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Fixed Point Theorems in Partial Metric Spaces with an Application

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Abstract. Matthews [12] introduced a new distance $\mathcal{P}$ on a nonempty set X, which he called a partial metric. The purpose of this paper is to present some fixed point results for weakly contractive type mappings in ordered partial metric space. An application to nonlinear fractional boundary value problem is also presented.

1. Introduction and Preliminaries

The Banach contraction mapping principle is one of the pivotal results of analysis. It is widely considered as the source of metric fixed point theory and its significance lies in its vast applicability in a number of branches of mathematics.
The weakly contractive single-valued maps were first defined by Alber and Guerre-Delabrire in [2]. Here we give a brief description of the basic known notations. If $(E, \|.\|)$ be a Banach space, a self-map F of E is said to satisfy the Banach contraction principle if there exists a constant k with $0 \leq k < 1$ such that, for $x, y \in E$,

$$\|Fx - Fy\| \leq k\|x - y\|.$$

As noted in introduction of [2], this inequality can be written in the form

$$\|Fx - Fy\| \leq \|x - y\| - q\|x - y\|,$$

where $k = 1 - q$ with $q \in (0, 1]$. The extension of the above inequality in the context of Banach spaces to what are called weakly contraction maps is a natural one. A self-map F of E is said to be weakly contractive if

$$\|Fx - Fy\| \leq \|x - y\| - \psi(\|x - y\|)$$

for every $x, y \in E$, where $\psi : [0, \infty) \to [0, \infty)$ is a continuous and nondecreasing function such that it is positive in $(0, \infty)$, $\psi(0) = 0$ and $\lim_{t\to\infty} \psi(t) = \infty$. This notation can easily be extended to a metric space E, that is, a map $F : E \to E$ is said to be weakly contractive if

$$d(Fx, Fy) \leq d(x, y) - \psi(d(x, y))$$

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for all $x, y \in E$, where $\psi : [0, \infty) \to [0, \infty)$ satisfies the above mentioned conditions.
Recently, there is a trend to weaken the requirement on the contraction by considering metric spaces endowed with partial order. In [13], [18] the Banach contraction principle was discussed in a metric space endowed with partial order. Also, existence of fixed point in partially ordered sets has been considered recently in [1-13]. The study on the existence of fixed points for single valued increasing operators is successful, the results obtained are widely used to investigate the existence of solutions to the ordinary and partial differential equations (see[7], [10]). Recently Bhaskar and Lakshmikantham [3], Nieto and Lopez [13], [14], Lakshmikantham and ćirić [11], Ran and Reurings [18] and Agarwal, El-Gebeily and O'Regan [1] presented some new results for contraction in partially ordered metric spaces. Bhaskar and Lakshmikantham [3] noted that their theorem can be used to investigate a large class of problems and have discussed the existence and uniqueness of solution for a periodic boundary value problem.
In this paper we will define an analogical weakly contractive type condition and obtain some results in ordered partial metric spaces.
In [4] authors were concerned with the existence and uniqueness of positive solutions for the following nonlinear fractional boundary value problem

$$D^{\alpha}_{0^+} u(t) + f(t, u(t)) = 0, \quad 0 \le t \le 1, 3 < \alpha \le 4$$

$$u(0) = u'(0) = u''(0) = u''(1) = 0$$

where $D^{\alpha}_{0^+}$ denotes the standard Rieman-Liouville fractional derivative. Their results can be derived from our fixed point theorems in partially ordered sets.

First, we recall some definitions of partial metric space and some properties of theirs [12], [15], [16] and [19].
A partial metric on a nonempty set X is a function $\mathcal{P} : X \times X \to \mathbb{R}^+$ such that for all $x, y, z \in X$:

(p_1) $x = y \Leftrightarrow \mathcal{P}(x, x) = \mathcal{P}(x, y) = \mathcal{P}(y, y)$,
(p_2) $\mathcal{P}(x, x) \le \mathcal{P}(x, y)$,
(p_3) $\mathcal{P}(x, y) = \mathcal{P}(y, x)$,
(p_4) $\mathcal{P}(x, y) \le \mathcal{P}(x, z) + \mathcal{P}(z, y) - \mathcal{P}(z, z)$.

A partial metric space is a pair $(X, \mathcal{P})$ such that X is a nonempty set and $\mathcal{P}$ is a partial metric on X. It is clear that, if $\mathcal{P}(x, y) = 0$, then from (p_1) and (p_2) $x = y$. But if $x = y$, $\mathcal{P}(x, y)$ may not be 0. A basic example of a partial metric space is the pair $(\mathbb{R}^+, \mathcal{P})$, where $\mathcal{P}(x, y) = \max\{x, y\}$ for all $x, y \in \mathbb{R}^+$. Other examples of partial metric spaces which are interesting from a computational point of view may be found in [12].
Each partial metric $\mathcal{P}$ on X generates a T_0 topology τ_P on X which has as a base the family open $\mathcal{P}$-balls $\{B_p(x, \epsilon) : x \in X, \epsilon > 0\}$, where $B_p(x, \epsilon) = \{y \in X : \mathcal{P}(x, y) < \mathcal{P}(x, x) + \epsilon\}$ for all $x \in X$ and $\epsilon > 0$.
If $\mathcal{P}$ is a partial metric on X, then the function $\mathcal{P}^s : X \times X \to \mathbb{R}^+$ given by

$$\mathcal{P}^s(x, y) = 2\mathcal{P}(x, y) - \mathcal{P}(x, x) - \mathcal{P}(y, y)$$

is a metric on X.
Let $(X, \mathcal{P})$ be a partial metric space.

A sequence $\{x_n\}$ in a partial metric space $(X, \mathcal{P})$ converges to a point $x \in X$ if and only if $\mathcal{P}(x, x) = \lim_{n\to\infty} \mathcal{P}(x, x_n)$.

A sequence $\{x_n\}$ in a partial metric space $(X, \mathcal{P})$ is called a Cauchy sequence if there exists (and is finite) $\lim_{n,m\to\infty} \mathcal{P}(x_n, x_m)$.

A partial metric space $(X, \mathcal{P})$ is said to be complete if every Cauchy sequence $\{x_n\}$ in X converges, with respect to τ_p, to a point $x \in X$ such that $\mathcal{P}(x, x) = \lim_{n,m\to\infty} \mathcal{P}(x_n, x_m)$.

A mapping $F : X \to X$ is said to be continuous at $x_0 \in X$, if for every $\epsilon > 0$, there exists $\delta > 0$ such that $F(B_p(x_0, \delta)) \subseteq B_p(Fx_0, \epsilon)$.

Lemma 1.1. *Let* $(X, \mathcal{P})$ *be a partial metric space.*

(a) $\{x_n\}$ *is a Cauchy sequence in* $(X, \mathcal{P})$ *if and only if it is a Cauchy sequence in the metric space* $(X, \mathcal{P}^s)$.

(b) A partial metric space $(X, \mathcal{P})$ *is complete if and only if the metric space* $(X, \mathcal{P}^s)$ *is complete. Furthermore,* $\lim_{n\to\infty} \mathcal{P}^s(x_n, x) = 0$ *if and only if*

$$\mathcal{P}(x, x) = \lim_{n\to\infty} \mathcal{P}(x_n, x) = \lim_{n,m\to\infty} \mathcal{P}(x_n, x_m).$$

2. Main Results

Definition 2.1. *Let* $\phi : [0, \infty) \to [0, \infty)$ *satisfy*

(i) $\phi(0) = 0$ *and* $\phi(t) > 0$ *for each* $t > 0$.

(ii) ϕ *is right lower semi-continuous, i.e. for any nonnegative nonincreasing sequence* $\{r_n\}$, $\liminf_{n\to\infty} \phi(r_n) \geq \phi(r)$, *provided* $\lim_{n\to\infty} r_n = r$.

(iii) For any sequence $\{r_n\}$ *with* $\lim_{n\to\infty} r_n = 0$, *there exists* $a \in (0, 1)$ *and* $n_0 \in \mathbb{N}$ *such that* $\phi(r_n) \geq ar_n$ *for each* $n \geq n_0$.

Define $\Phi = \{\phi : \phi \text{ satisfies}(i) - (iii) \text{ above}\}$.

Theorem 2.2. *Let* $(X, \leq)$ *be a partially ordered set and suppose that there is a partial metric* $\mathcal{P}$ *on* X *such that* $(X, \mathcal{P})$ *is a complete partial metric space. Suppose* $T : X \to X$ *is a continuous and nondecreasing mapping such that*

$$\mathcal{P}(Tx, Ty) \leq \mathcal{P}(x, y) - \phi(\mathcal{P}(x, y)), \tag{1}$$

for all $x, y \in X$ *which are comparable, where* $\phi \in \Phi$.
If there exists an $x_0 \in X$ *with* $x_0 \leq Tx_0$, *then there exists* $x \in X$ *such that* $x = Tx$. *Moreover,* $\mathcal{P}(x, x) = 0$.

Proof. If $Tx_0 = x_0$, then the proof is finished, so suppose $x_0 \neq Tx_0$. Now let $x_n = Tx_{n-1}$ for $n = 1, 2, \ldots$. If $x_{n_0} = x_{n_0+1}$ for some $n_0 \in \mathbb{N}$, then it is clear that x_{n_0} is a fixed point of T. Thus assume $x_n \neq x_{n+1}$ for all $n \in \mathbb{N}$. Notice that, since $x_0 \leq Tx_0$ and T is nondecreasing, we have

$$x_0 \leq x_1 \leq x_2 \leq \cdots \leq x_n \leq x_{n+1} \leq \ldots.$$

Now since $x_{n-1} \leq x_n$ we can use the inequality (1) for these points, then we have

$$\begin{aligned} \mathcal{P}(x_{n+1}, x_n) &= \mathcal{P}(Tx_n, Tx_{n-1}) \\ &\leq \mathcal{P}(x_n, x_{n-1}) - \phi(\mathcal{P}(x_n, x_{n-1})) \\ &< \mathcal{P}(x_n, x_{n-1}), \end{aligned}$$

it follows that $p_n = \{\mathcal{P}(x_{n+1}, x_n)\}$ is a nonnegative nonincreasing sequence and hence possesses a limit p. If $p > 0$, from the assumption (ii) of ϕ, there exists $n_0 \in \mathbb{N}$ such that

$$\phi(p_n) \geq \phi(p) > 0 \text{ for all } n > n_0.$$

In addition, we have

$$p_n \leq p_{n-1} - \phi(p_{n-1}) \leq p_{n-1} - \phi(p).$$

Taking the limit as $n \to \infty$, we have

$$p \leq p - \phi(p) < p,$$

this is a contradiction and hence we get $p = 0$.
Now, we show that $\{x_n\}$ is a Cauchy sequence. On the other hand, since $\lim_{n\to\infty} \mathcal{P}(x_n, x_{n-1}) = 0$, from the assumption (iii) of ϕ there exists $0 < a < 1$ and $n_0 \in \mathbb{N}$ such that

$$\phi(\mathcal{P}(x_n, x_{n-1})) \geq a\mathcal{P}(x_n, x_{n-1}) \text{ for all } n > n_0.$$

We have

$$\mathcal{P}(x_{n+1},x_n) \leq \mathcal{P}(x_n,x_{n-1}) - \phi(\mathcal{P}(x_n,x_{n-1})) \leq (1-a)\mathcal{P}(x_n,x_{n-1}). \tag{2}$$

By this inequality, we get

$$\mathcal{P}(x_{n+1},x_n) \leq (1-a)\mathcal{P}(x_n,x_{n-1}) \leq \cdots \leq (1-a)^n\mathcal{P}(x_1,x_0). \tag{3}$$

Set $\lambda = (1-a)$. Therefore,

$$\begin{aligned}\mathcal{P}^s(x_{n+1},x_n) &= 2\mathcal{P}(x_{n+1},x_n) - \mathcal{P}(x_{n+1},x_{n+1}) - \mathcal{P}(x_n,x_n)\\ &\leq 2\mathcal{P}(x_{n+1},x_n) + \mathcal{P}(x_{n+1},x_{n+1}) + \mathcal{P}(x_n,x_n)\\ &\leq 4\lambda^n\mathcal{P}(x_1,x_0).\end{aligned}$$

This shows that $\lim_{n\to\infty}\mathcal{P}(x_{n+1},x_n) = 0$. Now we have

$$\begin{aligned}\mathcal{P}^s(x_{n+k},x_n) &\leq \mathcal{P}^s(x_{n+k},x_{n+k-1}) + \cdots + \mathcal{P}^s(x_{n+1},x_n),\\ &\leq 4\lambda^{n+k-1}\mathcal{P}(x_1,x_0) + \cdots + 4\lambda^n\mathcal{P}(x_1,x_0),\\ &= 4\frac{\lambda^n(1-\lambda^k)}{1-\lambda}\mathcal{P}(x_1,x_0),\\ &\leq 4\frac{\lambda^n}{1-\lambda}\mathcal{P}(x_1,x_0).\end{aligned}$$

This shows that $\{x_n\}$ is a Cauchy sequence in the metric space $(X,\mathcal{P}^s)$. Since $(X,\mathcal{P})$ is complete then from Lemma 1.1, the sequence $\{x_n\}$ is converges in the metric space $(X,\mathcal{P}^s)$, say $\lim_{n\to\infty}\mathcal{P}^s(x_n,x) = 0$ for some $x \in X$. Again from Lemma 1.1, we have

$$\mathcal{P}(x,x) = \lim_{n\to\infty}\mathcal{P}(x_n,x) = \lim_{n,m\to\infty}\mathcal{P}(x_n,x_m). \tag{4}$$

Furthermore, since $\{x_n\}$ is a Cauchy sequence in the metric space $(X,\mathcal{P}^s)$, we have $\lim_{n,m\to\infty}\mathcal{P}^s(x_n,x_m) = 0$ and from (3) we have $\lim_{n\to\infty}\mathcal{P}(x_n,x_n) = 0$, thus from the definition $\mathcal{P}^s$ we have $\lim_{n,m\to\infty}\mathcal{P}(x_n,x_m) = 0$. Therefore from (4) we have

$$\mathcal{P}(x,x) = \lim_{n\to\infty}\mathcal{P}(x_n,x) = \lim_{n,m\to\infty}\mathcal{P}(x_n,x_m) = 0.$$

Now we claim that $Tx = x$. Suppose $\mathcal{P}(x,Tx) > 0$. Since T is continuous, then given $\epsilon > 0$, there exists $\delta > 0$ such that $T(B_p(x,\delta)) \subseteq B_p(Tx,\epsilon)$. Since $\mathcal{P}(x,x) = \lim_{n\to\infty}\mathcal{P}(x_n,x) = 0$, then there exists $k \in \mathbb{N}$ such that $\mathcal{P}(x_n,x) < \mathcal{P}(x,x) + \delta$ for all $n \geq k$. Therefore, we have $x_n \in B_p(x,\delta)$ for all $n \geq k$. Thus $Tx_n \in T(B_p(x,\delta)) \subseteq B_p(Tx,\epsilon)$ and so $\mathcal{P}(Tx_n,Tx) < \mathcal{P}(Tx,Tx) + \epsilon$ for all $n \geq k$. This shows that $\lim_{n\to\infty}\mathcal{P}(x_{n+1},Tx) = \mathcal{P}(Tx,Tx)$. Now

$$\begin{aligned}\mathcal{P}(x,Tx) &\leq \mathcal{P}(x,x_{n+1}) + \mathcal{P}(x_{n+1},Tx) - \mathcal{P}(x_{n+1},x_{n+1})\\ &\leq \mathcal{P}(x,x_{n+1}) + \mathcal{P}(x_{n+1},Tx),\end{aligned}$$

and letting $n \to \infty$, by (1) we have

$$\mathcal{P}(x,Tx) \leq \mathcal{P}(Tx,Tx) \leq \mathcal{P}(x,x) - \phi(\mathcal{P}(x,x)) = 0$$

Thus $\mathcal{P}(x,Tx) = 0$ and so $x = Tx$. $\square$

In the following theorem we remove the continuity of T:

Theorem 2.3. *Let $(X, \leq)$ be a partially ordered set and suppose that there is a partial metric $\mathcal{P}$ on X such that $(X, \mathcal{P})$ is a complete partial metric space. Suppose $T : X \to X$ is a nondecreasing mapping such that*

$$\mathcal{P}(Tx, Ty) \leq \mathcal{P}(x, y) - \phi(\mathcal{P}(x, y)), \tag{5}$$

for all $x, y \in X$ which are comparable, where $\phi \in \Phi$. Also, the condition
$\{x_n\}$ is a increasing sequence with $x_n \to x$ in X, then $x_n \leq x$ for all n,
hold. If there exists an $x_0 \in X$ with $x_0 \leq Tx_0$, then there exists $x \in X$ such that $x = Tx$. Moreover, $\mathcal{P}(x, x) = 0$.

Proof. As in the proof of Theorem 2.2, we can construct a sequence $\{x_n\}$ in X by $x_n = Tx_{n-1}$ such that

$$x_0 \leq x_1 \leq \cdots \leq x_n \leq x_{n+1} \leq \ldots.$$

Also, we can show that $\{x_n\}$ is a Cauchy sequence in the metric space $(X, \mathcal{P}^s)$ and therefore there exists $x \in X$ such that

$$\mathcal{P}(x, x) = \lim_{n\to\infty} \mathcal{P}(x_n, x) = \lim_{n,m\to\infty} \mathcal{P}(x_n, x_m) = 0.$$

Now we claim that $Tx = x$. Suppose $\mathcal{P}(x, Tx) > 0$, by the conditions of theorem we have

$$\begin{aligned}\mathcal{P}(x, Tx) &\leq \mathcal{P}(x, x_{n+1}) + \mathcal{P}(x_{n+1}, Tx) - \mathcal{P}(x_{n+1}, x_{n+1})\\ &\leq \mathcal{P}(x, x_{n+1}) + \mathcal{P}(Tx_n, Tx)\\ &\leq \mathcal{P}(x, x_{n+1}) + \mathcal{P}(x_n, x) - \phi(\mathcal{P}(x_n, x)),\end{aligned}$$

now letting $n \to \infty$, we have $\mathcal{P}(x, Tx) = 0$. □

In what follows, we give a sufficient condition for the uniqueness of the fixed point in Theorem 2.2 and Theorem 2.3. This condition is

$$\text{for every } x, y \in X \text{ there exists a lower bound or an upper bound.} \tag{6}$$

In [7] it is proved that condition (6) is equivalent to

$$\text{for } x, y \in X \text{ there exists } z \in X \text{ which is comparable to } x \text{ and } y. \tag{7}$$

Theorem 2.4. *Adding condition (7) to the hypothesis of Theorems 2.2 and 2.3 we obtain uniqueness of the fixed point of T.*

Proof. Suppose that there exists $z, y \in X$ are different fixed point of T. Then $\mathcal{P}(z, y) > 0$. Now we consider the following two cases:

(i) If z and y are comparable, then $T^n z = z$ and $T^n y = y$ are comparable for $n = 0, 1, \ldots$. So we can use the condition (1) then we have

$$\begin{aligned}\mathcal{P}(z, y) &= \mathcal{P}(T^n z, T^n y)\\ &\leq \mathcal{P}(T^{n-1} z, T^{n-1} y) - \phi(\mathcal{P}(T^{n-1} z, T^{n-1} y))\\ &= \mathcal{P}(z, y) - \phi(\mathcal{P}(z, y))\\ &< \mathcal{P}(z, y).\end{aligned}$$

This is a contradiction.

(ii) If z and y are not comparable, then there exists $x \in X$ comparable to z and y. Since T is nondecreasing, then $T^n x$ is comparable to $T^n z = z$ and $T^n y = y$ for $n = 0, 1, \ldots$. Moreover,

$$\begin{aligned}\mathcal{P}(z, T^n x) &= \mathcal{P}(T^n z, T^n x)\\ &\leq \mathcal{P}(T^{n-1} z, T^{n-1} x) - \phi(\mathcal{P}(T^{n-1} z, T^{n-1} x))\\ &= \mathcal{P}(z, T^{n-1} x) - \phi(\mathcal{P}(z, T^{n-1} x))\\ &< \mathcal{P}(z, T^{n-1} x).\end{aligned}$$

This shows that $\mathcal{P}(z, T^n x)$ is nonnegative and nondecreasing sequence and so has a limit, say $\alpha \geq 0$. From the last inequality we can obtain

$$\alpha \leq \alpha - \phi(\alpha) < \alpha,$$

and, hence $\alpha = 0$. Similarly it can be proved $\lim_{n\to\infty} \mathcal{P}(y, T^n x) = 0$. Finally

$$\begin{aligned}\mathcal{P}(z, y) &\leq \mathcal{P}(z, T^n x) + \mathcal{P}(T^n x, y) - \mathcal{P}(T^n x, T^n x)\\ &\leq \mathcal{P}(z, T^n x) + \mathcal{P}(T^n x, y),\end{aligned}$$

and the taking limit as $n \to \infty$ we have $\mathcal{P}(z, y) = 0$. This is a contradiction to $\mathcal{P}(z, y) > 0$.

□

Example 2.5. *Let $X = [0, +\infty)$. Then $(X, \leq)$ is a partially ordered set with the natural ordering of real numbers. Define $\mathcal{P} : X \times X \to \mathbb{R}$ by*

$$\mathcal{P}(x, y) = \max\{x, y\}.$$

Then $(X, \mathcal{P})$ is a complete partial metric space. Define $T : X \to X$ by

$$Tx = \begin{cases} 0, & if\ \ 0 \leq x < 1,\\ \frac{x+1}{3}, & if\ \ x \geq 1.\end{cases}$$

It is clear that T is a nondecreasing mapping. Let $\phi : [0, \infty) \to [0, \infty)$ be defined as

$$\phi(t) = \frac{1}{3}t, \ \ \textit{for } t \in [0, \infty).$$

It is easy to see that ϕ satisfies all of conditions of Definition 2.1.
Now, we verify the inequality (5) of Theorem 2.3. We consider the following cases:

Case-1 : $x, y \in [0, 1)$ and $x \leq y$:
Then,
$\mathcal{P}(Tx, Ty) = 0$ and obviously (5) is satisfied.

Case-2 : $x, y \geq 1$ and $x \leq y$:
Then,

$$\begin{aligned}\mathcal{P}(Tx, Ty) &= \mathcal{P}(\frac{x+1}{3}, \frac{y+1}{3})\\ &= \frac{y+1}{3}\\ &\leq \frac{x+y}{3}\\ &\leq \frac{2}{3}\max\{x, y\}\\ &= \max\{x, y\} - \frac{1}{3}\max\{x, y\}\\ &= \mathcal{P}(x, y) - \phi(\mathcal{P}(x, y)).\end{aligned}$$

Case-3 : $x \in [0, 1)$, $y \geq 1$ and $x < y$:

Then,

$$\begin{aligned}\mathcal{P}(Tx,Ty) &= \mathcal{P}(0,\frac{y+1}{3})\\ &= \frac{y+1}{3}\\ &\leq \frac{2}{3}y\\ &= \max\{x,y\} - \frac{1}{3}\max\{x,y\}\\ &= \mathcal{P}(x,y) - \phi(\mathcal{P}(x,y)).\end{aligned}$$

Thus it is verified that the nondecreasing mapping T satisfies all the conditions of Theorem 2.3. Here $x = 0$ is the unique fixed point of T in X

3. Application

Consider the nonlinear boundary value problem of fractional order:

$$D^{\alpha}_{0^+}u(t) + f(t,u(t)) = 0,\ \ 0 \leq t \leq 1, 3 < \alpha \leq 4 \tag{8}$$

with

$$u(0) = u'(0) = u''(0) = u''(1) = 0$$

The aim of this section is to present a recent existence theorem due to Caballero, Harjani and Sadarangani [4] for a solution of the above problem which can be derived by our Theorem 2.3.
Notice that the Banach space $C[0,1]$ can be equipped with a partial order given by

$$x, y \in C[0;1],\ x \leq y \Leftrightarrow x(t) \leq y(t),\ \ \text{for } t \in [0,1]$$

which satisfies the condition
if $\{x_n\} \subseteq C[0,1]$ is a increasing sequence with $x_n \to x$ in X, then $x_n \leq x$ for all n. Moreover, for $x, y \in C[0,1]$, as the function $\max(x,y) \in C[0,1]$, $(C[0,1], \leq)$ satisfies condition (7).
Here, we can consider the classical metric on $C[0,1]$ as a partial metric:

$$\mathcal{P}(x,y) = \sup_{0\leq t\leq 1}\{|x(t) - y(t)|\}.$$

By Ψ we denote the class of functions $\psi : [0,+\infty) \to [0,+\infty)$ satisfying that $I - \psi \in \Phi$, where I denotes the identity mapping on $[0,+\infty)$.

Theorem 3.1. *[4] Problem (8) has unique positive solution $u(t)$ if the following conditions are satisfied:*

(i) $f : [0,1]\times[0,\infty) \to [0,\infty)$ *is continuous and nondecreasing respect to the second argument;*
(ii) There exists $t_0 \in [0,1]$ *such that* $f(t_0,0) > 0$;
(iii) There exists $0 < \lambda \leq \frac{(\alpha-2)\Gamma(\alpha+1)}{2}$, *such that, for* $x, y \in [0,\infty)$ *with* $y \geq x$ *and* $t \in [0,1]$,

$$f(t,y) - f(t,x) \leq \lambda.\psi(y-x),$$

where $\psi \in \Psi$.

Proof. We omit the proof. For details, see [4]. □

References

[1] R. P. Agarwal, M. A. .El-Gebeily, D. O'Regan, *Generalized contractions in partially ordered metric spaces*, Appl. Anal. **87** (2008), 1-8.
[2] Ya.I. Alber, S. Guerre-Delabriere, *Principles of weakly contractive maps in Hilbert spaces, new results in operator theory, I. Gohberg, Yu Lyubich (Eds.)*, Advances and Appl. **98**, Birkhauser Verlag, Basel, (1997), 7-22.
[3] T. G. Bhaskar, V. Lakshmikantham, *Fixed point theorems in partially ordered metric spaces and applications*, Nonlinear Anal. **65** (2006), 1379-1393.
[4] J. Caballero, J. Harjani, K. Sadarangani, *Existence and uniqueness of positive solution for a boundary value problem of fractional order*, Abstr. Appl. Anal. 2011, Art. ID 165641, 12 pp.
[5] L. Ciric, N. Cakid, M. Rajovic, J. S. Uma, *Monotone generalized nonlinear contractions in partially ordered metric spaces*, Fixed Point Theory and Appl. Article ID 131294, (2008).
[6] B. S. Choudhury, A. Kunda, *A coupled coincidence point result in partially ordered metric spaces for compatible mappings*, Nonlinear Anal. **73** (2010), 2524-2531.
[7] D. J. Guo, *Partial order methods in nonlinear analysis*, Shandong Science and Technology Press, Jinan, 2000. [in Chines].
[8] J. Harjani, K. Sadarangani, *Fixed point theorems for weakly contractive mappings in partially ordered sets*, Nonlinear Anal. **71** (2009), 3403-3410.
[9] J. Harjani, K. Sadarangani, *Generalized contractions in partially ordered metric spaces and applications to ordinary differential equations*, Nonlinear Anal. **72** (2010), 1188-1197.
[10] S. Heikkil*ä*, V. Lakshmikantham, *Monotone iterative techniques for discontinuous nonlinear differential equations*, Dekker, New York, 1994.
[11] V. Lakshmikantham, L. Ciric, *Coupled fixed point theorems for nonlinear contractions in partially ordered metric spaces*, Nonlinear Anal. **70** (2009), 4341-4349.
[12] S. G. Matthews, *Partial metric topology*, Proc. 8th Summer Conference on General Topology and Applications, Ann. New York Acad. Sci., **728** (1994), 183-197.
[13] J. J. Nieto, R. Rodriguez-Lopez, *Contractive mapping theorems in partially ordered sets and applications to ordinary differential equations*, Order. **22** (2005), 223-239.
[14] J.J. Nieto, R. R. Lopez, *Existence and uniqueness of fixed point in partially ordered sets and applications to ordinary differential equations*, Acta Math Sinica, Engl. Ser. **23** (2007), 2205-2212.
[15] S. Oltra, O. Valero, *Banach's fixed point theorem for partial metric spaces*, Rend. lstit. Mat. Univ. Trieste., **36** (2004), 17-26.
[16] S. J. O'Neill, *Partial metrics, valuations and domain theory*, in: Proc. 11th Summer Conference on General Topology and its Applications, in: Ann. New York Acad. Sci., **806** (1996), 304-315.
[17] D. O'Regan, A. Petrusel, *Fixed point theorems for generalized contractions in ordered metric spaces*, J. Math. Anal. Appl. **341** (2008), 1241-1252.
[18] A. C. M. Ran, M. C. B. Reurings, *A fixed point theorem in partially ordered sets and some applications to matrix equations*, Proc. Amer. Math. Soc. **132** (2004), 1435-1443.
[19] O. Valero, *On Banach fixed point theorems for partial metric spaces*, Appl. Gen. Topol. **2** (2005), 229-240.