WEIGHTED COMPOSITION OPERATORS FROM MINIMAL MÖBIUS INVARIANT SPACES TO ZYGMUND SPACES

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Abstract. The boundedness and the compactness of weighted composition operators from minimal Möbius invariant spaces to Zygmund spaces on the unit disc are investigated in this paper.

1. Introduction

Let $\mathbb{D}$ denote the open unit disk in the complex plane $\mathbb{C}$ and $H(\mathbb{D})$ the space of all analytic functions in $\mathbb{D}$. Let H^∞ denote the bounded analytic function space in $\mathbb{D}$. An $f \in H(\mathbb{D})$ is said to belong to the Bloch space, denoted by $\mathcal{B} = \mathcal{B}(\mathbb{D})$, if

$$\|f\| = \sup_{z \in \mathbb{D}}(1 - |z|^2)|f'(z)| < \infty.$$

The space $\mathcal{B}$ becomes a Banach space with the norm $\|f\|_{\mathcal{B}} = |f(0)| + \|f\|$. Let $\mathcal{B}_0$ denote the subspace of $\mathcal{B}$ consisting of those $f \in \mathcal{B}$ such that $\lim_{|z| \to 1}(1 - |z|^2)|f'(z)| = 0$. This space is called the little Bloch space.

For $a \in \mathbb{D}$, let σ_a be the automorphism of $\mathbb{D}$ exchanging points 0 and a, namely

$$\sigma_a(z) = \frac{a - z}{1 - \bar{a}z}, \quad z \in \mathbb{D}.$$

The analytic Besov space B_1 is defined to be the set of all analytic functions f on $\mathbb{D}$ which can be written as

$$f(z) = \sum_{n=1}^{\infty} a_n \sigma_{\lambda_n}(z)$$

for some sequence $\{a_n\}$ in l^1 and $\lambda_n \in \mathbb{D}$. We norm B_1 by

$$\|f\|_{B_1} = \inf\left\{\sum_{n=1}^{\infty} |a_n| : f(z) = \sum_{n=1}^{\infty} a_n \sigma_{\lambda_n}(z)\right\}.$$

It is obvious that $B_1 \subset H^\infty$. In fact, functions in B_1 can extend continuously to the boundary, so B_1 is a "boundary regular" space (see [5]). B_1 is a subset of $\mathcal{B}_0$. Moreover as is the case for $\mathcal{B}_0$, the set of polynomials is dense in B_1 (see [2]). The space B_1 was extensively studied in [2], where it was shown that if one defines appropriately the notion of a "Möbius invariant space", then B_1 is the smallest one. Hence, B_1 is also called the minimal Möbius invariant space. Moreover, there exists a constant $C > 0$ such that for every $f \in B_1$ (see [2, 35]),

$$C^{-1} \int_{\mathbb{D}} |f''(z)| dA(z) \le \|f - f(0) - f'(0)z\|_{B_1} \le C \int_{\mathbb{D}} |f''(z)| dA(z), \tag{1}$$

where dA is the normalized area measure, that is $A(\mathbb{D}) = 1$.

An analytic function $f \in H(\mathbb{D}) \cap C(\overline{\mathbb{D}})$ is said to belong to the Zygmund space $\mathscr{Z}$ if

$$\sup \frac{|f(e^{i(\theta+h)}) + f(e^{i(\theta-h)}) - 2f(e^{i\theta})|}{h} < \infty,$$

where the supremum is taken over all $e^{i\theta} \in \partial \mathbb{D}$ and $h > 0$. By the Theorem 5.3 in [7] and the Closed Graph Theorem we have that $f \in \mathscr{Z}$ if and only if

$$\sup_{z \in \mathbb{D}} (1 - |z|^2) |f''(z)| < \infty.$$

It is easy to see that $\mathscr{Z}$ is a Banach space under the norm

$$\|f\|_{\mathscr{Z}} = |f(0)| + |f'(0)| + \sup_{z \in \mathbb{D}} (1 - |z|^2) |f''(z)|. \tag{2}$$

Let $\mathscr{Z}_0$ denote the subspace of $\mathscr{Z}$ consisting of those $f \in \mathscr{Z}$ for which

$$\lim_{|z| \to 1} (1 - |z|^2) |f''(z)| = 0.$$

The space $\mathscr{Z}_0$ is called the little Zygmund space.

Throughout the paper, $S(\mathbb{D})$ denotes the set of analytic self-map of $\mathbb{D}$. Associated with $\varphi \in S(\mathbb{D})$ is the composition operator C_φ defined by

$$C_\varphi f = f \circ \varphi$$

for $f \in H(\mathbb{D})$. Let $u \in H(\mathbb{D})$. Define a linear operator uC_φ on $H(\mathbb{D})$ as follows:

$$(uC_\varphi f)(z) = u(z) f(\varphi(z)), \quad f \in H(\mathbb{D}).$$

We call uC_φ the weighted composition operator and regard this operator as a generalization of a multiplication operator and a composition operator. We refer to the book [5] for the theory of composition operators.

Composition and some related operators from or to Zygmund spaces on the unit disk or the unit ball were studied, for example, in [4, 14–19, 26, 27]. Composition operators on the minimal Möbius invariant space was studied in [2, 3, 6, 32, 34]. For some extensions of Zygmund spaces and studying operators on them see, for example, [28–30] and the references therein. The boundedness and compactness of composition operators and weighted composition operators between different analytic function spaces on unit disc $\mathbb{D}$, as well as, on unit ball are studied, for example, in [1, 8–13, 20–25, 33, 37, 38] (see also related references therein).

In this paper, we investigate the boundedness and compactness of the weighted composition operator uC_φ from minimal Möbius invariant spaces to $\mathscr{Z}$ and $\mathscr{Z}_0$.

Throughout this paper C denotes a positive constant which may be different at different occurrences.

2. Main results and proofs

In this section we formulate and prove our main results in this paper. First, we quote several auxiliary results. The first one can be found in [31].

Lemma 1. *Assume f is a holomorphic function on $\mathbb{D}$ and continuous on $\overline{\mathbb{D}}$. Then the modulus of continuity on the closed unit disk is bounded by a constant times the modulus of continuity on the unit circle.*

Using Lemma 1 and the fact that $B_1 \subset A(\mathbb{D})$, the disk algebra of analytic functions in $C(\overline{\mathbb{D}})$, the following criterion for compactness follows by similar arguments as in the proof of Lemma 3.7 in [22].

Lemma 2. *Suppose that $\varphi \in S(\mathbb{D})$ and $u \in H(\mathbb{D})$. The operator $uC_\varphi : B_1 \to \mathscr{Z}$ is compact if and only if $uC_\varphi : B_1 \to \mathscr{Z}$ is bounded and for any bounded sequence $(f_n)_{n\in\mathbb{N}}$ in B_1 which converges to zero uniformly on $\overline{\mathbb{D}}$, we have $\|uC_\varphi f_n\|_{\mathscr{Z}} \to 0$ as $n \to \infty$.*

Lemma 3. *[15] A closed set K in $\mathscr{Z}_0$ is compact if and only if it is bounded and satisfies*

$$\lim_{|z|\to 1} \sup_{f\in K} (1 - |z|^2)|f''(z)| = 0.$$

Now, we are in a position to formulate and prove the main results of this paper.

Theorem 1. *Suppose that $\varphi \in S(\mathbb{D})$ and $u \in H(\mathbb{D})$. Then $uC_\varphi : B_1 \to \mathscr{Z}$ is bounded if and only if $u \in \mathscr{Z}$,*

$$M_1 := \sup_{z\in\mathbb{D}} \frac{(1 - |z|^2)|v(z)|}{1 - |\varphi(z)|^2} < \infty \tag{3}$$

and

$$M_2 := \sup_{z\in\mathbb{D}} \frac{(1 - |z|^2)|u(z)||\varphi'(z)|^2}{(1 - |\varphi(z)|^2)^2} < \infty, \tag{4}$$

where $v(z) = 2u'(z)\varphi'(z) + u(z)\varphi''(z)$.

Proof. Suppose that $u \in \mathscr{Z}$, (3) and (4) hold. Note that $B_1 \subset H^\infty \cap \mathcal{B}_0$ and H^∞ is Möbius invariant (see [2, 35]), for $h = \sum_{n=1}^{\infty} a_n \sigma_{t_n} \in B_1$, with $(a_n)_{n\in\mathbb{N}} \in \ell^1$ and $t_n \in \mathbb{D}$, $n \in \mathbb{N}$,

$$\|h\| \leq \|h\|_\infty \leq \sum_{n=1}^{\infty} |a_n|\|\sigma_{t_n}\|_\infty = \sum_{n=1}^{\infty} |a_n|.$$

Taking the greatest lower bound over all such sequences $(a_n)_{n\in\mathbb{N}}$ in the above representation of h, we obtain

$$\|h\| \leq \|h\|_\infty \leq \|h\|_{B_1}.$$

Thus, for an arbitrary z in $\mathbb{D}$ and $f \in B_1$, we have

$$
\begin{aligned}
&(1 - |z|^2)|(uC_\varphi f)''(z)| \\
={} &(1 - |z|^2)\left|u''(z)f(\varphi(z)) + v(z)f'(\varphi(z)) + u(z)(\varphi'(z))^2 f''(\varphi(z))\right| \\
\leq{} &(1 - |z|^2)|u''(z)||f(\varphi(z))| + (1 - |z|^2)|u(z)(\varphi'(z))^2||f''(\varphi(z))| + (1 - |z|^2)|v(z)||f'(\varphi(z))| \\
\leq{} &C\|f\|_{B_1}\left[(1 - |z|^2)|u''(z)| + \frac{(1 - |z|^2)|u(z)(\varphi'(z))^2|}{(1 - |\varphi(z)|^2)^2} + \frac{(1 - |z|^2)|v(z)|}{1 - |\varphi(z)|^2}\right],
\end{aligned}
\tag{5}
$$

where in the last inequality we have used the following well known characterization for Bloch functions (see [36])

$$\sup_{z\in\mathbb{D}}(1 - |z|^2)|f'(z)| \asymp |f'(0)| + \sup_{z\in\mathbb{D}}(1 - |z|^2)^2|f''(z)|.$$

Taking the supremum in (5) over $\mathbb{D}$ and then using the assumption conditions we obtain that $uC_\varphi : B_1 \to \mathscr{Z}$ is bounded.

Conversely, suppose that $uC_\varphi : B_1 \to \mathscr{Z}$ is bounded. By using test functions $1, z \in B_1$, we get $u \in \mathscr{Z}$ and

$$\sup_{z \in \mathbb{D}}(1 - |z|^2)|u''(z)\varphi(z) + v(z)| < \infty,$$

respectively. Hence, by the fact that $u \in \mathscr{Z}$, we obtain

$$\sup_{z \in \mathbb{D}}(1 - |z|^2)|v(z)| < \infty, \tag{6}$$

which implies that

$$\sup_{|\varphi(z)| \le \frac{1}{2}} \frac{(1 - |z|^2)|v(z)|}{1 - |\varphi(z)|^2} < \infty. \tag{7}$$

By using test function $p(z) = z^2 \in B_1$, we have

$$\infty > \|p\|_{B_1}\|uC_\varphi\| \ge \|uC_\varphi p\|_{\mathscr{Z}} \ge \sup_{z \in \mathbb{D}}(1 - |z|^2)|(uC_\varphi p)''(z)|$$

$$= \sup_{z \in \mathbb{D}}(1 - |z|^2)\left|u''(z)(\varphi(z))^2 + 2u(z)(\varphi'(z))^2 + 2v(z)\varphi(z)\right|,$$

which with $u \in \mathscr{Z}$ and (6) implies that

$$\sup_{z \in \mathbb{D}}(1 - |z|^2)|u(z)(\varphi'(z))^2| < \infty. \tag{8}$$

Hence by (8), we get

$$\sup_{|\varphi(z)| \le \frac{1}{2}} \frac{(1 - |z|^2)|u(z)(\varphi'(z))^2|}{(1 - |\varphi(z)|^2)^2} < \infty. \tag{9}$$

If $a \in \mathbb{D}$ such that $|\varphi(a)| > \frac{1}{2}$, take

$$f_a(z) = \frac{(1 - |\varphi(a)|^2)(\varphi(a) - z)}{(1 - \overline{\varphi(a)}z)^2} - \frac{\varphi(a) - z}{1 - \overline{\varphi(a)}z} = (-\overline{\varphi(a)}) \times \frac{\varphi(a) - z}{1 - \overline{\varphi(a)}z} \times \frac{\varphi(a) - z}{1 - \overline{\varphi(a)}z}. \tag{10}$$

From Theorem 10 of [2], we see that B_1 is an algebra and

$$\|fg\|_{B_1} \le 7\|f\|_{B_1}\|g\|_{B_1}, \quad \text{for all} \quad f, g \in B_1. \tag{11}$$

Hence $f_a \in B_1$. It is easy to check that $f_a(\varphi(a)) = 0$,

$$f_a'(z) = \frac{2\overline{\varphi(a)}(1 - |\varphi(a)|^2)(\varphi(a) - z)}{(1 - \overline{\varphi(a)}z)^3} \quad \text{and} \quad f_a''(z) = \frac{2\overline{\varphi(a)}(1 - |\varphi(a)|^2)(3|\varphi(a)|^2 - 1 - 2\overline{\varphi(a)}z)}{(1 - \overline{\varphi(a)}z)^4}.$$

Thus

$$f_a'(\varphi(a)) = 0 \quad \text{and} \quad f_a''(\varphi(a)) = \frac{-2\overline{\varphi(a)}}{(1 - |\varphi(a)|^2)^2}.$$

Therefore, we have

$$\infty > \|uC_\varphi f_a\|_{\mathscr{Z}} \ge \sup_{z \in \mathbb{D}}(1 - |z|^2)|(uC_\varphi f_a)''(z)|$$

$$= \sup_{z \in \mathbb{D}}(1 - |z|^2)\left|u''(z)f_a(\varphi(z)) + u(z)(\varphi'(z))^2 f_a''(\varphi(z)) + v(z)f_a'(\varphi(z))\right|$$

$$\ge (1 - |a|^2)\left|u''(a)f_a(\varphi(a)) + u(a)(\varphi'(a))^2 f_a''(\varphi(a)) + v(a)f_a'(\varphi(a))\right|$$

$$= (1 - |a|^2)\left|u(a)(\varphi'(a))^2 \frac{-2\overline{\varphi(a)}}{(1 - |\varphi(a)|^2)^2}\right|. \tag{12}$$

It follows that

$$\sup_{|\varphi(a)|>1/2} (1-|a|^2)\left|u(a)(\varphi'(a))^2 \frac{2\overline{\varphi(a)}}{(1-|\varphi(a)|^2)^2}\right| < \infty, \tag{13}$$

which implies

$$\sup_{|\varphi(a)|>1/2} \frac{(1-|a|^2)|u(a)||\varphi'(a)|^2}{(1-|\varphi(a)|^2)^2} < \infty. \tag{14}$$

By (9) and (14) we obtain (4).

Next, set $g_a(z) = \frac{\varphi(a)-z}{1-\overline{\varphi(a)}z}$. Then $g_a(z) \in B_1$,

$$g_a'(z) = -\frac{1-|\varphi(a)|^2}{(1-\overline{\varphi(a)}z)^2} \text{ and } g_a''(z) = -\frac{2\overline{\varphi(a)}(1-|\varphi(a)|^2)}{(1-\overline{\varphi(a)}z)^3}. \tag{15}$$

Thus, by the boundedness of uC_φ and (15), we have

$$\begin{aligned}
\infty &> \|uC_\varphi g_a\|_{\mathscr{Z}} \geq \sup_{z\in\mathbb{D}}(1-|z|^2)|(uC_\varphi g_a)''(z)| \\
&= \sup_{z\in\mathbb{D}}(1-|z|^2)\left|u''(z)g_a(\varphi(z)) + u(z)(\varphi'(z))^2 g_a''(\varphi(z)) + v(z)g_a'(\varphi(z))\right| \\
&\geq (1-|a|^2)\left|u''(a)g_a(\varphi(a)) + u(a)(\varphi'(a))^2 g_a''(\varphi(a)) + v(a)g_a'(\varphi(a))\right| \\
&= (1-|a|^2)\left|\frac{2u(a)(\varphi'(a))^2\overline{\varphi(a)}}{(1-|\varphi(a)|^2)^2} + \frac{v(a)}{1-|\varphi(a)|^2}\right|.
\end{aligned} \tag{16}$$

Employing (12) and (16), it follows that

$$\sup_{|\varphi(a)|>1/2} \frac{(1-|a|^2)|v(a)|}{1-|\varphi(a)|^2} < \infty. \tag{17}$$

From (17) and (7), we get (3). The proof is completed.

Theorem 2. *Suppose that $\varphi \in S(\mathbb{D})$ and $u \in H(\mathbb{D})$. Then $uC_\varphi : B_1 \to \mathscr{Z}$ is compact if and only if $uC_\varphi : B_1 \to \mathscr{Z}$ is bounded,*

$$\lim_{|\varphi(z)|\to 1} \frac{(1-|z|^2)|v(z)|}{1-|\varphi(z)|^2} = 0 \tag{18}$$

and

$$\lim_{|\varphi(z)|\to 1} \frac{(1-|z|^2)|u(z)||\varphi'(z)|^2}{(1-|\varphi(z)|^2)^2} = 0. \tag{19}$$

Proof. First assume that $uC_\varphi : B_1 \to \mathscr{Z}$ is bounded, (18) and (19) hold. By the boundedness of $uC_\varphi : B_1 \to \mathscr{Z}$ and the proof of Theorem 1, we see that $u \in \mathscr{Z}$,

$$I_1 := \sup_{z\in\mathbb{D}}(1-|z|^2)|v(z)| < \infty, \quad I_2 := \sup_{z\in\mathbb{D}}(1-|z|^2)|u(z)(\varphi'(z))^2| < \infty. \tag{20}$$

From (18) and (19), for any $\varepsilon > 0$, there is a positive number δ, $0 < \delta < 1$, such that

$$\frac{(1-|z|^2)|v(z)|}{1-|\varphi(z)|^2} < \varepsilon, \quad \frac{(1-|z|^2)|u(z)||\varphi'(z)|^2}{(1-|\varphi(z)|^2)^2} < \varepsilon, \tag{21}$$

when $\delta < |\varphi(z)| < 1$.

Let $(f_n)_{n\in\mathbb{N}}$ be a bounded sequence in B_1 and suppose that f_n converges to zero uniformly on $\overline{\mathbb{D}}$. Employing (2), (20) and (21), we have

$$\sup_{z\in\mathbb{D}}(1-|z|^2)|(uC_\varphi f_n)''(z)|$$

$$= \sup_{z\in\mathbb{D}}(1-|z|^2)\left|u''(z)f_n(\varphi(z)) + u(z)(\varphi'(z))^2 f_n''(\varphi(z)) + v(z)f_n'(\varphi(z))\right|$$

$$\leq \sup_{z\in\mathbb{D}}(1-|z|^2)|u''(z)f_n(\varphi(z))| + \sup_{|\varphi(z)|\leq\delta}(1-|z|^2)|u(z)(\varphi'(z))^2 f_n''(\varphi(z))|$$

$$+ \sup_{\delta<|\varphi(z)|<1}(1-|z|^2)|u(z)(\varphi'(z))^2 f_n''(\varphi(z))| + \sup_{|\varphi(z)|\leq\delta}(1-|z|^2)\left|v(z)f_n'(\varphi(z))\right| + \sup_{\delta<|\varphi(z)|<1}(1-|z|^2)\left|v(z)f_n'(\varphi(z))\right|$$

$$\leq \|u\|_{\mathscr{Z}}\sup_{z\in\mathbb{D}}|f_n(\varphi(z))| + \sup_{|\varphi(z)|\leq\delta}(1-|z|^2)|u(z)(\varphi'(z))^2||f_n''(\varphi(z))| + C\|f_n\|_{B_1}\sup_{\delta<|\varphi(z)|<1}\frac{(1-|z|^2)|u(z)||\varphi'(z)|^2}{(1-|\varphi(z)|^2)^2}$$

$$+ \sup_{|\varphi(z)|\leq\delta}(1-|z|^2)|v(z)||f_n'(\varphi(z))| + C\|f_n\|_{B_1}\sup_{\delta<|\varphi(z)|<1}\frac{(1-|z|^2)|v(z)|}{1-|\varphi(z)|^2}$$

$$\leq \|u\|_{\mathscr{Z}}\sup_{w\in\overline{\mathbb{D}}}|f_n(w)| + I_1\sup_{|w|\leq\delta}|f_n'(w)| + I_2\sup_{|w|\leq\delta}|f_n''(w)| + \varepsilon C\|f_n\|_{B_1},$$

i.e., we get

$$\|uC_\varphi f_n\|_{\mathscr{Z}} = \sup_{z\in\mathbb{D}}(1-|z|^2)|(uC_\varphi f_n)''(z)| + |u(0)f_n(\varphi(0))| + |u'(0)f_n(\varphi(0)) + u(0)f_n'(\varphi(0))\varphi'(0)|$$

$$\leq \|u\|_{\mathscr{Z}}\sup_{w\in\overline{\mathbb{D}}}|f_n(w)| + I_1\sup_{|w|\leq\delta}|f_n'(w)| + I_2\sup_{|w|\leq\delta}|f_n''(w)| + \varepsilon C\|f_n\|_{B_1}$$

$$+\Big((|u(0)| + |u'(0)|)|f_n(\varphi(0))| + |u(0)f_n'(\varphi(0))\varphi'(0)|\Big). \tag{22}$$

From the fact that $f_n \to 0$ uniformly on $\overline{\mathbb{D}}$ as $n\to\infty$, we have $f_n' \to 0$ and $f_n'' \to 0$ uniformly on any compact subset of $\mathbb{D}$ as $n\to\infty$. Therefore

$$\|u\|_{\mathscr{Z}}\sup_{w\in\overline{\mathbb{D}}}|f_n(w)| + I_1\sup_{|w|\leq\delta}|f_n'(w)| + I_2\sup_{|w|\leq\delta}|f_n''(w)| \to 0$$

and $(|u(0)| + |u'(0)|)|f_n(\varphi(0))| + |u(0)f_n'(\varphi(0))\varphi'(0)| \to 0$ as $n\to\infty$. Let $n\to\infty$ in (22) and notice that ε is an arbitrary positive number it follows that

$$\lim_{n\to\infty}\|uC_\varphi f_n\|_{\mathscr{Z}} = 0.$$

By Lemma 2, we have that $uC_\varphi : B_1 \to \mathscr{Z}$ is compact.

Conversely, suppose $uC_\varphi : B_1 \to \mathscr{Z}$ is compact. Then it is clear that $uC_\varphi : B_1 \to \mathscr{Z}$ is bounded. Let $(z_k)_{k\in\mathbb{N}}$ be a sequence in $\mathbb{D}$ such that $|\varphi(z_k)| \to 1$ as $k\to\infty$ (If there are no such sequences, then (18) and (19) automatically hold). Set

$$f_k(z) = \frac{(1-|\varphi(z_k)|^2)(\varphi(z_k) - z)}{(1-\overline{\varphi(z_k)}z)^2} - \Big(\frac{\varphi(z_k) - z}{1-\overline{\varphi(z_k)}z} - \varphi(z_k)\Big), \tag{23}$$

$$g_k(z) = \frac{\varphi(z_k) - z}{1-\overline{\varphi(z_k)}z} - \varphi(z_k) \quad\text{and}\quad h_k(z) = \frac{(1-|\varphi(z_k)|^2)(\varphi(z_k) - z)}{(1-\overline{\varphi(z_k)}z)^2}. \tag{24}$$

We see that $f_k, g_k, h_k \in B_1$. Moreover, f_k, g_k, h_k converges to zero uniformly on $\overline{\mathbb{D}}$ as $k\to\infty$. Since uC_φ is compact, by Lemma 2 we have

$$\|uC_\varphi f_k\|_{\mathscr{Z}} \to 0, \quad \|uC_\varphi g_k\|_{\mathscr{Z}} \to 0, \quad \|uC_\varphi h_k\|_{\mathscr{Z}} \to 0 \quad\text{as}\quad k\to\infty.$$

After some calculations, we have

$$(1 - |z_k|^2)\left|u''(z_k)\varphi(z_k) + \frac{v(z_k)}{1 - |\varphi(z_k)|^2} + \frac{2\overline{\varphi(z_k)}u(z_k)(\varphi'(z_k))^2}{(1 - |\varphi(z_k)|^2)^2}\right| \le \|uC_\varphi g_k\|_{\mathscr{X}} \to 0, \tag{25}$$

$$(1 - |z_k|^2)\left|u''(z_k)\varphi(z_k) + \frac{2\overline{\varphi(z_k)}u(z_k)(\varphi'(z_k))^2}{(1 - |\varphi(z_k)|^2)^2}\right| \le \|uC_\varphi f_k\|_{\mathscr{X}} \to 0 \tag{26}$$

and

$$(1 - |z_k|^2)\left|\frac{v(z_k)}{1 - |\varphi(z_k)|^2} + \frac{2\overline{\varphi(z_k)}u(z_k)(\varphi'(z_k))^2}{(1 - |\varphi(z_k)|^2)^2}\right| \le \|uC_\varphi h_k\|_{\mathscr{X}} \to 0 \tag{27}$$

as $k \to \infty$. From (25) and (27), we get

$$(1 - |z_k|^2)|u''(z_k)\varphi(z_k)| \to 0 \quad \text{as} \quad k \to \infty. \tag{28}$$

From (26) and (28), we obtain

$$(1 - |z_k|^2)\left|\frac{2\overline{\varphi(z_k)}u(z_k)(\varphi'(z_k))^2}{(1 - |\varphi(z_k)|^2)^2}\right| \to 0 \quad \text{as} \quad k \to \infty. \tag{29}$$

From (27) and (29), we see that

$$(1 - |z_k|^2)\left|\frac{v(z_k)}{1 - |\varphi(z_k)|^2}\right| \to 0 \quad \text{as} \quad k \to \infty. \tag{30}$$

Then the result follows from (29) and (30). This finish the proof.

Theorem 3. *Suppose that $\varphi \in S(\mathbb{D})$ and $u \in H(\mathbb{D})$. Then $uC_\varphi : B_1 \to \mathscr{Z}_0$ is bounded if and only if $uC_\varphi : B_1 \to \mathscr{Z}$ is bounded, $u \in \mathscr{Z}_0$,*

$$\lim_{|z| \to 1}(1 - |z|^2)|v(z)| = 0 \tag{31}$$

and

$$\lim_{|z| \to 1}(1 - |z|^2)|u(z)(\varphi'(z))^2| = 0. \tag{32}$$

Proof. Suppose that $uC_\varphi : B_1 \to \mathscr{Z}_0$ is bounded, then $uC_\varphi : B_1 \to \mathscr{Z}$ is bounded. By using test functions $1, z, z^2 \in B_1$, we get $u \in \mathscr{Z}_0$,

$$\lim_{|z| \to 1}(1 - |z|^2)|u''(z)\varphi(z) + v(z)| = 0 \tag{33}$$

and

$$\lim_{|z| \to 1}(1 - |z|^2)|u''(z)(\varphi(z))^2 + 2\varphi(z)v(z) + 2u(z)(\varphi'(z))^2| = 0. \tag{34}$$

From $u \in \mathscr{Z}_0$, (33) and (34), we see that (31) and (32) hold.

Conversely, suppose that $uC_\varphi : B_1 \to \mathscr{Z}$ is bounded, $u \in \mathscr{Z}_0$, (31) and (32) hold. For each polynomial p, we have

$$(1 - |z|^2)|(uC_\varphi p)''(z)| = (1 - |z|^2)\left|u''(z)p(\varphi(z)) + v(z)p'(\varphi(z)) + u(z)(\varphi'(z))^2 p''(\varphi(z))\right|.$$

The conditions imply $uC_\varphi p \in \mathscr{Z}_0$. Since the set of all polynomials is dense in B_1, we see that, for every $f \in B_1$, there is a sequence of polynomials $(p_n)_{n \in N}$ such that $\|f - p_n\|_{B_1} \to 0$, as $n \to \infty$. Hence, by the boundedness of the operator $uC_\varphi : B_1 \to \mathscr{Z}$, we have

$$\|uC_\varphi f - uC_\varphi p_n\|_{\mathscr{X}} \le \|uC_\varphi\|\|f - p_n\|_{B_1} \to 0, \quad \text{as} \quad n \to \infty.$$

Since $\mathscr{Z}_0$ is a closed subset of $\mathscr{Z}$, we obtain $uC_\varphi(B_1) \subseteq \mathscr{Z}_0$. Therefore $uC_\varphi : B_1 \to \mathscr{Z}_0$ is bounded. The proof is completed.

Theorem 4. *Suppose that $\varphi \in S(\mathbb{D})$ and $u \in H(\mathbb{D})$. Then $uC_\varphi : B_1 \to \mathscr{Z}_0$ is compact if and only if $u \in \mathscr{Z}_0$,*

$$\lim_{|z| \to 1} \frac{(1 - |z|^2)|v(z)|}{1 - |\varphi(z)|^2} = 0 \tag{35}$$

and

$$\lim_{|z| \to 1} \frac{(1 - |z|^2)|u(z)(\varphi'(z))^2|}{(1 - |\varphi(z)|^2)^2} = 0. \tag{36}$$

Proof. Assume that $uC_\varphi : B_1 \to \mathscr{Z}_0$ is compact, then $uC_\varphi : B_1 \to \mathscr{Z}$ is compact and $uC_\varphi : B_1 \to \mathscr{Z}_0$ is bounded. Applying Theorem 2, we obtain that (18) and (19) hold. From Theorem 3, we see that $u \in \mathscr{Z}_0$, (31) and (32) hold. Similarly to the proof of Lemma 4.2 of [12], from (18) and (31) we conclude that (35) holds. Similarly, from (19) and (32), we see that (36) holds.

Assume that $u \in \mathscr{Z}_0$, (35) and (36) hold. From (5), for every $f \in B_1$, we have

$$(1 - |z|^2)|(uC_\varphi f)''(z)| \leq C\|f\|_{B_1}\left[(1 - |z|^2)|u''(z)| + \frac{(1 - |z|^2)|u(z)(\varphi'(z))^2|}{1 - |\varphi(z)|^2} + \frac{(1 - |z|^2)|v(z)|}{1 - |\varphi(z)|^2}\right]. \tag{37}$$

Taking the supremum in (37) over the unit ball $\{f \in B_1 : \|f\|_{B_1} \leq 1\}$, and taking $|z| \to 1$, from the assumption conditions it follows that

$$\lim_{|z| \to 1} \sup_{\|f\|_{B_1} \leq 1} (1 - |z|^2)|(uC_\varphi f)''(z)| = 0.$$

By Lemma 3, we see that $uC_\varphi : B_1 \to \mathscr{Z}_0$ is compact. The proof is completed.

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