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SPECTRUM OF CLASS ABSOLUTE -$*$-k-PARANORMAL OPERATORS FOR $0 \leq k \leq 1$

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Abstract. In this paper, we shall introduce a new class absolute-$*$-k-paranormal operators given by a norm inequality and $*$-$A(k)$ operator by operator inequality, we will discuss the inclusion relation of them. And we study spectral properties of class absolute-$*$-k-paranormal operators. We show that if T belongs to class absolute-$*$-k-paranormal operators, then its point spectrum and joint point spectrum are identical, its approximate point spectrum and joint approximate point spectrum are identical. Next as an application of them, for Weyl spectrum $w(\cdot)$ and essential approximate point spectrum $\sigma_{ea}(\cdot)$, we will show that if T or T^* is absolute-$*$-k-paranormal for $0 \leq k \leq 1$, then $w(f(T)) = f(w(T))$, $\sigma_{ea}(f(T)) = f(\sigma_{ea}(T))$ for every $f \in H(\sigma(T))$ where $H(\sigma(T))$ denotes the set of all analytic functions on an open neighborhood of $\sigma(T)$.

1. Introduction

Let H be an infinite dimensional separable Hilbert space, let $B(H)$ and $K(H)$ denote, respectively, the algebra of all bounded linear operators and the ideal of compact operators on H. If $T \in B(H)$, write $N(T)$ and $R(T)$ for the null space and range space of T; $\alpha(\mathrm{T}) := \dim N(T)$; $\beta(T) := \dim N(T^*)$; $\sigma(T)$, $\sigma_a(\mathrm{T})$, $\sigma_p(\mathrm{T})$, $\sigma_{jp}(T)$, $\sigma_{ja}(T)$ for the spectrum of T, the approximate point spectrum of T, the point spectrum of T, the joint point spectrum of T, the joint approximate point spectrum of T, respectively.

An operator $T \in B(H)$ is called Fredholm if it has closed range with finite dimension null space and its range of finite co-dimension. The index of a Fredholm operator $T \in B(H)$ is given by

$$i(T) := \alpha(T) - \beta(T).$$

An operator $T \in B(H)$ is called Weyl if it is Fredholm of index zero. An operator $T \in B(H)$ is called Browder if it is Fredholm of finite ascent and descent: equivalently ([1], Theorem 7.9.3) if T is Fredholm and $T - \lambda$ is invertible for sufficiently small $\lambda \neq 0$ in C; The essential spectrum $\sigma_e(T)$, The Weyl spectrum $w(T)$ and the Browder spectrum $\sigma_b(T)$ of $T \in B(H)$ are defined in [1] or [2]:

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$$\sigma_e(T) := \{\lambda \in C : T - \lambda \text{ is not Fredholm}\};$$
$$w(T) := \{\lambda \in C : T - \lambda \text{ is not Weyl}\};$$
$$\sigma_b(T) := \{\lambda \in C : T - \lambda \text{ is not Browder}\} :$$

Evidently
$$\sigma_e(T) \subseteq w(T) \subseteq \sigma_b(T) = \sigma_e(T) \cup \text{ acc } \sigma(T),$$
where *accK* denotes the accumulation points of $K \subseteq C$.

We consider the sets

$$\Phi_+(H) := \{T \in B(H) : R(T) \text{ is closed and } \alpha(T) < \infty\};$$
$$\Phi_-(H) := \{T \in B(H) : R(T) \text{ is closed and } \alpha(T^*) < \infty\};$$
$$\Phi_+^-(H) := \{T \in B(H) : T \in \Phi_+(H) \text{ and } i(T) \leq 0\}.$$

On the other hand, $\sigma_{ea}(T) := \{\lambda \in C : T - \lambda \notin \Phi_+^-(H)\}$ is the essential approximate point spectrum and $\sigma_{ab}(T) := \cap\{\sigma_a(T + K) : TK = KT\ K \in K(H)\}$ is the Browder essential approximate point spectrum.

We say that α-Browder's theorem holds for $T \in B(H)$ if there is equality $\sigma_{ea}(T) = \sigma_{ab}(T)$.

Recall ([3]) that $S, T \in B(H)$ are said to be quasisimilar if there exist injections $X, Y \in B(H)$ with the dense range such that $XS = TX$ and $YT = SY$, respectively, and this relation of S and T is denoted by $S \sim T$. We say that $T \in B(H)$ has the single valued extension property (abbev. SVEP) if for every open set U of C the only analytic solution $f : U \to H$ of the equation

$$(T - \lambda)f(\lambda) = 0$$

for all $\lambda \in U$ is the zero function on U.

A complex number $\lambda \in C$ is said to be in the point spectrum $\sigma_p(T)$ of the operator T if there is a unit vector x satisfying $(T - \lambda)x = 0$. If in addition, $(T^* - \bar{\lambda})x = 0$, then λ is said to be in the joint point spectrum $\sigma_{jp}(T)$ of T.

A complex number $\lambda \in C$ is said to be in the approximate point spectrum $\sigma_a(T)$ of the operator T if there is a sequence $\{x_n\}$ of unit vectors satisfying $(T - \lambda)x_n \to 0$. If in addition, $(T^* - \bar{\lambda})x_n \to 0$, then λ is said to be in the joint approximate point spectrum $\sigma_{ja}(T)$ of T. The boundary $\partial\sigma(T)$ of the spectrum $\sigma(T)$ of the operator T is known to be a subset of $\sigma_a(T)$. Although, in general, one has $\sigma_{jp}(T) \subset \sigma_p(T)$, $\sigma_{ja}(T) \subset \sigma_a(T)$, there are many classes of operators T for which

$$\sigma_{jp}(T) = \sigma_p(T). \tag{1}$$
$$\sigma_{ja}(T) = \sigma_a(T). \tag{2}$$

For example, if T is either normal or hyponormal, then T satisfies (1) (2). More generally, (1) (2) hold if T is semi-hyponormal[4], p-hyponormal[5] or log-hyponormal [6], [7, Corollary 4.5]. In [8], Duggal introduced a class $K(p)$ of operators which contains the p-hyponormal operators and showed [8,Theorem 4] that operators T in the class $K(p)$ satisfy (1), (2).

In this paper, we proof that absolute-$*$-k-paranormal operators satisfy (1), (2).

2. Main results

Definition 1. *An operator $T \in B(H)$ is said to be $*$-paranormal if*
$||T^*x||^2 \leq ||T^2x||$ *for every unit vector $x \in H$.*

Definition 2. *For each $k > 0$, an operator T belongs to class $* - A(k)$ if*
$(T^* \mid T\mid^{2k} T)^{\frac{1}{k+1}} \geq \mid T^* \mid^2$.

Definition 3. *For each $k > 0$, an operator T is absolute-$*$-k-paranormal if*
$\| T^*x \|^{k+1} \leq \| \mid T\mid^k Tx \|$ *for every unit vector $x \in H$.*

To prove the inclusion relation between $*$-$A(k)$ operator and absolute-$*$-k-paranormal operator, we need the following lemma.

Lemma 1. [12] *Let A be a positive linear operator on a Hilbert space H. Then the following properties (1), (2) and (3) hold.*

(1) $(A^\lambda x, x) \geq (Ax, x)^\lambda$ *for any $\lambda > 1$ and any unit vector x.*
(2) $(A^\lambda x, x) \leq (Ax, x)^\lambda$ *for any $\lambda \in [0,\ 1]$ and any unit vector x.*
(3) If A is invertible, then
$(A^\lambda x, x) \geq (Ax, x)^\lambda$ *for any $\lambda < 0$ and any unit vector x.*

Moreover (1), (2) and (3) are equivalent to the following (1'), (2') and (3'), respectively.
(1') $(A^\lambda x, x) \geq (Ax, x)^\lambda \parallel x \parallel^{2(1-\lambda)}$ *for any $\lambda > 1$ and any vector x.*
(2') $(A^\lambda x, x) \leq (Ax, x)^\lambda \parallel x \parallel^{2(1-\lambda)}$ *for any $\lambda \in [0,\ 1]$ and any vector x.*
(3') If A is invertible, then
$(A^\lambda x, x) \geq (Ax, x)^\lambda \parallel x \parallel^{2(1-\lambda)}$ *for any $\lambda < 0$ and any vector x.*

We obtain the following inclusion relation.

Theorem 2. *For each $k > 0$, every class $*$-$A(k)$ operator is an absolute-$*$-k-paranormal operator.*

Proof. Suppose that T belongs to class $*$- $A(k)$ for $k > 0$, i.e.,
$(T^* \mid T\mid^{2k} T)^{\frac{1}{k+1}} \geq \mid T^* \mid^2$ for $k > 0$.
Then for every unit vector $x \in H$,

$$\begin{aligned} \| \mid T\mid^k Tx \|^2 &= (T^* \mid T\mid^{2k} Tx\ ,\ x) \\ &\geq ((T^* \mid T\mid^{2k} T)^{\frac{1}{k+1}} x\ ,\ x)^{k+1} \\ &\geq (\mid T^* \mid^2 x\ ,\ x)^{k+1} \\ &= \| T^*x \|^{2(k+1)}. \end{aligned}$$

Hence we have
$\| T^*x \|^{k+1} \leq \| \mid T\mid^k Tx \|$ for every unit vector $x \in H$,
so that T is absolute-$*$-k-paranormal for $k > 0$. Whence the proof is complete.

But the inverse of Theorem 2 is not correct, we will give a counterexample, and we need the following theorem and lemma.

Lemma 3. *Let a and b be positive real numbers.Then*
$a^\lambda b^\mu \leq \lambda a + \mu b$ *holds for $\lambda > 0$ and $\mu > 0$ such that $\lambda + \mu = 1$.*

Theorem 4. *For each $k > 0$, an operator T is absolute-$*$-k-paranormal if and only if*

$$T^* \mid T\mid^{2k} T - (k+1)\lambda^k \mid T^* \mid^2 + k\lambda^{k+1} \geq 0 \quad \lambda > 0.$$

Proof. Suppose that T is absolute-$*$-k-paranormal for $k > 0$, i.e.,

$$\| T^*x \|^{k+1} \leq \| \mid T\mid^k Tx \| \tag{3}$$

for every unit vector $x \in H$.

(3) holds if and only if

$$\| | T |^k Tx \|^{\frac{1}{k+1}} \| x \|^{\frac{k}{k+1}} \geq \| T^* x \|$$

or equivalently,

$$(T^* | T |^{2k} Tx, x)^{\frac{1}{k+1}} (x, x)^{\frac{k}{k+1}} \geq (| T^* |^2 x, x) \tag{4}$$

for all $x \in H$.

By Lemma 3, for all $x \in H$ and $\lambda > 0$

$$\begin{aligned} &(T^* | T |^{2k} Tx, x)^{\frac{1}{k+1}} (x, x)^{\frac{k}{k+1}} \\ &= \{(\tfrac{1}{\lambda})^k (T^* | T |^{2k} Tx, x)\}^{\frac{1}{k+1}} \{\lambda(x, x)\}^{\frac{k}{k+1}} \\ &\leq \tfrac{1}{k+1} \cdot \tfrac{1}{\lambda^k} (T^* | T |^{2k} Tx, x) + \tfrac{k}{k+1} \lambda(x, x) \end{aligned} \tag{5}$$

so that (4) ensures the following (6) by (5).

$$\frac{1}{k+1} \cdot \frac{1}{\lambda^k} (T^* | T |^{2k} Tx, x) + \frac{k}{k+1} \lambda(x, x) \geq (| T^* |^2 x, x) \tag{6}$$

for all $x \in H$ and $\lambda > 0$.

Conversely, (6) implies (4) by putting $\lambda = \{\frac{(T^*|T|^{2k}Tx,x)}{(x,x)}\}^{\frac{1}{k+1}}$.

(In case $(T^* | T |^{2k} Tx, x) = 0$, let $\lambda \to 0$, we have $(| T^* |^2 x, x) = 0$). Hence (6) holds if and only if (3) is valid, so the proof of Theorem 4 is complete.

By computing, we have the following Lemma 5.

Lemma 5. [12] *Let $K = \oplus_{n=-\infty}^{+\infty} H_n$, where $H_n \cong H$. For given positive operators A and B on H, define the operator T on K as follows:*

$$T = \begin{pmatrix} \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \\ \cdots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ \cdots & B & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ \cdots & 0 & B & (0) & 0 & 0 & 0 & 0 & \cdots \\ \cdots & 0 & 0 & B & 0 & 0 & 0 & 0 & \cdots \\ \cdots & 0 & 0 & 0 & A & 0 & 0 & 0 & \cdots \\ \cdots & 0 & 0 & 0 & 0 & A & 0 & 0 & \cdots \\ & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

where ( ) shows the place of the (0, 0) matrix element. Then the following assertions hold:

(1)For each $k > 0$, T belongs to class $*$-$A(k)$ if and only if

$$(BA^{2k}B)^{\frac{1}{k+1}} \geq B^2.$$

(2)For each $k > 0$, T is absolute-$*$-k-paranormal if and only if

$$BA^{2k}B - (k+1)\lambda^k B^2 + k\lambda^{k+1} \geq 0$$

for all $\lambda > 0$.

Example 1: A non-class $*$-$A(2)$ and absolute-$*$-2-paranormal operator.

Take A and B as

$$A = \begin{pmatrix} 4 & 0 \\ 0 & 20 \end{pmatrix}^{\frac{1}{4}}, B = \frac{1}{2} \begin{pmatrix} 1+\sqrt{3} & 1-\sqrt{3} \\ 1-\sqrt{3} & 1+\sqrt{3} \end{pmatrix}.$$

Then

$$(BA^4B)^{\frac{1}{3}} - B^2 = \begin{pmatrix} -0.0091543\ldots & 0.44289\cdots \\ 0.44289\cdots & 1.2774\cdots \end{pmatrix}.$$

The eigenvalues of $(BA^4B)^{\frac{1}{3}} - B^2$ are 1.4151...and −0.14687..., so that $(BA^4B)^{\frac{1}{3}} \ngeq B^2$. Hence T is a non-class $*$-$A(2)$ operator by (1) of Lemma 5.

On the other hand, for $\lambda > 0$, define $X_2(\lambda)$ as follow:

$$X_2(\lambda) = BA^4B - 3\lambda^2B^2 + 2\lambda^3 = \begin{pmatrix} 24 - 8\sqrt{3} - 6\lambda^2 + 2\lambda^3 & -12 + 3\lambda^2 \\ -12 + 3\lambda^2 & 24 + 8\sqrt{3} - 6\lambda^2 + 2\lambda^3 \end{pmatrix}.$$

Put $p_2(\lambda) = \mathrm{tr}X_2(\lambda)$ and $q_2(\lambda) = \det X_2(\lambda)$. Then

$p_2(\lambda) = 4\lambda^3 - 12\lambda^2 + 48$

and

$q_2(\lambda) = (24 - 8\sqrt{3} - 6\lambda^2 + 2\lambda^3)(24 + 8\sqrt{3} - 6\lambda^2 + 2\lambda^3) - (-12 + 3\lambda^2)^2$
$= 4\lambda^6 - 24\lambda^5 + 27\lambda^4 + 96\lambda^3 - 216\lambda^2 + 240.$

We easily obtain $p_2(\lambda) > 0$ for all $\lambda > 0$. And we have

$q_2'(\lambda) = 24\lambda^5 - 120\lambda^4 + 108\lambda^3 + 288\lambda^2 - 432\lambda$
$= 12\lambda(\lambda - 2)(2\lambda^3 - 6\lambda^2 - 3\lambda + 18).$

So $q_2'(\lambda) = 0$ if and only if $\lambda = 0, 2$, since $2\lambda^3 - 6\lambda^2 - 3\lambda + 18 > 0$ for all $\lambda > 0$ by an easy calculation, that is, $q_2(\lambda) \geq q_2(2) = 64 > 0$ for all $\lambda > 0$.

Hence $X_2(\lambda) \geq 0$ for all $\lambda > 0$. Since $\mathrm{tr}X_2(\lambda) = p_2(\lambda) > 0$ and $\det X_2(\lambda) = q_2(\lambda) > 0$ for all $\lambda > 0$. Therefore T is absolute-$*$-2-paranormal by (2) of Lemma 5.

We will give some spectral properties of absolute-$*$-k-paranormal operator and $*$-$A(k)$ operator.

Theorem 6. *If T is absolute-$*$-k-paranormal for $0 \leq k \leq 1$, then $Tx = \lambda x$ implies $T^*x = \bar{\lambda}x$.*

Proof. It is suffice to show that $N(T-\lambda) \subseteq N(T^* - \overline{\lambda})$. Let $\lambda \in C$ and suppose $x \in N(T-\lambda)$. Then $Tx = \lambda x$. Since T is absolute-$*$-k-paranormal, $\| T^*x \|^{k+1} \leq \| |T|^k Tx \|$ for every unit vector $x \in H$. So$\| T^*x \|^{k+1} \leq \| |T|^k Tx \| = |\lambda| (|T|^{2k} x \ x)^{\frac{1}{2}} \leq |\lambda| (|T|^2 x \ x)^{\frac{k}{2}} = |\lambda| \| Tx \|^k = |\lambda|^{k+1}$, and so $\| T^*x \| \leq |\lambda|$ for all $x \in N(T - \lambda)$ with $\| x \| = 1$. Therefore if $x \in N(T - \lambda)$, then $((T - \lambda)^*x \ (T - \lambda)^*x)) = \| T^*x \|^2 - (x \ \overline{\lambda}Tx) - (\overline{\lambda}Tx \ x) + |\lambda|^2 \| x \|^2 \leq |\lambda|^2 - |\lambda|^2 - |\lambda|^2 + |\lambda|^2 = 0$. Thus $\| T^*x - \overline{\lambda}x \| = 0$, and hence $x \in N(T - \lambda)^*$. Thus $N(T - \lambda) \subseteq N(T^* - \overline{\lambda})$.

Corollary 7. *If T is absolute-$*$-k-paranormal for $0 \leq k \leq 1$, then (1) $\sigma_{jp}(T) = \sigma_p(T)$. (2) If $Tx = \lambda x$, $Ty = \mu y$ and $\lambda \neq \mu$, then $(x, y)=0$.*

Proof. (1)It is obvious from Theorem 6.
(2)As $\lambda(x, y) = (Tx, y) = (x, T^*y) = \mu(x, y)$ and $\lambda \neq \mu$, then $(x, y) = 0$.

Corollary 8. *If T is $*$-$A(k)$ operator or $*$-paranormal operator, then $\sigma_{jp}(T) = \sigma_p(T)$.*

Proof. It is clear from Corollary 7 and Theorem 2.

Corollary 9. *If T^* is absolute-$*$-k-paranormal for $0 \leq k \leq 1$, then*
$\beta(T - \lambda) \leq \alpha(T - \lambda)$ for all $\lambda \in C$.

Proof. It is obvious from Theorem 6.

Theorem 10. *If T or T^* is absolute-$*$-k-paranormal for $0 \leq k \leq 1$, then*
$w(f(T)) = f(w(T))$ for every $f \in H(\sigma(T))$,
where $H(\sigma(T))$ denotes the set of all analytic functions on an open neighborhood of $\sigma(T)$.

Proof. Since $w(f(T)) \subseteq f(w(T))$ holds for any operator T, we need only prove

$$f(w(T)) \subseteq w(f(T)) . \qquad (7)$$

Note that (7) clearly holds if f is constant on G. Thus assume f is nonconstant on G. Let $\lambda \notin w(f(T))$ and write

$$f(z) - \lambda = (z - \lambda_1)......(z - \lambda_n)g(z),$$

where λ_j, $j = 1, ..., n$ are the zeros of $f(z) - \lambda$ in G, listed according to multiplicity, and $g(z) \neq 0$ for all $z \in G$. Thus

$$f(T) - \lambda = (T - \lambda_1)......(T - \lambda_n)g(T). \qquad (8)$$

Clearly, $\lambda \in f(w(T))$ if and only if $\lambda_j \in w(T)$ for some j. Therefore, to prove (7), we need only establish $\lambda_j \notin w(T)$ for all j. Since $f(T) - \lambda$ is Weyl and the operators on the right side of (8) commute, each $T - \lambda_j$ is Fredholm. Moreover, since $N(T - \lambda_j) \subseteq N(f(T) - \lambda)$ and $N((T - \lambda_j)^*) \subseteq N((f(T) - \lambda)^*)$, both $N(T - \lambda_j)$ and $N(T - \lambda_j)^*$ are finite dimensional. Then $i(T - \lambda_j) \leq 0$ by Theorem 6. Since $i(f(T) - \lambda) = i(g(T)) = 0$, it follow from (8) that $i(T - \lambda_j) = 0$ for all j. Consequently, $T - \lambda_j$ is Weyl, and $\lambda_j \notin w(T)$. Suppose that T^* is absolute-∗-k-paranormal, then by Corollary 10 $i(T - \lambda_j) \geq 0$ for each $j = 1, 2, ..., n$. However,

$$\sum_{j=1}^{n} i(T - \lambda_j) = i(f(T) - \lambda) = 0,$$

and so $T - \lambda_j$ is Weyl for each $j = 1, 2, ..., n$. Hence $\lambda \notin f(w(T))$, and so $w(f(T)) = f(w(T))$. This completes the proof of theorem.

Theorem 11. *If T or T^* is absolute-∗-k-paranormal for $0 \leq k \leq 1$, then*
$\sigma_{ea}(f(T)) = f(\sigma_{ea}(T))$ for every $f \in H(\sigma(T))$,
where $H(\sigma(T))$ denotes the set of all analytic functions on an open neighborhood of $\sigma(T)$.

Proof. Let $f \in H(\sigma(T))$. It suffices to show that $f(\sigma_{ea}(T)) \subseteq \sigma_{ea}(f(T))$. Suppose that $\lambda \notin \sigma_{ea}(f(T))$. Then $f(T) - \lambda \in \Phi_+^-(H)$ and

$$f(T) - \lambda = c(T - \lambda_1)...(T - \lambda_n)g(T) \qquad (9)$$

where $c, \lambda_1, \lambda_2...\lambda_n \in C$ and $g(T)$ is invertible. Since the operators on the right side of (9) commute, $T - \lambda_i \in \Phi_+(H)$. Suppose T is absolute-∗-k-paranormal. Then by Theorem 6 $i(T - \lambda_j) \leq 0$ for each $j = 1, 2, ..., n$. Therefore $\lambda \notin f(\sigma_{ea}(T))$. If T^* is absolute-∗-k-paranormal, it follows from Corollary 9 that $i(T - \lambda_j) \geq 0$ for each $j = 1, 2, ..., n$. Therefore

$$0 \leq \sum_{j=1}^{n} i(T - \lambda_j) = i(f(T) - \lambda) \leq 0,$$

and so $T - \lambda_j$ is Weyl for each $j = 1, 2, ..., n$. Therefore $\lambda \notin f(\sigma_{ea}(T))$, and so $\sigma_{ea}(f(T)) = f(\sigma_{ea}(T))$. This completes the proof of theorem.

Lemma 12. *If T is absolute-∗-k-paranormal for $0 \leq k \leq 1$, then $T - \lambda$ has finite ascent for each $\lambda \in C$.*

Proof. Suppose that T is absolute-∗-k-paranormal for $0 \leq k \leq 1$. Then $N(T - \lambda) \subseteq N(T^* - \overline{\lambda})$ for each $\lambda \in C$. Thus we can represent $T - \lambda$ as the following 2×2 operator matrix with respect to the decomposition $N(T - \lambda) \oplus (N(T - \lambda))^{\perp}$:

$$T - \lambda = \begin{pmatrix} 0 & 0 \\ 0 & S \end{pmatrix}.$$

Let $x \in N((T - \lambda)^2)$. Write $x = y + z$, where $y \in N(T - \lambda)$ and $z \in (N(T - \lambda))^{\perp}$. Then $0 = (T - \lambda)^2 x = (T - \lambda)^2 z$, so that $(T - \lambda)z \in N(T - \lambda) \bigcap N(T - \lambda)^{\perp} = \{0\}$. Which implies that $z \in N(T - \lambda)$, and hence $x \in N(T - \lambda)$. Therefore $N(T - \lambda) = N(T - \lambda)^2$.

Theorem 13. *If T is absolute-∗-k-paranormal for $0 \leq k \leq 1$ and suppose that $S \sim T$. Then S has SVEP.*

Proof. Since T is absolute-∗-k-paranormal for $0 \leq k \leq 1$, it follows from Lemma 12 that $T - \lambda$ has finite ascent for each $\lambda \in C$. So by [9, proposition 1.8], T has SVEP. Let U be any open set and $f : U \to H$ be any analytic function such that $(S - \lambda)f(\lambda) = 0$ for all $\lambda \in U$. Since $S \sim T$, there exists an injective operator A with dense range such that $AS = TA$. So $A(S - \lambda) = (T - \lambda)A$ for all $\lambda \in U$. Since $(S - \lambda)f(\lambda) = 0$ for all $\lambda \in U$, $0 = A(S - \lambda)f(\lambda) = (T - \lambda)Af(\lambda)$ for all $\lambda \in U$. But T has SVEP; hence $Af(\lambda) = 0$ for all $\lambda \in U$. Since A is injective, $f(\lambda) = 0$ for all $\lambda \in U$. Therefore S has SVEP.

Theorem 14. *If T is absolute-∗-k-paranormal for $0 \leq k \leq 1$ and suppose that $S \sim T$. Then a-Browder's theorem holds for f(S) for every $f \in H(\sigma(S))$.*

Proof. Since T is absolute-$*$-k-paranormal for $0 \le k \le 1$ and $S \sim T$, it follows from Theorem 13 that S has SVEP. Next we show that a-Browder's theorem holds for S. It is well known that $\sigma_{ea}(T) \subseteq \sigma_{ab}(T)$. Conversely, suppose that $\lambda \in \sigma_a(S) \backslash \sigma_{ea}(S)$. Then $S - \lambda \in \Phi_+^-(H)$ and $S - \lambda$ is not bounded below. Since S has SVEP and $S - \lambda \in \Phi_+^-(H)$, it follows from [10, Theorem 2.6] that $S - \lambda$ has finite ascent. Therefore by [11, Theorem2.1], $\lambda \in \sigma_a(S) \backslash \sigma_{ab}(S)$. Thus a-Browder's theorem holds for S. Let $f \in H(\sigma(S))$. Then it follows from Theorem 11 that $\sigma_{ab}(f(S)) = f(\sigma_{ab}(S)) = f(\sigma_{ea}(S)) = \sigma_{ea}(f(S))$. Hence a-Browder's theorem holds for $f(S)$.

Lemma 15. [8] *Let $T = U \mid T \mid$ be the polar decomposition of the operator T, $\lambda = \mid \lambda \mid e^{i\theta} \neq 0$, and $\{x_n\}$ be a sequence of vectors. The following assertions are equivalent.*

(*a*) $(T - \lambda)x_n \longrightarrow 0$ *and* $(T^* - \overline{\lambda})x_n \longrightarrow 0, (n \longrightarrow \infty)$;

(*b*) $(\mid T \mid - \mid \lambda \mid)x_n \longrightarrow 0$ *and* $(U - e^{i\theta})x_n \longrightarrow 0, (n \longrightarrow \infty)$;

(*c*) $(\mid T^* \mid - \mid \lambda \mid)x_n \longrightarrow 0$ *and* $(U^* - e^{-i\theta})x_n \longrightarrow 0, (n \longrightarrow \infty)$.

Theorem 16. *If T is absolute-$*$-k-paranormal for $0 \le k \le 1$, then $\sigma_a(T) = \sigma_{ja}(T)$*

Proof. It is suffice to show $(T^* - \bar{\lambda})x_n \to 0$ when $(T - \lambda)x_n \to 0$ for any unit vectors sequence $\{x_n\}$. Since T is absolute-$*$-k-paranormal for $0 \le k \le 1$, then

$$\| T^*x_n \|^{k+1} \le \| \mid T \mid^k Tx_n \|$$

for any unit vectors sequence $\{x_n\}$.

Since $(T - \lambda)x_n \to 0$, thus $(T^2 - \lambda^2)x_n \to 0$.

Because

$$\| T^2x_n \|^k \le (\| (T^2 - \lambda^2)x_n \| + \mid \lambda \mid^2)^k.$$

$$\| Tx_n \|^{2(1-k)} \le (\| (T - \lambda)x_n \| + \mid \lambda \mid)^{2(1-k)}.$$

Thus

$$\begin{aligned}
\| T^*x_n \|^{k+1} &\le \| \mid T \mid^k Tx_n \| \\
&= (\mid T \mid^{2k} Tx_n \ \ Tx_n)^{\frac{1}{2}} \\
&\le (\mid T \mid^2 Tx_n, \ Tx_n)^{\frac{k}{2}} \ \| Tx_n \|^{(1-k)} \\
&= \| T^2x_n \|^k \| Tx_n \|^{(1-k)} \\
&\le (\| (T^2 - \lambda^2)x_n \| + \mid \lambda^2 \mid)^k (\| (T - \lambda)x_n \| + \mid \lambda \mid)^{1-k}.
\end{aligned}$$

But

$$\begin{aligned}
0 &\le ((T - \lambda)^*x_n \ \ (T - \lambda)^*x_n) \\
&= \| T^*x_n \|^2 - (\bar{\lambda}Tx_n \ \ x_n) - (x_n \ \ \bar{\lambda}Tx_n) + \mid \lambda \mid^2 \\
&\le (\| (T^2 - \lambda^2)x_n \| + \mid \lambda^2 \mid)^{\frac{2k}{k+1}} (\| (T - \lambda)x_n \| + \mid \lambda \mid)^{\frac{2(1-k)}{k+1}} \\
&\quad - (\bar{\lambda}Tx_n \ \ x_n) - (x_n \ \ \bar{\lambda}Tx_n) + \mid \lambda \mid^2 \\
&\to 0.
\end{aligned}$$

Thus $(T^* - \bar{\lambda})x_n \to 0$, $\sigma_a(T)) = \sigma_{ja}(T)$.

Corollary 17. *If T is absolute-$*$-k-paranormal for $0 \le k \le 1$, and $T = U \mid T \mid$ is the polar decomposition of the operator T.*

(a) If $\lambda \in \sigma_a(T)$, then $\mid \lambda \mid \in \sigma_a(\mid T \mid) \cap \sigma_a(\mid T^* \mid)$. In particular, if $\lambda \in \partial\sigma(T)$, then $\mid \lambda \mid \in \sigma_a(\mid T \mid) \cap \sigma_a(\mid T^* \mid)$.

(b) If $\lambda = \mid \lambda \mid e^{i\theta} \neq 0$ is such that $\lambda \in \sigma_a(T)$, then $e^{i\theta} \in \sigma_{ja}(U)$.

Proof. (a) Since T is absolute-$*$-k-paranormal for $0 \le k \le 1$, then $\sigma_a(T)) = \sigma_{ja}(T)$ by Theorem 17. Since $\lambda \in \sigma_a(T)$, thus $\lambda \in \sigma_{ja}(T)$, then there exists a unit vector sequence $\{x_n\}$ such that $(T - \lambda)x_n \to 0$ and $(T^* - \bar{\lambda})x_n \to 0$. Then $(\mid T \mid - \mid \lambda \mid)x_n \to 0$, $(\mid T^* \mid - \mid \lambda \mid)x_n \to 0$ by Lemma 15, thus $\mid \lambda \mid \in \sigma_a(\mid T \mid) \cap \sigma_a(\mid T^* \mid)$.

(b)Since T is absolute-$*$-k-paranormal for $0 \le k \le 1$, then $\sigma_a(T)) = \sigma_{ja}(T)$ by Theorem 16. Since $\lambda \in \sigma_a(T)$, thus $\lambda \in \sigma_{ja}(T)$, then there exists a unit vector sequence $\{x_n\}$ such that $(T - \lambda)x_n \to 0$ and $(T^* - \bar{\lambda})x_n \to 0$. Then $(U - e^{i\theta})x_n \to 0$, $(U^* - e^{-i\theta})x_n \to 0$ by Lemma 15, thus $e^{i\theta} \in \sigma_{ja}(U)$.

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