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Characterization of compact operators between certain BK spaces

Eberhard Malkowsky

Department of Mathematics, Faculty of Arts and Sciences, Fatih University,Büyükçekmece 34500, Istanbul, Turkey
Department of Mathematics, University of Giessen, Arndtstrasse 2, D–35392 Giessen, Germany

Abstract. We give a survey of the known results concerning the sets $c_0(\Lambda)$, $c(\Lambda)$ and $c_\infty(\Lambda)$ including their basic topological properties, their first and second dual spaces, and the characterizations of matrix transformations from them into the spaces ℓ_∞, c and c_0. Furthermore, we establish some new results such as the representations of the general bounded linear operators from $c(\Lambda)$ into the spaces ℓ_∞, c and c_0, and estimates for their Hausdorff measures of noncompactness. Finally, we apply our results to characterize some classes of compact operators on $c_0(\Lambda)$, $c(\Lambda)$ and $c_\infty(\Lambda)$. We also generalize a classical result by Cohen and Dunford which states that a regular matrix operator cannot be compact.

1. Introduction, notations and known results

The set $c(\Lambda)$ of Λ–strongly convergent sequences was first introduced and studied by Móricz [19]; this set generalizes the concept of strong convergence of Hyslop [10], and Kuttner and Thorpe [15]. We also consider the sets $c_0(\Lambda)$ and $c_\infty(\Lambda)$ of sequences that are Λ–strongly convergent to 0 and Λ–strongly bounded.

First, we give a survey of known results in Sections 2–4. They include the most important topological properties of the spaces $c_0(\Lambda)$, $c(\Lambda)$ and $c_\infty(\Lambda)$ ([16]), their first and second dual spaces ([17]), and the complete list of characterizations of the classes of matrix transformations from them into the spaces ℓ_∞, c and c_0 of bounded, convergent and null sequences ([3]).

In Section 5, we start with a short introduction to measures of noncompactness, in particular, the Hausdorff measure of noncompactness of bounded sets in metric spaces and of operators between normed spaces, and list their most important properties.

After this we establish several new results. We extend our studies from the normally considered matrix transformations to general bounded linear operators on $c(\Lambda)$. First, we establish the representations of the general bounded linear operators from $c(\Lambda)$ into ℓ_∞, c and c_0, and prove some estimates for their Hausdorff measures of noncompactness. Then we use our results to characterize the classes of compact operators

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Email address: Eberhard.Malkowsky@math.uni-giessen.de (Eberhard Malkowsky)

from $c_0(\Lambda)$ and $c(\Lambda)$ into c_0 and c. As an application, we also obtain the characterizations of compact matrix operators between those spaces. Finally, we show that the matrix operator of a Λ–strongly regular matrix cannot be compact. This generalizes a classical result of Cohen and Dunford [5] which states that a regular matrix operator cannot be compact.

This paper is intended to serve as both a survey and research paper.

1.1. The basic notations

Here we list the basic notations and concepts that are used throughout the paper; this is done for the convenience of readers who may not be too familiar with the topics of this paper. We also refer to [18, 20]

A sequence $(b_n)_{n=1}^\infty$ in a linear metric space X is a *Schauder basis* if, for every $x \in X$, there exists a unique sequence $(\lambda_n)_{n=1}^\infty$ of scalars such that $x = \sum_{n=1}^\infty \lambda_n b_n$.

Let X be a normed space. We write $S = S_X = \{x \in X : \|x\| = 1\}$, $B = B_X = \{x \in X : \|x\| < 1\}$ and $\overline{B} = \overline{B_X} = \{x \in X : \|x\| \leq 1\}$ for the unit sphere and the open and closed unit balls in X. If X and Y are normed spaces then we write $\mathcal{B}(X, Y)$ for the set of all bounded linear operators $L : X \to Y$ which is a Banach space with the operator norm $\|L\| = \sup\{\|L(x)\| : x \in S_X\}$ whenever Y is a Banach space. In particular, if $Y = \mathbb{C}$ then X^* denotes the space of all continuous linear functionals on X with the norm $\|f\| = \sup\{|f(x)| : x \in S\}$; we will refer to X^* as the *continuous dual of* X, for short.

As usual, let ω denote the set of all complex sequences $x = (x_k)_{k=1}^\infty$ which is a complete linear metric space with the algebraic operations defined termwise, and its natural metric d given by

$$d(x, y) = \sum_{k=1}^\infty \frac{1}{2^k} \cdot \frac{|x_k - y_k|}{1 + |x_k - y_k|} \text{ for all } x = (x_k)_{k=1}^\infty, y = (y_k)_{k=1}^\infty \in \omega.$$

It is known that convergence in (ω, d) and coordinatewise convergence are equivalent, that is, $d(x^n, x) \to 0$ $(n \to \infty)$ if and only if $x_k^{(n)} \to x_k$ $(n \to \infty)$ for each k.

Let e and $e^{(n)}$ for $n = 1, 2, \dots$ denote the sequences with $e_k = 1$ $(k = 1, 2, \dots)$, and $e_n^{(n)} = 1$ and $e_k^{(n)} = 0$ for $k \neq n$. For any sequence $x = (x_k)_{k=1}^\infty$, let $x^{[n]} = \sum_{k=1}^n x_k e^{(k)}$ be its *n–section*.

We write ℓ_∞, c, c_0 and ϕ for the sets of all bounded, convergent, null and finite sequences; also ℓ_1, cs and bs denote the sets of all absolutely convergent, convergent and bounded series.

A subspace X of ω is said to be an *FK space* if it is a Fréchet space, that is, a complete locally convex linear metric space, with continuous coordinates $P_n : X \to \mathbb{C}$ $(n = 1, 2, \dots)$ where $P_n(x) = x_n$ for all $x \in X$; an *FK* space whose metric is given by a norm is said to be a *BK space*. An *FK* space $X \supset \phi$ is said have *AK* if $x = \lim_{n\to\infty} x^{[n]}$ for every sequence $x = (x_k)_{k=1}^\infty \in X$.

Let $A = (a_{nk})_{k=1}^\infty$ be an infinite matrix of complex numbers, $A_n = (a_{nk})_{k=1}^\infty$ denote the sequence in the n^{th} row of A, X and Y be subsets of ω, and $x \in \omega$. Then we write $A_n x = \sum_{k=1}^\infty a_{nk} x_k$ for the n^{th} *A transform of* x, $Ax = (A_n x)_{n=1}^\infty$ for the *A transform of* x (provided all the series converge), $X_A = \{x \in \omega : Ax \in X\}$ for the *matrix domain of A in X*, and (X, Y) for the class of all infinite matrices that map X into Y, that is, $A \in (X, Y)$ if and only if $X \subset Y_A$.

1.2. Some basic results

Here we list some basic results which are needed in this paper.

It is known that ω is an *FK* space with its natural metric d, ℓ_∞, c and c_0 are *BK* spaces with their natural norm $\|x\|_\infty = \sup_k |x_k|$, c_0 is a closed subspace of c, c is a closed subspace of ℓ_∞; ℓ_1 is a *BK* space with its natural norm $\|x\|_1 = \sum_{k=1}^\infty |x_k|$, cs and bs are *BK* spaces with their natural norm $\|x\|_{bs} = \sup_n |\sum_{k=1}^n x_k|$, and cs is a closed subspace of bs; furthermore, ω, c_0, ℓ_1 and cs have *AK*, every sequence $x = (x_k)_{k=1}^\infty \in c$ has a unique representation $x = \xi \cdot e + \sum_{k=1}^\infty (x_k - \xi) e^{(k)}$ where $\xi = \lim_{k\to\infty} x_k$; ℓ_∞ and bs have no Schauder basis.

One of the most important results in the theory of sequence spaces states that matrix transformations between *FK* spaces are continuous [20, Theorem 4.2.8]. In particular, the following results hold for *BK* spaces.

Theorem 1.1. *Let X and Y be BK spaces.*
(a) Then we have $(X, Y) \subset \mathcal{B}(X, Y)$, that is, every matrix $A \in (X, Y)$ defines an operator $L_A \in \mathcal{B}(X, Y)$ where $L_A(x) = Ax$ for all $x \in X$ ([18, Theorem 1.23]).
(b) If X has AK then $\mathcal{B}(X, Y) \subset (X, Y)$, that is, every operator $L \in \mathcal{B}(X, Y)$ is given by a matrix $A \in (X, Y)$ such that $Ax = L(x)$ for all $x \in X$ ([13, Theorem 1.9]).

2. The spaces of Λ–strongly convergent and bounded sequences

The spaces $c_0(\Lambda)$, $c(\Lambda)$ and $c_\infty(\Lambda)$ were defined and studied for exponentially bounded sequences Λ in [16, 17].

A nondecreasing sequence $\Lambda = (\lambda_n)_{n=1}^\infty$ of positive reals is said to be *exponentially bounded* [16] if there exists an integer $m \geq 2$ such that for each $\nu \in \mathbb{N}_0$ there is at least one λ_n in the interval $[m^\nu, m^{\nu+1})$. The following result is a useful characterization of exponentially bounded sequences.

Lemma 2.1. ([16, Lemma 1]) *A nondecreasing sequence Λ of positive reals is exponentially bounded if and only if the following condition holds:*

(I) There are real numbers s and t with $0 < s \leq t < 1$ such that for some subsequence $(\lambda_{n(\nu)})_{\nu=0}^\infty$

$$s \leq \frac{\lambda_{n(\nu)}}{\lambda_{n(\nu+1)}} \leq t \text{ for all } \nu = 0, 1, \ldots.$$

If Λ is an exponentially bounded sequence, then we can always determine a subsequence $(\lambda_{n(\nu)})_{\nu=0}^\infty$ which satisfies the condition in (I); such a subsequence will be referred to as an *associated subsequence*.

Throughout, let μ be a nondecreasing sequence of real numbers tending to infinity, Λ be an exponentially bounded sequence and $(\lambda_{n(\nu)})_{\nu=0}^\infty$ be an associated subsequence; we always write $K(s, t) = (s(1 - t))^{-1}$ with $s, t \in (0, 1)$ from condition (I). If $(n(\nu))_{\nu=0}^\infty$ is a strictly increasing sequence of nonnegative integers then $\sum_\nu$ and $\max_\nu$ denote the sum and maximum over all k with $n(\nu) \leq k \leq n(\nu + 1) - 1$. Let Δ be the matrix with the rows $\Delta_n = e^{(n)} - e^{(n-1)}$. If $x, y \in \omega$ then we write $x \cdot y = (x_k y_k)_{k=1}^\infty$. Now we define the sets

$$\tilde{c}_0(\mu) = \left\{ x \in \omega : \lim_{n\to\infty} \left(\frac{1}{\mu_n} \sum_{k=1}^n \left| \Delta_k(\mu \cdot x) \right| \right) = 0 \right\}, \qquad \tilde{c}_\infty(\mu) = \left\{ x \in \omega : \sup_n \left(\frac{1}{\mu_n} \sum_{k=1}^n \left| \Delta_k(\mu \cdot x) \right| \right) < \infty \right\},$$

$$c_0(\Lambda) = \left\{ x \in \omega : \lim_{\nu\to\infty} \left(\frac{1}{\lambda_{n(\nu+1)}} \sum\nolimits_\nu |\Delta_k(\Lambda \cdot x)| \right) = 0 \right\}, \quad c_\infty(\Lambda) = \left\{ x \in \omega : \sup_\nu \left(\frac{1}{\lambda_{n(\nu+1)}} \sum\nolimits_\nu |\Delta_k(\Lambda \cdot x)| \right) < \infty \right\},$$

$\tilde{c}(\mu) = \{x \in \omega : x - \xi \cdot e \in \tilde{c}_0(\mu) \text{ for some } \xi \in \mathbb{C}\}$, and $c(\Lambda) = \{x \in \omega : x - \xi \cdot e \in c_0(\Lambda) \text{ for some } \xi \in \mathbb{C}\}$.

If $x \in c(\Lambda)$ then a number $\xi \in \mathbb{C}$ with $x - \xi \cdot e \in c_0(\Lambda)$ is called Λ*–strong limit*. The Λ–strong limit of a sequence $x \in c(\Lambda)$ is unique [16, Lemma 2]; ξ will always denote the Λ–strong limit of a sequence $x \in c(\Lambda)$.

Our spaces have the following fundamental topological properties similar to those of c_0, c and ℓ_∞.

Theorem 2.2. ([16, Theorem 2]) *Let $\Lambda = (\lambda_n)_{n=1}^\infty$ be an exponentially bounded sequence and $(\lambda_{n(\nu)})_{\nu=0}^\infty$ be an associated subsequence.*
(a) Then $c_0(\Lambda) = \tilde{c}_0(\Lambda)$, $c(\Lambda) = \tilde{c}(\Lambda)$ and $c_\infty(\Lambda) = \tilde{c}_\infty(\Lambda)$.
(b) The block and sectional norms $\|\cdot\|_b$ and $\|\cdot\|_s$ defined by

$$\|x\|_b = \sup_\nu \frac{1}{\lambda_{n(\nu+1)}} \sum\nolimits_\nu |\Delta_k(\Lambda \cdot x)| \quad \text{and} \quad \|x\|_s = \sup_n \frac{1}{\lambda_n} \sum_{k=1}^n |\Delta_k(\Lambda \cdot x)|$$

are equivalent on $c_\infty(\Lambda)$, $c(\Lambda)$ and $c_\infty(\Lambda)$, more precisely

$$\|x\|_b \leq \|x\|_s \leq K(s, t) \cdot \|x\|_b. \tag{1}$$

(c) Each of the spaces $c_0(\Lambda)$, $c(\Lambda)$ and $c_\infty(\Lambda)$ is a BK space, $c_0(\Lambda)$ is a closed subspace of $c(\Lambda)$, $c(\Lambda)$ is a closed subspace of $c_\infty(\Lambda)$, $c_0(\Lambda)$ has AK, and every sequence $x = (x_k)_{k=1}^\infty \in c(\Lambda)$ has a unique representation $x = \xi \cdot e + \sum_{k=1}^\infty (x_k - \xi) e^{(k)}$.
(d) The space $c_\infty(\Lambda)$ has no Schauder basis.

Throughout the paper, we will always assume that the spaces $c_0(\Lambda)$, $c(\Lambda)$ and $c_\infty(\Lambda)$ have the block norm, unless explicitly stated otherwise.

Example 2.3. ([16, Example 1]) (a) If $\lambda_n = 2^{n-1}$ $(n = 1, 2, \cdots)$ then we may choose the sequence $(\lambda_n)_{n=1}^\infty$ itself as an associated subsequence, $\lambda_n/\lambda_{n+1} = 1/2$ $(n = 0, 1, \cdots)$ and we obtain, for instance,

$$c_0(\Lambda) = \left\{x \in \omega : \frac{1}{\lambda_{n+1}}|\lambda_{n-1}x_{n-1} - \lambda_n x_n| \to 0 \ (n \to \infty)\right\}.$$

(b) Let $\alpha > 0$, $\lambda_1 = 1$ and $\lambda_{n+1} = n^\alpha$ for $n = 1, 2, \cdots$. Then we may choose $(\lambda_{2^\nu})_{\nu=0}^\infty$ as an associated subsequence , $\lambda_{2^\nu}/\lambda_{2^{\nu+1}} = 2^{-\alpha}$ for $\nu = 0, 1, \cdots$ and we obtain, for instance,

$$c_0(\Lambda) = \left\{x \in \omega : \frac{1}{(2^{\nu+1})^\alpha}\sum_{k=2^\nu}^{2^{\nu+1}-1} |\lambda_{k-1}x_{k-1} - \lambda_k x_k| \to 0 \ (\nu \to \infty)\right\}.$$

If $\alpha = 1$ then the sets $c_0(\Lambda)$, $c(\Lambda)$ and $c_\infty(\Lambda)$ reduce to the sets $[c_0]_1$, $[c]_1$ and $[c_\infty]_1$ introduced and studied by Hyslop, Kuttner and Thorpe [10, 15].
(c) Let $\alpha > 0$, $\lambda_1 = 1$ and $\lambda_{n+1} = (\log_2 n)^\alpha$ for $n \geq 1$. Then we may choose $(\lambda_{2^{(2^\nu)}})_{\nu=0}^\infty$ as an associated subsequence.

3. The dual spaces

Here we give the dual spaces of the sets $c_0(\Lambda)$, $c(\Lambda)$ and $c_\infty(\Lambda)$.

We need the following notations. Let X and Y be subsets of ω and $z \in \omega$. Then we write $z^{-1} * X = \{x \in \omega : x \cdot z = (x_k z_k)_{k=1}^\infty \in X\}$. The set $M(X, Y) = \bigcap_{x \in X} x^{-1} * Y$ is called the *multiplier space of X and Y*. The special cases $X^\alpha = M(X, \ell_1)$, $X^\beta = M(X, cs)$ and $X^\gamma = M(X, bs)$ are called the *α–, β– and γ–duals of X*. If $X \supset \phi$ then the set $X^f = \{(f(e^{(n)}))_{n=1}^\infty : f \in X^*\}$ is called the *functional* or *f–dual of X*.

If X is a normed sequence space and $a \in \omega$ then we write $\|a\| = \|a\|_X^* = \sup\{|\sum_{k=1}^\infty a_k x_k| : x \in S_X\}$ provided the last term exists and is finite which is the case whenever $X \supset \phi$ is a BK space and $a \in X^\beta$ ([20, Theorem 7.2.9]).

The following result gives the first and second α–duals of the sets $c_0(\Lambda)$, $c(\Lambda)$ and $c_\infty(\Lambda)$.

Theorem 3.1. ([17, Theorem 1]) *We have* $(c_0(\Lambda))^\alpha = (c(\Lambda))^\alpha = (c_\infty(\Lambda))^\alpha = \ell_1$ *and* $(c_0(\Lambda))^{\alpha\alpha} = (c(\Lambda))^{\alpha\alpha} = (c_\infty(\Lambda))^{\alpha\alpha} = \ell_\infty$.

We put

$$C(\Lambda) = \left\{a \in \omega : \sum_{\nu=0}^\infty \lambda_{n(\nu+1)} \max{}_\nu \left|\sum_{j=k}^\infty \frac{a_j}{\lambda_j}\right| < \infty\right\} \text{ and write } \|a\|_{C(\Lambda)} = \sum_{\nu=0}^\infty \lambda_{n(\nu+1)} \max{}_\nu \left|\sum_{j=k}^\infty \frac{a_j}{\lambda_j}\right|.$$

Finally, we give the first and second β–, γ– and f–duals of the sets $c_0(\Lambda)$, $c(\Lambda)$, $c_\infty(\Lambda)$, and the continuous duals of $c_0(\Lambda)$ and $c(\Lambda)$.

Theorem 3.2. ([17, Theorem 2]) *(a) We have* $X^\dagger = C(\Lambda)$ *for* $X = c_0(\Lambda), c(\Lambda), c_\infty(\Lambda)$ *and* $\dagger = \beta, \gamma, f$.
(b) The continuous dual $(c_0(\Lambda))^*$ *is norm isomorphic to* $C(\Lambda)$*; also* $\|\cdot\|_{c_\infty(\Lambda)}^* = \|\cdot\|_{C(\Lambda)}$ *on* $(c(\Lambda))^\beta$ *and* $(c_\infty(\Lambda))^\beta$.
(c) We have $f \in (c(\Lambda))^\beta$ *if and only if*

$$f(x) = \xi \cdot \chi_f + \sum_{n=1}^\infty a_n x_n \text{ for all } x \in c(\Lambda) \text{ where } a = (f(e^{(n)}))_{n=1}^\infty \in C(\Lambda), \text{ and } \chi_f = f(e) - \sum_{n=1}^\infty a_n; \tag{2}$$

furthermore, we have

$$|\chi_f| + \|a\|_{C(\Lambda)} \leq \|f\| \leq K(s,t) \cdot \left(|\chi_f| + \|a\|_{C(\Lambda)}\right) \text{ for all } f \in (c(\Lambda))^*. \tag{3}$$

Theorem 3.3. ([17, Theorem 3]) *We have*
(a) $(C(\Lambda), \|\cdot\|_{C(\Lambda)})$ *is a BK space with AK;*
(b) $c_\infty(\Lambda)$ *is* β*–perfect, that is,* $(c_\infty(\Lambda))^{\beta\beta} = c_\infty(\Lambda)$*; also* $\|\cdot\|^*_{C(\Lambda)} = \|\cdot\|_b$ *on* $(C(\Lambda))^\beta$*;*
(c) $(C(\Lambda))^\beta = (C(\Lambda))^\gamma = (C(\Lambda))^f$*;*
(d) $(C(\lambda))^*$ *is norm isomorphic to* $c_\infty(\Lambda)$*.*

4. Matrix transformations

Here we give the complete list of characterizations of the classes (X, Y) for $X = c_0(\Lambda), c(\Lambda), c_\infty(\Lambda)$ and $Y = c_0, c, \ell_\infty$

Theorem 4.1. ([3, Theorem 2.4])
The necessary and sufficient conditions for $A \in (X, Y)$ *when* $X \in \{c_0(\Lambda), c(\Lambda), c_\infty(\Lambda)\}$ *and* $Y \in \{\ell_\infty, c, c_0\}$ *can be read from the following table*

From / *To*	$c_\infty(\Lambda)$	$c_0(\Lambda)$	$c(\Lambda)$
ℓ_∞	**1.**	**1.**	**1.**
c_0	**2.**	**3.**	**4.**
c	**5.**	**6.**	**7.**

where
1. $(1.1)^*$ $\sup_n \|A_n\|_{C(\Lambda)} < \infty$*;*
2. $(2.1)^*$ $\lim_{n\to\infty} \|A_n\|_{C(\Lambda)} = 0$*;*
3. $(1.1)^*$ *and* $(3.1)^*$ $\lim_{n\to\infty} a_{nk} = 0$ *for all k;*
4. $(1.1)^*$, $(3.1)^*$ *and* $(4.1)^*$ $\lim_{n\to\infty} \sum_{k=1}^\infty a_{nk} = 0$*;*
5. $(5.1)^*$ $\alpha_k = \lim_{n\to\infty} a_{nk}$ *exists for all k,*
$(5.2)^*$ $(\alpha_k)_{k=1}^\infty, A_n \in C(\Lambda)$ *for all n,*
$(5.3)^*$ $\lim_{n\to\infty}; \|A_n - (\alpha_k)_{k=1}^\infty\|_{C(\Lambda)} = 0$*;*
6. $(1.1)^*$ *and* $(5.1)^*$*;*
7. $(1.1)^*$, $(5.1)^*$ *and* $(7.1)^*$ $\alpha = \lim_{n\to\infty} \sum_{k=1}^\infty a_{nk}$ *exists.*

Remark 4.2. The conditions for $A \in (c_\infty(\Lambda), c_0)$ and $A \in (c_\infty(\Lambda), c)$ can be replaced by
2′. $(3.1)^*$ and $(2.1')^*$ $\|A_n\|_{C(\Lambda)}$ converges uniformly in n,
5′. $(2.1')^*$ and $(5.1)^*$.

5. Compact operators

Here we characterize some classes of compact operators on the spaces $c_0(\Lambda)$, $c(\Lambda)$ and $c_\infty(\Lambda)$. This is most effectively achieved by applying the *Hausdorff measure of noncompactness*. The results concerning general bounded linear operators and their compactness are new.

5.1. Compact operators and measures of noncompactness

We recall a few necessary notions and results, and refer the reader to [1, 4, 18] for further studies.

If X and Y are infinite–dimensional complex Banach spaces, then a linear operator $L : X \to Y$ is said to be *compact* if the domain of L is all of X and, for every bounded sequence (x_n) in X, the sequence $(L(x_n))$ has a convergent subsequence. We write $C(X, Y)$ for the class of all compact operators from X into Y. We recall that a set in a topological space is said to be *precompact* or *relatively compact* if its closure is compact.

The first measure of noncompactness, the function α, was defined and studied by Kuratowski [14] in 1930. In 1955, Darbo [6] was the first who continued to use the function α. He proved that if T is a continuous self–mapping of a nonempty, bounded, closed and convex subset C of a Banach space X such that there exists a constant $K \in (0, 1)$ such that $\alpha(T(Q)) \le K\alpha(Q)$ for all subsets Q of C, then T has at least one fixed point in the set C. Darbo's fixed point theorem is a very important generalization of Schauder's fixed point theorem, and includes the existence part of Banach's fixed point theorem.

Other measures of noncompactness were introduced and studied by Gohberg, Goldenštein and Markus in 1957 [9], the *ball* or *Hausdorff measure of noncompactness*, and by Istrăţesku in 1972 [11, 12].

Now we recall the definition of a measure of noncompactness on bounded sets in a complete metric space.

Definition 5.1. Let (X, d) be a complete metric space and $\mathcal{M}_X$ be the class of all bounded subsets of X. A map $\phi : \mathcal{M}_X \to [0, \infty)$ is called a *measure of noncompactness on X* if it satisfies the following properties for all $Q, Q_1, Q_2 \in \mathcal{M}_X$

(i) $\phi(Q) = 0$ if and only if Q is precompact (*regularity*);

(ii) $\phi(Q) = \phi(\overline{Q})$ (*invariance under closure*);

(iii) $\phi(Q_1 \cup Q_2) = \max\{\phi(Q_1), \phi(Q_2)\}$ (*semi–additivity*).

It is easy to see that any measure of noncompactness satisfies the following properties for all $Q, Q_1, Q_2 \in \mathcal{M}_X$

(iv) $Q_1 \subset Q_2$ implies $\phi(Q_1) \leq \phi(Q_2)$ (*monotonicity*);

(v) $\phi(Q_1 \cap Q_2) \leq \min\{\phi(Q_1), \phi(Q_2)\}$;

(vi) if Q is a finite set then $\phi(Q) = 0$ (*non–singularity*).

(vii) If (Q_n) is a decreasing sequence of nonempty, closed and bounded subsets of X, and $\lim_{n\to\infty} \phi(Q_n) = 0$ then $\bigcap_n Q_n$ is nonempty and compact (*generalized Cantor intersection property*).

We also need the definition of a measure of noncompactness of operators between Banach spaces.

Definition 5.2. Let X and Y be Banach spaces, and ϕ_1 and ϕ_2 be measures of noncompactness on X and Y. The operator $L : X \to Y$ is called (ϕ_1, ϕ_2)*–bounded* if

(i) $L(Q) \in \mathcal{M}_Y$ for every $Q \in \mathcal{M}_X$,

(ii) there exists a constant $C > 0$ such that $\phi_2(L(Q)) \leq C\phi_1(Q)$ for every $Q \in \mathcal{M}_X$.

If an operator L is (ϕ_1, ϕ_2)–bounded, then the number $\|L\|_{(\phi_1,\phi_2)} = \inf\left\{C \geq 0 : \phi_2(L(Q)) \leq C\phi_1(Q) \text{ for all } Q \in \mathcal{M}_X\right\}$ is called the (ϕ_1, ϕ_2)*– measure of noncompactness of L*. In particular, if $\phi_1 = \phi_2 = \phi$, then we write $\|L\|_\phi$ instead of $\|L\|_{(\phi,\phi)}$.

We recall the definitions of the Hausdorff measure of noncompactness of bounded sets in a metric space, and of linear operators between normed spaces. We write $B(x, r) = \{y \in X : d(y, x) < r\}$ for the open ball of radius $r > 0$ and centre in x in a metric space (X, d).

Definition 5.3. (a) Let (X, d) be a metric space and $Q \in \mathcal{M}_X$. The *Hausdorff measure of noncompactness* $\chi(Q)$ of the set Q is defined by

$$\chi(Q) = \inf\left\{\epsilon \geq 0 : Q \subset \bigcup_{k=1}^{n} B(x_k, r_k);\ x_k \in X,\ r_k < \epsilon\ (k = 1, \dots, n),\ n \in \mathbb{N}\right\}.$$

(b) Let X and Y be Banach spaces, χ_1 and χ_2 be Hausdorff measures of noncompactness on X and Y, and $L : X \to Y$ be an operator. We write $\|L\|_{(\chi_1,\chi_2)}$ for the (χ_1, χ_2)–measure of noncompactness of L. In particular, if $\chi_1 = \chi_2 = \chi$, then we write $\|L\|_\chi$ instead of $\|L\|_{(\chi,\chi)}$.

It is easy to see that χ is a measure of noncompactness, that is, regular, invariant under closure, and semi–additive. Consequently, χ also satisfies conditions (iv)–(vii) ([18, Lemma 2.11]). Also if X is a Banach space then we have for all $Q, Q_1, Q_2 \in \mathcal{M}_X$ [18, Theorem 2.12]

$\chi(Q_1 + Q_2) \leq \chi(Q_1) + \chi(Q_2)$,

$\chi(x + Q) = \chi(Q)$ for all $x \in X$,

$\chi(\lambda Q) = |\lambda|\chi(Q)$ for each scalar λ,

$\chi(\mathrm{conv}(Q)) = \chi(Q)$ where $\mathrm{conv}(Q)$ denotes the convex hull of Q.

The following result due to Gohberg, Goldenštein and Markus gives an explicit estimate for the Hausdorff measure of noncompactness of any bounded set in a Banach space with a Schauder basis.

Theorem 5.4. (Gohberg, Goldenštein, Markus) ([9] or [18, Theorem 2.23]) *Let X be a Banach space with a Schauder basis $(b_n)_{n=1}^{\infty}$, $\mathcal{P}_n : X \to X$ be the projector onto the linear span* $\mathrm{span}(\{b_1, b_2, \cdot, b_n\})$ *of $b_1, b_2, \cdots, b_n$, $\mathcal{R}_n = I - \mathcal{P}_n$, where I is the identity on X, and $Q \in \mathcal{M}_X$. Then we have, writing $\mu_n(Q) = \sup_{x\in Q} \|\mathcal{R}_n(x)\|$ for $n = 1, 2, \dots,$*

$$\frac{1}{a} \cdot \limsup_{n\to\infty} \mu_n(Q) \le \chi(Q) \le \limsup_{n\to\infty} \mu_n(Q), \text{ where } a = \limsup_{n\to\infty} \|\mathcal{R}_n\|. \tag{4}$$

We obtain an explicit formula for the Hausdorff measure of noncompactness of bounded subsets of special BK spaces, and of c. We say that a norm $\|\cdot\|$ on a sequence space X is *monotonous*, if $x, x' \in X$ with $|x_k| \le |x'_k|$ for all k implies $\|x\| \le \|x'\|$, and call such a space X monotonous.

Theorem 5.5. ([8, Lemma 3.4]) *(a) Let X be a monotonous BK space with AK and $\mathcal{P}_n : X \to X$ be the projectors onto* $\mathrm{span}(\{e^{(1)}, e^{(2)}, \dots, e^{(n)}\})$ *for $n \in \mathbb{N}$. Then we have*

$$\chi(Q) = \lim_{n\to\infty} \mu_n(Q) \text{ for all } Q \in \mathcal{M}_X. \tag{5}$$

(b) Let $\mathcal{P}_n : c \to c$ be the projectors onto the linear span of $\{e, e^{(1)}, \cdots, e^{(n)}\}$. Then the limit in (5) exists for all $Q \in \mathcal{M}_c$, and

$$a = \lim_{n\to\infty} \|\mathcal{R}_n\| = 2. \tag{6}$$

Next we state some important properties of $\|\cdot\|_{\chi}$.

Theorem 5.6. ([18, Theorem 2.25, Corollary 2.26]) *Let X and Y be Banach spaces and $L \in \mathcal{B}(X, Y)$.*
(a) Then $\|L\|_{\chi} = \chi(L(S_X)) = \chi(L(B_X))$.
(b) Then $\|\cdot\|_{\chi}$ is a seminorm on $\mathcal{B}(X, Y)$, and

$$\|L\|_{\chi} = 0 \text{ if and only if } L \in \mathcal{C}(X, Y). \tag{7}$$

Now we give an estimate for $\|L\|_{\chi}$. If X is a BK space with AK and $L \in \mathcal{B}(X, Y)$, let $A \in (X, Y)$ denote the matrix with $Ax = L(x)$ for all $x \in X$ (Theorem 1.1 (b)).

Theorem 5.7. *Let X be a BK space with AK.*
(a) Let $L \in \mathcal{B}(X, c)$ and $\tilde{A}$ be the matrix with the rows $\tilde{A}_n = A_n - (\alpha_k)_{k=1}^{\infty}$ $(n = 1, 2, \dots)$, where

$$\alpha_k = \lim_{n\to\infty} a_{nk} \text{ for every } k \in \mathbb{N} \text{ and } (\alpha_k)_{k=1}^{\infty} \in X^{\beta}. \tag{8}$$

Then we have

$$\frac{1}{2} \cdot \lim_{r\to\infty} \left(\sup_{n\ge r} \|\tilde{A}_n\|_X^* \right) \le \|L\|_{\chi} \le \lim_{r\to\infty} \left(\sup_{n\ge r} \|\tilde{A}_n\|_X^* \right) \text{ ([7, Theorem 3.4]).}$$

(b) If $L \in \mathcal{B}(X, c_0)$, then we have

$$\|L\|_{\chi} = \lim_{r\to\infty} \left(\sup_{n\ge r} \|A_n\|_X^* \right) \text{ ([2, Corollary 3.4]).}$$

Remark 5.8. It was shown in the proof of Theorem 5.7 that if $L \in \mathcal{B}(X, c)$, then

$$\lim_{n\to\infty} A_n x = \sum_{k=1}^{\infty} \alpha_k x_k \text{ for all } x \in X. \tag{9}$$

Now we establish the representations of the general operators $L \in \mathcal{B}(c(\Lambda), Y)$ when $Y = \ell_\infty, c, c_0$, estimates for their norms, and a formula for the limit of $(L(x))_n$ $(n \to \infty)$ for $x \in c(\Lambda)$ when $L \in \mathcal{B}(c(\Lambda), c)$. These results are new.

Theorem 5.9. *We have*
(a) $L \in \mathcal{B}(c(\Lambda), \ell_\infty)$ if and only if there exists a matrix $A \in (c_0(\Lambda), \ell_\infty)$ and a sequence $b \in \ell_\infty$ such that

$$L(x) = b \cdot \xi + Ax \text{ for all } x \in c(\Lambda); \tag{10}$$

(b) $L \in \mathcal{B}(c(\Lambda), c)$ if and only if there exists a matrix $A \in (c_0(\Lambda), c)$ and a sequence $b \in \ell_\infty$ for which the limit

$$\beta = \lim_{n\to\infty} \left(b_n + \sum_{k=1}^{\infty} a_{nk} \right) \text{ exists} \tag{11}$$

such that (10) holds;
(c) $L \in \mathcal{B}(c(\Lambda), c_0)$ if and only if there exists a matrix $A \in (c_0(\Lambda), c_0)$ and a sequence $b \in \ell_\infty$ with $\beta = 0$ in (11) such that (10) holds.
(d) If $L \in \mathcal{B}(c(\Lambda), Y)$ for $Y = \ell_\infty, c, c_0$ then we have

$$\sup_n \left(|b_n| + \|A_n\|_{C(\Lambda)} \right) \le \|L\| \le K(s,t) \cdot \sup_n \left(|b_n| + \|A_n\|_{C(\Lambda)} \right). \tag{12}$$

(e) Let $L \in \mathcal{B}(c(\Lambda), c)$, $x \in c(\Lambda)$ and ξ be the Λ–strong limit of the sequence x. Then we have

$$\eta = \lim_{n\to\infty} (L(x)) = \beta \cdot \xi + \sum_{k=1}^{\infty} \alpha_k (x_k - \xi) = \left(\beta - \sum_{k=1}^{\infty} \alpha_k \right) \xi + \sum_{k=1}^{\infty} \alpha_k x_k. \tag{13}$$

Proof. (a)–(c) First we assume $L \in \mathcal{B}(c(\Lambda), Y)$ for $Y = \ell_\infty, c, c_0$. We write $L_n = P_n \circ L$ for $n = 1, 2, \dots$, where P_n is the n^{th} coordinate. Since $c(\Lambda)$ is a BK space, we have $L_n \in (c(\Lambda))^*$ for each n, and it follows from (2) that

$$L_n(x) = b_n \xi + A_n x \text{ for each } x \in c(\Lambda), \text{ where } \xi \text{ is the } \Lambda\text{–strong limit of } x, \tag{14}$$

$$A_n = (a_{nk})_{k=1}^{\infty} = \left(L_n \left(e^{(k)} \right) \right)_{k=1}^{\infty} \in C(\Lambda) \text{ and } b_n = L_n(e) - \sum_{k=1}^{\infty} L_n \left(e^{(k)} \right) \text{ for all } n. \tag{15}$$

This yields (10) in Parts (a)–(c).
Also since $L(x^{(0)}) = Ax^{(0)}$ for all $x^{(0)} \in c_0(\Lambda)$, we have $A \in (c_0(\Lambda), Y)$.
If $Y = \ell_\infty$, then $(c_0(\Lambda), \ell_\infty) = (c(\Lambda), \ell_\infty)$ by Theorem 4.1 **1.**, and it follows from $e \in c(\Lambda)$ that $b = L(e) - Ae \in \ell_\infty$.
If $Y = c$, then $L \in \mathcal{B}(c(\Lambda), c) \subset \mathcal{B}(c(\Lambda), \ell_\infty)$, and $b \in \ell_\infty$ and $A \in (c_0(\Lambda), \ell_\infty)$, as before. Since $e^{(k)} \in c(\Lambda)$ for each k, we obtain $L(e^{(k)}) = (a_{nk})_{n=1}^{\infty} \in c$ for each k, that is, the limits α_k in (8) exist for all k. This and $A \in (c_0(\Lambda), \ell_\infty)$ imply $A \in (c_0(\Lambda), c)$ by Theorem 4.1 **6.**. Also $L(e) = b + Ae \in c$ implies (11).
If $Y = c_0$, then we analogously obtain $b \in \ell_\infty$, $\lim_{n\to\infty} a_{nk} = 0$, for each k, $A \in (c_0(\Lambda), c_0)$ and $\beta = 0$.
Conversely, we assume that $A \in (c_0(\Lambda), Y)$ for $Y = c_0, c, \ell_\infty$ and $b \in \ell_\infty$ such that (10) holds. Since $A \in (c_0(\Lambda), Y)$ implies $A_n \in (c_0(\Lambda))^\beta$ for each n, and $(c_0(\Lambda))^\beta = (c(\Lambda))^\beta$ by Theorem 3.2 (a), it follows by Theorem 3.2 (c) that $L_n = P_n \circ L \in (c(\Lambda))^*$ for each n. Since $c(\Lambda)$ and Y are BK spaces, $L : c(\Lambda) \to Y$ is clearly linear, and $L_n \in (c(\Lambda))^*$ for each n, it follows by [20, Corollary 4.2.3] that L is continuous, so $L \in \mathcal{B}(c(\Lambda), Y)$.

(d) We assume $L \in \mathcal{B}(c(\Lambda), Y)$ and write $L_n = P_n \circ L$ for all $n = 1, 2, \dots$. Then we have for all $x \in S_{c(\Lambda)}$, by the second inequality in (3), $\|L(x)\|_\infty = \sup_n |L_n(x)| \le K(s,t) \cdot \sup_n (|b_n| + \|A_n\|_{C(\Lambda)})$. This yields the second inequality in (12). Also $L \in \mathcal{B}(c(\Lambda), c)$ implies $|L_n(x)| \leq \|L(x)\|_\infty \leq \|L\|$ for all $n \in \mathbb{N}$ and all $x \in S_{c(\Lambda)}$, and since $L_n \in (c(\Lambda))^*$ for all n, we have $\|L_n\| \leq \|L\|$ for all n, and so, by the first inequality in (3), $(|b_n| + \|A_n\|_{C(\Lambda)}) \leq \|L\|$ for all n, which yields the first inequality in (12).

(e) Let $L \in \mathcal{B}(c(\Lambda), c)$. Then it follows from Part (b) that $A \in (c_0(\Lambda), c)$ and so the limits α_k in (8) exist for all k, and $(\alpha_k)_{k=1}^\infty \in (c_0(\Lambda))^\beta$, but $(c_0(\Lambda))^\beta = (c(\Lambda))^\beta = C(\Lambda)$ by Theorem 3.2 (a). Hence $e \in c(\Lambda)$ implies $(\alpha_k)_{k=1}^\infty \in cs$, and so we have $\beta\xi + \sum_{k=1}^\infty \alpha_k(x_k - \xi) = (\beta - \sum_{k=1}^\infty \alpha_k)\xi + \sum_{k=1}^\infty \alpha_k x_k$ and (9) implies $\lim_{n\to\infty} L_n(x^{(0)}) = \lim_{n\to\infty} A_n x^{(0)} = \sum_{k=1}^\infty \alpha_k x_k^{(0)}$ for all $x^{(0)} \in c_0(\Lambda)$. Now let $x \in c(\Lambda)$. Then we have $x^{(0)} = x - \xi \cdot e \in c_0(\Lambda)$, and

$$\eta = \lim_{n\to\infty} L_n(x) = \xi \cdot \lim_{n\to\infty} L_n(e) + \lim_{n\to\infty} L_n(x^{(0)}) = \beta \cdot \xi + \sum_{k=1}^\infty \alpha_k(x_k - \xi) = \left(\beta - \sum_{k=1}^\infty \alpha_k\right)\xi + \sum_{k=1}^\infty \alpha_k x_k.$$

□

Now we give an estimate for $\|L\|_\chi$ when $L \in \mathcal{B}(c(\Lambda), c)$. We use the notations of Theorem 5.9.

Theorem 5.10. *If $L \in \mathcal{B}(c(\Lambda), c)$, then we have*

$$\frac{1}{2} \cdot \lim_{n\to\infty}\left(\left|b_n - \beta + \sum_{k=1}^\infty \alpha_k\right| + \left\|\tilde{A}_n\right\|_{C(\Lambda)}\right) \leq \|L\|_\chi \leq K(s,t) \cdot \lim_{n\to\infty}\left(\left|b_n - \beta + \sum_{k=1}^\infty \alpha_k\right| + \left\|\tilde{A}_n\right\|_{C(\Lambda)}\right), \tag{16}$$

where $\tilde{A} = (\tilde{a}_{nk})_{n,k=1}^\infty$ is the matrix with the rows $\tilde{A}_n = A_n - (\alpha_k)_{k=1}^\infty$ for $n = 1, 2, \dots$ and α_k $(k = 1, 2, \dots)$ from (8).

Proof. Let $L \in \mathcal{B}(c(\Lambda), c)$. Then, by Theorem 5.9 (b), L can be represented by (10) where the sequence $b \in \ell_\infty$ and the matrix $A \in (c_0(\Lambda), c)$ are given by (15), and the limits β in (11) and α_k $(k = 1, 2, \dots)$ in (8) exist; we also saw in the proof of Theorem 5.9 (e) that $(\alpha_k)_{k=1}^\infty \in cs$. So $\gamma_n = b_n - \beta + \sum_{k=1}^\infty \alpha_k$ exists for each $n = 1, 2, \dots$. Since every sequence $y = (y_n)_{n=1}^\infty \in c$ has a unique representation $y = \eta \cdot e + \sum_{n=1}^\infty (y_n - \eta)e^{(n)}$ with $\eta = \lim_{n\to\infty} y_n$, we obtain $\mathcal{R}_m(y) = \sum_{n=m+1}^\infty (y_n - \eta)e^{(n)}$ for $m = 1, 2, \dots$, and so $\|\mathcal{R}_m(y)\|_\infty = \sup_{n\geq m+1} |y_n - \eta|$. Writing $y = L(x)$, we obtain from (10) and (13)

$$y_n - \eta = b_n \cdot \xi + A_n x - \left(\left(\beta - \sum_{k=1}^\infty \alpha_k\right)\xi + \sum_{k=1}^\infty \alpha_k x_k\right) = \gamma_n \cdot \xi + \tilde{A}_n x \text{ for } n = 1, 2, \dots,$$

and so by (12)

$$\sup_{n\geq m+1}\left|\gamma_n + \left\|\tilde{A}_n\right\|_{C(\Lambda)}\right| \leq \sup_{x \in S_{c(\Lambda)}} \|\mathcal{R}_m(L(x))\|_\infty \leq K(s,t) \cdot \sup_{n\geq m+1}\left|\gamma_n + \left\|\tilde{A}_n\right\|_{C(\Lambda)}\right|. \tag{17}$$

Finally, since $\lim_{m\to\infty} \|\mathcal{R}_m\| = 2$ by (6), the inequalities in (16) now follow from (4) in Theorem 5.4, and from Theorems 5.5 (b) and 5.6 (a). □

We obtain the following corollaries from Theorem 5.10.

Corollary 5.11. *Let $A \in (c(\Lambda), c)$. Then we have*

$$\frac{1}{2} \lim_{n\to\infty}\left(\left|\sum_{k=1}^\infty \alpha_k - \alpha\right| + \|\tilde{A}_n\|_{C(\Lambda)}\right) \leq \|A\|_\chi \leq \lim_{n\to\infty}\left(\left|\sum_{k=1}^\infty \alpha_k - \alpha\right| + \|\tilde{A}_n\|_{C(\Lambda)}\right),$$

where $\alpha_k = \lim_{n\to\infty} a_{nk}$ for each $k \in \mathbb{N}$, $\alpha = \lim_{n\to\infty} \sum_{k=1}^\infty a_{nk}$, and $\tilde{A}_n = A_n - (\alpha_k)_{k=1}^\infty$ for all $n \in \mathbb{N}$.

Proof. This is an immediate consequence of Theorem 5.10 with $b_n = 0$ for $n = 1, 2, \dots$ and the fact that for $A \in (c(\Lambda), c) \subset (c_0(\Lambda), c)$, we have $\|(L_A)_n\| = \|A_n\|_{C(\Lambda)}$ for all n by Theorem 3.2 (b), so we get $K(s,t) = 1$ and equality in (17). □

Corollary 5.12. *Let $L \in \mathcal{B}(c(\Lambda), c_0)$. Then we have*

$$\lim_{n\to\infty}\left(|b_n| + \|A_n\|_{C(\Lambda)}\right) \le \|L\|_\chi \le K(s,t)\cdot \lim_{n\to\infty}\left(|b_n| + \|A_n\|_{C(\Lambda)}\right).$$

Proof. This is an immediate consequence of Theorem 5.10 with $\beta = 0$ and $\alpha_k = 0$ for all k □

Corollary 5.13. *Let $A \in (c(\Lambda), c_0)$. Then we have*

$$\lim_{n\to\infty}\|A_n\|_{C(\Lambda)} \le \|L_A\|_\chi \le K(s,t)\cdot \lim_{n\to\infty}\|A_n\|_{C(\Lambda)}.$$

Proof. This is an immediate consequence of Corollary 5.11 with $\alpha_k = 0$ for all k and $\alpha = 0$ and of the fact that $\lim_{m\to\infty}\|\mathcal{R}_m\| = 1$. □

We also obtain from Theorem 5.7 an estimate for $\|L\|_\chi$ when $L \in \mathcal{B}(c_0(\Lambda), c)$, and an identity when $L \in \mathcal{B}(c_0(\Lambda), c_0)$.

Corollary 5.14. *(a) Let $L \in \mathcal{B}(c_0(\Lambda), c)$. Then we have*

$$\frac{1}{2}\cdot \lim_{r\to\infty}\left(\sup_{n\ge r}\|\tilde{A}_n\|_{C(\Lambda)}\right) \le \|L\|_\chi \le \lim_{r\to\infty}\left(\sup_{n\ge r}\|\tilde{A}_n\|_{C(\Lambda)}\right).$$

(b) Let $L \in \mathcal{B}(c_0(\Lambda), c_0)$. Then we have

$$\|L\|_\chi = \lim_{r\to\infty}\left(\sup_{n\ge r}\|A_n\|_{C(\Lambda)}\right).$$

Proof. Corollary 5.14 is an immediate consequence of Theorem 5.7 and the fact that $(c_0(\Lambda))^*$ is norm isomorphic to $C(\Lambda)$ by Theorem 3.2 (b). □

Now we apply our results on the Hausdorff measure of noncompactness of operators and (7) to characterize the classes $\mathcal{C}(X, Y)$ for $X = c_0(\Lambda), c(\Lambda)$ and $Y = c_0, c$.

Corollary 5.15. *Let $L \in \mathcal{B}(X, Y)$ where $X = c_0(\Lambda), c(\Lambda)$ and $Y = c_0, c$. Then the necessary and sufficient conditions for $L \in \mathcal{C}(X, Y)$ can be read from the table*

From / To	$c_0(\Lambda)$	$c(\Lambda)$
c_0	**1.**	**2.**
c	**3.**	**4.**

where

1. $\lim_{r\to\infty}(\sup_{n\ge r}\|A_n\|_{C(\Lambda)}) = 0$; **2.** $\lim_{n\to\infty}(|b_n + \|A_n\|_{C(\Lambda)}) = 0$;

3. $\lim_{r\to\infty}(\sup_{n\ge r}\|\tilde{A}_n\|_{C(\Lambda)}) = 0$; **4.** $\lim_{n\to\infty}(|b_n - \beta + \sum_{k=1}^\infty \alpha_k| + \|\tilde{A}_n\|_{C(\Lambda)}) = 0$.

Proof. Using (7), we obtain the conditions in **1.–4.** from Corollaries 5.14 (b), 5.12 and 5.14 (a), and Theorem 5.10, respectively. □

Putting $b_n = 0$ for all n in Corollary 5.15 **2.** and **4.**, we obtain the following characterizations of compact matrix operators in the classes $(c(\Lambda), c_0)$ and $(c(\Lambda), c)$.

Corollary 5.16. *(a) Let $A \in (c(\Lambda), c_0)$. Then we have $L_A \in \mathcal{C}(c(\Lambda), c_0)$ if and only if $\lim_{n\to\infty}\|A_n\|_{C(\Lambda)} = 0$.*
(b) Let $A \in (c(\Lambda), c)$. Then we have $L_A \in \mathcal{C}(c(\Lambda), c)$ if and only if $\lim_{n\to\infty}(|\sum_{k=1}^\infty \alpha_k - \alpha| + \|\tilde{A}_n\|_{C(\Lambda)}) = 0$.

Using the fact that $\|\cdot\|^*_{c_\infty(\Lambda)} = \|\cdot\|_{C(\Lambda)}$ on $(c_\infty(\Lambda))^\beta$ by Theorem 3.2 (b), we obtain similarly as above

Corollary 5.17. *(a) Let* $A \in (c_\infty(\Lambda), c_0)$*. Then we have* $L_A \in \mathcal{C}(c_\infty(\Lambda), c_0)$ *if and only if* **1.** *in Corollary 5.15 holds.*
(b) Let $A \in (c_\infty(\Lambda), c)$*. Then we have* $L_A \in \mathcal{C}(c_\infty(\Lambda), c_0)$ *if and only if* **3.** *in Corollary 5.15 holds.*

We close with an application to Λ–strong regularity. We call an operator $L \in \mathcal{B}(c(\Lambda), c)$ Λ*–strongly regular*, if $\lim_{n\to\infty}(L(x))_n = \xi$ for all $x \in c(\Lambda)$, where ξ is the Λ–strong limit of x. A matrix $A \in (c(\Lambda), c)$ is said to be Λ*–strongly regular*, if the operator L_A is Λ–strongly regular.

We apply our results to the characterization of compact Λ–strongly regular operators.

Corollary 5.18. *Let* $L \in \mathcal{B}(c(\Lambda), c)$ *be* Λ*–strongly regular. Then we have* $L \in \mathcal{C}(c(\Lambda), c)$ *if and only if*

$$\lim_{n\to\infty}\left(|b_n - 1| + \|A_n\|_{C(\Lambda)}\right) = 0 \tag{18}$$

Proof. By Theorem 5.9 (e), $L \in \mathcal{B}(c(\Lambda), c)$ is strongly regular if and only if $\beta = \lim_{n\to\infty}(b_n + \sum_{k=1}^{\infty} a_{nk}) = 1$ and $\alpha_k = 0$ for $k = 1, 2, \ldots$. So it follows from Corollary 5.15 **4.** that L is compact if and only if the condition in (18) holds. □

It turns out that a Λ–strongly regular matrix cannot be compact; this is analogous to the classical result by Cohen and Dunford [5] which states that a regular matrix cannot be compact.

Remark 5.19. If A is a Λ–strongly regular matrix then L_A cannot be compact, since we have with $b_n = 0$ for all $n \in \mathbb{N}_0$ and $1 + \|A_n\|_{C(\Lambda)} \geq 1 \neq 0$ for all n, and so *(18)* in Corollary 5.18 cannot hold.

References

[1] J.M. Ayerbe Toledano, T. Dominguez Benavides, G. Lopez Acedo, Measures of Noncompactness in Metric Fixed Point Theory, Operator Theory, Advances and Applications 99, Birkhäuser, Basel, Boston, Berlin, 1997.
[2] F. Başar, E. Malkowsky, The characterisation of compact operators on spaces of strongly summable and bounded sequences, Appl. Math. Comput. 217 (2011) 5199–5207.
[3] F. Başar, E. Malkowsky, B. Altay, Matrix transformations on the matrix domains of triangles in the spaces of strongly C_1–summable and bounded sequences, Publ. Math. Debrecen 73 (2008) 193–213.
[4] J. Banás, K. Goebl, Measures of Noncompactness in Banach spaces, Lecture Notes in Pure and Appl. Math. 60, Marcel Dekker, New York and Basel, 1980.
[5] L.W. Cohen, N. Dunford, Transformations on sequence spaces, Duke Math. J. 3 (1937) 689–701.
[6] G. Darbo, Punti uniti in transformazioni a condominii non compatto, Rend. Sem. Mat. Univ. Padova 24 (1955) 84–92.
[7] I. Djolović, E. Malkowsky, A note on compact operators on matrix domains, J. Math. Anal. Appl. 340 (2008) 291–307.
[8] I. Djolović, E. Malkowsky, The Hausdorff measure of noncompactness of operators on the matrix domains of triangles in the spaces of strongly C_1 summable and bounded sequences, Appl. Math. Comput. 216 (2010) 1122–1130.
[9] I.T. Gohberg, L.S. Goldenštein, A.S. Markus, Investigations of some properties of bounded linear operators with their q–norms, Učen. Zap. Kishinevsk. Univ.29 (1957) 29–36 (Russian).
[10] J.M. Hyslop, Note on the strong summability of series, Proc. Glasgow Math. Assoc. 1 (1951/53) 16–20.
[11] V. Istrăţesku, On a measure of noncompactness, Bull. Math. Soc. Sci. Mat. R. S. Roumanie (N.S) 16 (1972) 195–197.
[12] V. Istrăţesku, Fixed Point Theory, An Introduction, Reidel, Dordrecht, Boston and London, 1981.
[13] A.M. Jarrah, E. Malkowsky, Ordinary, absolute and strong summability and matrix transformations, Filomat 17 (2003) 59–78.
[14] K. Kuratowski, Sur les espaces complets, Fund. Math. 15 (1930) 301–309.
[15] B. Kuttner, B. Thorpe, Strong convergence, J. Reine Angew. Math. 311/312 (1979) 42–55.
[16] E. Malkowsky, The continuous duals of the spaces $c_0(\Lambda)$ and $c(\Lambda)$ for exponentially bounded sequences, Acta Sci. Math. (Szeged) 61 (1995) 241–250.
[17] E. Malkowsky, The dual spaces of the sets of Λ–strongly convergent and bounded sequences, Novi Sad J. Math. 30:3 (2000) 99–110.
[18] E. Malkowsky, V. Rakočević, An Introduction Into the Theory of Sequence Spaces and Measures of Noncompactness, Zbornik radova, Matematički institut SANU, Beograd 9(17) (2000) 143–234.
[19] F. Móricz, On Λ–strong convergence of numerical sequences and Fourier series, Acta Math. Hung. 54 (1989) 319–327.
[20] A. Wilansky, Summability through Functional Analysis, North–Holland Mathematics Studies No. 85, 1984.