\int_0^{1-\epsilon} x^{n-1}(1-qx)_q^{-1}\,d_qx = \sum_{i=n}^\infty \frac{1}{[i]_q}.$$

Hence

$$B_q(n,0) = \sum_{i=n}^\infty \frac{1}{[i]_q}$$

for $n \geq 1$.

Definition 4.7. The q-analogue of the beta function for negative integers is defined by

$$B_q(-n,-m) = \mathrm{N}-\lim_{\epsilon\to 0}\int_\epsilon^{1-\epsilon} x^{-n-1}(1-qx)_q^{-m-1}\,d_qx \tag{14}$$

for $n, m = 1, 2, \ldots$.

In the following theorem, we will say that the function $B_q(-n, -m)$ exists for $n, m = 1, 2, \ldots$. First of all we need the following lemma.

Lemma 4.8. *The neutrix limit as ϵ tends to zero of the q-integral*

$$\int_\epsilon^{1-\epsilon} x^{-n-1}(1 - qx)_q^{-1} \, d_q x$$

exists for $n = 1, 2, \ldots$.

Proof. Since we have

$$\int_\epsilon^{1-\epsilon} x^{-n-1}(1 - qx)_q^{-1} \, d_q x$$

$$= \int_\epsilon^{1-\epsilon} (1 - x)^{-1} \, d_q x + \int_\epsilon^{1-\epsilon} x^{-1} \, d_q x + \int_\epsilon^{1-\epsilon} \sum_{j=2}^{n+1} x^{-j} \, d_q x =$$

$$= \sum_{j=0}^{\infty} \frac{(1 - \epsilon)^{j+1}}{[j + 1]_q} - \sum_{j=0}^{\infty} \frac{\epsilon^{j+1}}{[j + 1]_q} + \frac{q - 1}{\ln q} [\ln(1 - \epsilon) - \ln \epsilon] + \sum_{j=2}^{n+1} \frac{1}{[-j + 1]_q}[(1 - \epsilon)^{-j+1} - \epsilon^{-j+1}],$$

then it follows that

$$\underset{\epsilon \to 0}{\text{N}-\lim} \int_\epsilon^{1-\epsilon} x^{-n-1}(1 - qx)_q^{-1} \, d_q x = \sum_{j=0}^{\infty} \frac{1}{[j + 1]_q} + \sum_{j=2}^{n+1} \frac{1}{[-j + 1]_q} =$$

$$= \sum_{i=1}^{\infty} \frac{1}{[i]_q} + \sum_{i=1}^{n} \frac{1}{[-i]_q} = \sum_{i=-n_{i \neq 0}}^{\infty} \frac{1}{[i]_q}$$

This completes the proof of the theorem. □

Theorem 4.9. *The neutrix limit as ϵ tends to zero of the q-integral*

$$\int_\epsilon^{1-\epsilon} x^{-n-1}(1 - qx)_q^{-m-1} \, d_q x$$

exists for $n, m = 1, 2, \ldots$.

Proof. Assume the existence of the neutrix limit as ϵ tends to zero of the function

$$\int_\epsilon^{1-\epsilon} x^{-n-1}(1 - qx)_q^{-m} \, d_q x$$

for $n = 1, 2 \ldots$ and some positive integer m. Lemma 4.8 shows us that this is certainly true when $m = 1$. Then using the q-integration by parts we have

$$\int_\epsilon^{1-\epsilon} x^{-n-1}(1 - qx)_q^{-m-1} \, d_q x =$$

$$= -\frac{1}{[-m]_q} \int_\epsilon^{1-\epsilon} x^{-n-1} D_q (1 - x)_q^{-m} =$$

$$= \frac{1}{[-m]_q}\left(\epsilon^{-n-1}(1 - \epsilon)_q^{-m} - (1 - \epsilon)^{-n-1}\epsilon^{-m}\right) + \frac{[-n - 1]_q}{[-m]_q} \int_\epsilon^{1-\epsilon} x^{-n-2}(1 - qx)_q^{-m} \, d_q x =$$

$$= \frac{1}{[-m]_q}\left(\sum_{j=0}^{n+1} \begin{bmatrix} -m \\ j \end{bmatrix}_q q^{j(j-1)/2} \epsilon^{j-n-1} - \sum_{j=0}^{m} \frac{(n + j)!}{n! \, j!} \epsilon^{j-m}\right) +$$

$$+ \frac{[-n - 1]_q}{[-m]_q} \int_\epsilon^{1-\epsilon} x^{-n-2}(1 - qx)_q^{-m} \, d_q x + O(\epsilon).$$

It follows from our assumption that

$$\text{N}-\lim_{\epsilon \to 0} \int_{\epsilon}^{1-\epsilon} x^{-n-1}(1 - qx)_q^{-m-1} \, d_q x =$$

$$= \left[\begin{array}{c} -m-1 \\ n+1 \end{array} \right]_q q^{n(n+1)/2} - \frac{(n+m)!}{[-m]_q \, n! \, m!} + \frac{[-n-1]_q}{[-m]_q} B_q(-n-1, m+1).$$

The existence of $B_q(-n, -m)$ now follows by induction, and the proof is complete. □

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