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Lipschitz-type spaces of pluriharmonic mappings

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Abstract. In this paper, we discuss the pluriharmonic mappings in the n-dimensional complex space $\mathbb{C}^n$. Several characterizations for pluriharmonic mappings to be in Lipschitz-type spaces are given, which are generalizations of the corresponding results for harmonic functions. Our proofs are related to the corresponding results of Gehring and Martio, Lappalainen, Mateljević, Dyakonov and Pavlović.

1. Introduction and main results

Let $\mathbb{C}^n = \{z = (z_1, \ldots, z_n) : z_1, \ldots, z_n \in \mathbb{C}\}$ denote the complex vector space of dimension n. For $a = (a_1, \ldots, a_n) \in \mathbb{C}^n$, we define the Euclidean inner product $\langle \cdot, \cdot \rangle$ by

$$\langle z, a \rangle = z_1\overline{a}_1 + \cdots + z_n\overline{a}_n,$$

where $\overline{a}_k$ ($k \in \{1, \cdots, n\}$) denotes the complex conjugate of a_k. Then the Euclidean length of z is defined by

$$|z| = \langle z, z \rangle^{1/2} = (|z_1|^2 + \cdots + |z_n|^2)^{1/2}.$$

Denote a ball in $\mathbb{C}^n$ with center z' and radius $r > 0$ by

$$\mathbb{B}^n(z', r) = \{z \in \mathbb{C}^n : |z - z'| < r\}.$$

In particular, $\mathbb{B}^n$ denotes the unit ball $\mathbb{B}^n(0, 1)$ and $\mathbb{S}^{n-1}$ the sphere $\{z \in \mathbb{C}^n : |z| = 1\}$. Set $\mathbb{D} = \mathbb{B}^1$, the open unit disk in $\mathbb{C}$, and let $T = \mathbb{S}^0$, the unit circle in $\mathbb{C}$.

A continuous complex-valued function f defined in a domain $\Omega \subset \mathbb{C}^n$ is said to be *pluriharmonic* if for fixed $z \in \Omega$ and $\theta \in \mathbb{S}^{n-1}$, the function $f(z + \theta\zeta)$ is harmonic in $\{\zeta \in \mathbb{C} : |\theta\zeta - z| < d_\Omega(z)\}$, where $d_\Omega(z)$ denotes the distance from z to the boundary $\partial\Omega$ of Ω. It is easy to verify that the real part of any holomorphic function is pluriharmonic (cf. [20, P59]). It is known that for every real function u, pluriharmonic in a simply connected domain, there must exist a real function v such that $f = u + iv$ is holomorphic (cf. [2, 21, 22]). Hence for evry pluriharmonic mapping f in a simply connected domain Ω,

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there exists holomorphic functions h and g such that $f = h + \overline{g}$. In this paper, the main aim is to investigate pluriharmonic mappings. In order to state the main results, some preparations are needed.

In $\mathbb{B}^n$, we consider the Poisson kernel

$$Q(z, \zeta) = \frac{1 - |z|^2}{|\zeta - z|^{2n}}$$

for $\zeta \in \mathbb{S}^{n-1}$. It is known that for a continuous function f in $\mathbb{S}^{n-1}$,

$$\mathcal{P}f(z) = \int_{\mathbb{S}^{n-1}} \frac{1 - |z|^2}{|\zeta - z|^{2n}} f(\zeta) dm(\zeta)$$

is harmonic in $\mathbb{B}^n$ (see, for example, [20]), i.e. $\Delta \mathcal{P}f = 0$, where Δ is the complex Laplacian operator defined by (cf. [20])

$$\Delta = \sum_{i=1}^{n} \Big(\frac{\partial^2}{\partial x_i^2} + \frac{\partial^2}{\partial y_i^2} \Big) = 4 \sum_{j=1}^{n} \frac{\partial^2}{\partial z_i \partial \overline{z}_i}$$

and dm denotes the surface measure on $\mathbb{S}^{n-1}$ with $m(\mathbb{S}^{n-1}) = 1$.

Let $\omega : [0, +\infty) \to [0, +\infty)$ with $\omega(0) = 0$ be a continuous function. We call that ω is a *majorant* if

1. $\omega(t)$ is increasing, and
2. $\omega(t)/t$ is nonincreasing for $t > 0$.

If, in addition, there is a constant $C > 0$ depending only on ω such that

$$\int_0^{\delta} \frac{\omega(t)}{t} dt \leq C\omega(\delta), \quad 0 < \delta < \delta_0 \tag{1}$$

and

$$\delta \int_{\delta}^{\infty} \frac{\omega(t)}{t^2} dt \leq C\omega(\delta), \quad 0 < \delta < \delta_0 \tag{2}$$

for some δ_0, then we say that ω is a *regular majorant*. A majorant is called *fast* (resp. *slow*) if the condition (1) (resp. (2)) is fulfilled.

Given a majorant ω, we define $\Lambda_\omega(\Omega)$ (resp. $\Lambda_\omega(\partial\Omega)$) to be the *Lipschitz-type* space consisting of all complex-valued functions f for which there exists a constant C such that for all z and $w \in \Omega$ (resp. z and $w \in \partial\Omega$),

$$|f(z) - f(w)| \leq C\omega(|z - w|). \tag{3}$$

Obviously, for $f \in \Lambda_\omega(\Omega)$,

$$\|f\|_{\Lambda_\omega(\Omega)} \triangleq \sup_{z \neq w \in \Omega} \Big\{ \frac{|f(z) - f(w)|}{\omega(|z - w|)} \Big\} < \infty,$$

and for $f \in \Lambda_\omega(\partial\Omega)$,

$$\|f\|_{\Lambda_\omega(\partial\Omega)} \triangleq \sup_{z \neq w \in \partial\Omega} \Big\{ \frac{|f(z) - f(w)|}{\omega(|z - w|)} \Big\} < \infty.$$

For $z, w \in \Omega$, let

$$d_{\omega,\Omega}(z, w) \triangleq \inf \int_{\gamma} \frac{\omega(d_\Omega(z))}{d_\Omega(z)} ds(z),$$

where ds stands for the arclength measure on γ, and the infimum is taken over all rectifiable curves $\gamma \subset \Omega$ joining z and w. We say that $f \in \Lambda_{\omega,int}(\Omega)$ if for all $z, w \in \Omega$,

$$|f(z) - f(w)| \leq Cd_{\omega,\Omega}(z, w),$$

where C is a positive constant which depends only on f, ω and n (cf. [12]).

Let Ω be a proper subdomain of $\mathbb{C}^n$. We say that a function f belongs to the *local Lipschitz space* $\mathrm{loc}\Lambda_\omega(\Omega)$ if there is a constant $C > 0$ satisfying (3) for all z, $w \in \Omega$ with $|w - z| < \frac{1}{2}d_\Omega(z)$. Moreover, Ω is said to be a Λ_ω*-extension domain* if $\Lambda_\omega(\Omega) = \mathrm{loc}\Lambda_\omega(\Omega)$. The geometric characterization of Λ_ω-extension domains was first given by Gehring and Martio [9]. Later, Lappalainen [13] extended it to the general case, and proved that Ω is a Λ_ω-extension domain if and only if each pair of points $z, w \in \Omega$ can be joined by a rectifiable curve $\gamma \subset \Omega$ satisfying

$$\int_\gamma \frac{\omega(d_\Omega(z))}{d_\Omega(z)}\, ds(z) \le C\omega(|z - w|) \tag{4}$$

with some fixed positive constant $C = C(\Omega, \omega, n)$. Furthermore, Lappalainen [13, Theorem 4.12] proved that Λ_ω-extension domains exist only for fast majorants and that each bounded uniform domain is a Λ_ω-extension domain. Clearly, $\mathbb{B}^n$ is a Λ_ω-extension domain.

Dyakonov [7] characterized the holomorphic functions in $\Lambda_\omega(\mathbb{D})$ in terms of their modulus. Later, in [19], Pavlović came up with a relatively simple proof of the results of Dyakonov. Recently, many authors considered this topic and generalized the work of Dyakonov to the cases of holomorphic functions, quasi-conformal mappings, pseudo-holomorphic functions and real harmonic functions with several variables, see [1, 3–6, 8, 10, 14]. Using version of Koebe theorem for analytic functions or Bloch theorem, a simple proof and generalization of Dyakonov are given in [15–18]. By using the Garsia-type norm in $\mathbb{B}^n$, the authors in [6] got some characterizations for holomorphic functions to be in $\Lambda_\omega(\mathbb{B}^n)$. In [8], Dyakonov himself discussed the implication from $|f| \in \Lambda_\omega(\Omega)$ to $f \in \Lambda_\omega(\Omega)$, where f is a holomorphic function in a Λ_ω-extension domain Ω. And, he also proved that $|f| \in \Lambda_{\omega,int}(\Omega)$ if and only if $f \in \Lambda_{\omega,int}(\Omega)$, where f is holomorphic function in a domain Ω. In this paper, we mainly discuss pluriharmonic mappings in Ω. By using a different approach, we give several characterizations for a pluriharmonic mapping to be in $\Lambda_\omega(\Omega)$ or $\Lambda_{\omega,int}(\Omega)$. The following three theorems are generalizations of the corresponding ones in [8].

Theorem 1. *Let ω be a fast majorant, and let $f = h + \overline{g}$ be a pluriharmonic mapping in a simply connected Λ_ω-extension domain Ω. Then the following are equivalent:*

1. $f \in \Lambda_\omega(\Omega)$;
2. $h \in \Lambda_\omega(\Omega)$ *and* $g \in \Lambda_\omega(\Omega)$;
3. $|h| \in \Lambda_\omega(\Omega)$ *and* $|g| \in \Lambda_\omega(\Omega)$;
4. $|h| \in \Lambda_\omega(\Omega, \partial\Omega)$ *and* $|g| \in \Lambda_\omega(\Omega, \partial\Omega)$,

where $\Lambda_\omega(\Omega, \partial\Omega)$ denotes the class of all continuous functions f on $\Omega \cup \partial\Omega$ which satisfy (3) *with some positive constant C, whenever $z \in \Omega$ and $w \in \partial\Omega$.*

Theorem 2. *Let ω be a fast majorant, and let $f = h + \overline{g}$ be pluriharmonic in a simply connected domain Ω. Then the following are equivalent:*

1. $f \in \Lambda_{\omega,int}(\Omega)$;
2. $h \in \Lambda_{\omega,int}(\Omega)$ *and* $g \in \Lambda_{\omega,int}(\Omega)$;
3. $|h| \in \Lambda_{\omega,int}(\Omega)$ *and* $|g| \in \Lambda_{\omega,int}(\Omega)$.

Theorem 3. *Suppose that ω is a regular majorant, and that $f = h + \overline{g}$ is pluriharmonic in $\mathbb{B}^n$, where h and g are holomorphic functions. Then $f \in \Lambda_\omega(\mathbb{B}^n)$ if and only if*

1. $|h| \in \Lambda_\omega(\mathbb{S}^{n-1})$, $|g| \in \Lambda_\omega(\mathbb{S}^{n-1})$; *and*
2. $\mathcal{P}|h|(z) - |h(z)| \le C\omega(1 - |z|)$, $\mathcal{P}|g|(z) - |g(z)| \le C\omega(1 - |z|)$

for some constant C.

We remark that if f is holomorphic in $\mathbb{B}^n$ and continuous up to $\mathbb{S}^{n-1}$, then $|f|$ is subharmonic (cf. [20, Proposition 1.5.4]) and therefore, the Poisson integral, $\mathcal{P}|f|$, of the boundary function of $|f|$ satisfies $\mathcal{P}|f| - |f| \ge 0$ in $\mathbb{B}^n$. The following theorem is a generalization of [19, Lemma 1].

Theorem 4. *Let ω be a regular majorant. A function f pluriharmonic in $\mathbb{B}^n$ belongs to $\Lambda_\omega(\mathbb{B}^n)$ if and only if for each $i \in \{1, \cdots, n\}$,*

$$\left|\frac{\partial f}{\partial z_i}(z)\right| \leq C\frac{\omega(1-|z|)}{1-|z|} \quad \text{and} \quad \left|\frac{\partial f}{\partial \overline{z}_i}(z)\right| \leq C\frac{\omega(1-|z|)}{1-|z|}$$

for some constant C depending only on f, ω and n.

2. The proofs of Theorems 1 and 2

In what follows, we always use C to denote a positive constant which is independent of the variables z and may vary with each occurrence (even within a single calculation).

We begin this section with some lemmas which will be useful in the proofs of the main results.

Lemma 1. *Let f be a real pluriharmonic function of $\mathbb{B}^n$ with $|f(z)| < 1$. Then*

$$|\nabla f(z)| \leq \frac{2\sqrt{n}}{\pi}\left(\frac{1-|f(z)|^2}{1-|z|^2}\right),$$

where $\nabla f = (f_{z_1}, \cdots, f_{z_n})$.

Proof. For $z = (z_1, \cdots, z_n) \in \mathbb{B}^n$, let $z_1 = \sqrt{1-|z_2|^2 - \cdots - |z_n|^2}\lambda_0$, where $\lambda_0 \in \mathbb{D}$, and let

$$f_0(\lambda) = f(\sqrt{1-|z_2|^2 - \cdots - |z_n|^2}\lambda, z_2, \cdots, z_n)$$

in $\mathbb{D}$. By [11, Theorem 1.8],

$$\left|(f_0)_\lambda(\lambda)\right| \leq \frac{2}{\pi}\frac{1-|f_0(\lambda)|^2}{1-|\lambda|^2},$$

and then

$$\begin{aligned}
|f_{z_1}(z)| &= \frac{\left|(f_0)_\lambda(\lambda_0)\right|}{\sqrt{1-|z_2|^2-\cdots-|z_n|^2}}\\
&\leq \frac{2}{\pi}\frac{1-|f_0(\lambda_0)|^2}{(1-|\lambda_0|^2)\sqrt{1-|z_2|^2-\cdots-|z_n|^2}}\\
&\leq \frac{2}{\pi}\frac{1-|f(z)|^2}{(1-|\lambda_0|^2)(1-|z_2|^2-\cdots-|z_n|^2)}\\
&= \frac{2}{\pi}\frac{1-|f(z)|^2}{1-|z|^2}.
\end{aligned}$$

Similarly, for $i \in \{1, \cdots, n\}$ and $i \neq 1$,

$$|f_{z_i}(z)| \leq \frac{2}{\pi}\frac{1-|f(z)|^2}{1-|z|^2}.$$

Hence

$$|\nabla f(z)| \leq \frac{2\sqrt{n}}{\pi}\left(\frac{1-|f(z)|^2}{1-|z|^2}\right).$$

□

Lemma 2. *Let ω be a fast majorant, and let Ω be a Λ_ω-extension domain. If $h = u + iv$ is a holomorphic function in Ω and $u \in \Lambda_\omega(\Omega)$, then $h \in \Lambda_\omega(\Omega)$.*

Proof. For fixed $z \in \Omega$, let

$$M_z = \sup\{|u(\xi)| : |\xi - z| < d_\Omega(z)\}$$

and

$$U(w) = u(z + d_\Omega(z)w)/M_z, \quad w \in \mathbb{B}^n.$$

Obviously, U is pluriharmonic in $\mathbb{B}^n$ and $|U(w)| < 1$. By Lemma 1,

$$|\nabla U(0)| \leq \frac{2\sqrt{n}}{\pi}(1-|U(0)|^2) \leq \frac{4\sqrt{n}}{\pi}(1-|U(0)|).$$

Since $\nabla U(0) = \frac{d_\Omega(z)}{M_z}\nabla u(z)$ and $U(0) = u(z)/M_z$, it follows that

$$d_\Omega(z)|\nabla u(z)| \le \frac{4\sqrt{n}}{\pi}(M_z - |u(z)|).$$

Hence

$$d_\Omega(z)|\nabla h(z)| \le \frac{8\sqrt{n}}{\pi}(M_z - |u(z)|).$$

For $\xi \in \mathbb{B}^n(z, d_\Omega(z))$, we have

$$\Big||u(\xi)| - |u(z)|\Big| \le |u(\xi) - u(z)| \le C\omega(d_\Omega(z)),$$

and so

$$M_z - |u(z)| = \sup_{\xi\in\mathbb{B}^n(z,d_\Omega(z))} \Big||u(\xi)| - |u(z)|\Big| \le C\omega(d_\Omega(z)),$$

whence

$$d_\Omega(z)|\nabla h(z)| \le \frac{8C\sqrt{n}}{\pi}\omega(d_\Omega(z)). \tag{5}$$

For a pair of points $z_1, z_2 \in \Omega$, we let $\gamma \subset \Omega$ be a rectifiable curve which joins z_1 and z_2 satisfying (4). Integrating (5) along γ, we obtain that

$$|h(z_1) - h(z_2)| \le \frac{8C\sqrt{n}}{\pi}\int_\gamma \frac{\omega(d_\Omega(z))}{d_\Omega(z)}\,ds(z).$$

Hence (4) yields

$$|h(z_1) - h(z_2)| \le C\omega(|z_1 - z_2|),$$

which completes the proof. □

We remark that Lemma 2 is a generalization of [8, Remark A] to the case of pluriharmonic mappings, and our result shows that the condition "f being bounded" in [8, Remark A] is not necessary.

Lemma 3. *Let ω be a fast majorant, and let $f = h+\overline{g}$ be a pluriharmonic mapping in a simply connected Λ_ω-extension domain Ω. Then $f \in \Lambda_\omega(\Omega)$ if and only if $h \in \Lambda_\omega(\Omega)$ and $g \in \Lambda_\omega(\Omega)$.*

Proof. Let $f = h + \overline{g} = u + iv \in \Lambda_\omega(\Omega)$, where u and v are real pluriharmonic functions. Then there are some holomorphic functions h_1 and g_1 such that

$$u = \frac{h_1 + \overline{h_1}}{2} \text{ and } v = \frac{g_1 + \overline{g_1}}{2}.$$

It follows from

$$f = h + \overline{g} = \frac{h_1 + ig_1}{2} + \frac{\overline{h_1} + i\overline{g_1}}{2}$$

that

$$h = \frac{h_1 + ig_1}{2} + c_1 \text{ and } g = \frac{h_1 - ig_1}{2} + c_2,$$

where c_1 and c_2 are constants with $c_1+\overline{c}_2 = 0$. Since both f and $\overline{f}$ belong to $\Lambda_\omega(\Omega)$, it follows that $u, v \in \Lambda_\omega(\Omega)$. By Lemma 2, we know $h_1, g_1 \in \Lambda_\omega(\Omega)$. Hence both $h, g \in \Lambda_\omega(\Omega)$.

The proof for the sufficiency is obvious. □

Lemma 4. *Let ω be a fast majorant and Ω be a domain in $\mathbb{C}^n$. If $h = u + iv$ is a holomorphic function in Ω and $u \in \Lambda_{\omega,int}(\Omega)$, then $h \in \Lambda_{\omega,int}(\Omega)$.*

Proof. For fixed $z \in \Omega$, let M_z be the same as in the proof of Lemma 2. By the similar reasoning as in the proof of Lemma 2, we have

$$d_\Omega(z)|\nabla h(z)| \leq \frac{8\sqrt{n}}{\pi}(M_z - |u(z)|\,). \tag{6}$$

Let $w \in B_z = \mathbb{B}^n(z, d_\Omega(z))$. Then

$$\Big||u(w)| - |u(z)|\Big| \leq |u(w) - u(z)| \leq C d_{\omega,\Omega}(w, z) \leq C \int_{[w,z]} \frac{\omega\big(d_\Omega(\eta)\big)}{d_\Omega(\eta)} ds(\eta),$$

where $[w, z]$ denotes the line segment with endpoints w and z. It follows that

$$d_{B_z}(\eta) \leq d_\Omega(\eta) \;\; \text{for} \;\; \eta \in B_z,$$

and then

$$\frac{\omega\big(d_\Omega(\eta))\big)}{d_\Omega(\eta)} \leq \frac{\omega\big(d_{B_z}(\eta)\big)}{d_{B_z}(\eta)}.$$

Note that $d_{B_z}(\eta) = d_\Omega(z) - |\eta - z|$. We have

$$\Big||u(w)| - |u(z)|\Big| \leq C \int_0^{d_\Omega(z)} \frac{\omega(t)}{t} dt \leq C\omega(d_\Omega(z)),$$

whence

$$M_z - |u(z)| \leq C\omega(d_\Omega(z))$$

and so (6) implies

$$d_\Omega(z)|\nabla h(z)| \leq C\omega(d_\Omega(z)). \tag{7}$$

For any two points $z_1, z_2 \in \Omega$, we let $\gamma \subset \Omega$ be a rectifiable curve which joins z_1 and z_2. Integrating (7) along γ leads to

$$|h(z_1) - h(z_2)| \leq C \int_\gamma \frac{\omega(d_G(z))}{d_G(z)}\, ds(z),$$

which shows that $h \in \Lambda_{\omega,int}(\Omega)$. □

Lemma 5. *Let ω be a fast majorant, and let $f = h + \overline{g}$ be a pluriharmonic mapping in a simply connected domain Ω. Then $f \in \Lambda_{\omega,int}(\Omega)$ if and only if $h \in \Lambda_{\omega,int}(\Omega)$ and $g \in \Lambda_{\omega,int}(\Omega)$.*

Proof. The proof follows from Lemma 4 and the similar reasoning as in the proof of Lemma 3. □

2.1. The proof of Theorem 1

The proof follows from Lemma 3 and [8, Theorem 1]. □

2.2. The proof of Theorem 2

The proof follows from Lemma 5 and [8, Theorem 3]. □

3. The proofs of Theorems 3 and 4

We begin this section with two lemmas.

Lemma 6. *If ω is a regular majorant and $\alpha \in (0,1)$, then $\omega_\alpha(t) = \omega(\alpha t)$ is a regular majorant.*

Lemma 7. *Let ω be a regular majorant, and let f be a harmonic mapping in $\mathbb{D}$. Then $f \in \Lambda_\omega(\mathbb{D})$ if and only if*

$$|f_z(z)| \le C\frac{\omega(1-|z|)}{1-|z|} \quad \text{and} \quad |f_{\bar z}(z)| \le C\frac{\omega(1-|z|)}{1-|z|}$$

for some constant C depending only on ω and f.

Proof. Assume that f belongs to $\Lambda_\omega(\mathbb{D})$. Since for $z \in \{z : |z| < r < 1\}$,

$$f(z) = \int_T \mathrm{Re}\Big(\frac{r\zeta + z}{r\zeta - z}\Big) f(r\zeta)\, dm(\zeta),$$

where dm denotes the normalized arc length measure on T, it follows that

$$f_z(z) = \int_T \frac{r\zeta}{(r\zeta - z)^2} f(r\zeta)\, dm(\zeta),$$

and so

$$\begin{aligned} |f_z(z)| &= \Big|\int_T \frac{r\zeta}{(r\zeta - z)^2}\Big(f(r\zeta) - f(z)\Big) dm(\zeta)\Big| \\ &\le \frac{1}{r^2 - |z|^2}\int_T \frac{r^2 - |z|^2}{|r\zeta - z|^2}|f(r\zeta) - f(z)|\, dm(\zeta). \end{aligned} \tag{8}$$

Obviously,

$$\begin{aligned} |f(r\zeta) - f(z)| &\le \Big|f(r\zeta) - f\Big(\tfrac{r}{|z|}z\Big)\Big| + \Big|f\Big(\tfrac{r}{|z|}z\Big) - f(z)\Big| \\ &\le \|f\|_{\Lambda_\omega(\mathbb{D})}\,\omega\left(\Big|r\zeta - \frac{r}{|z|}z\Big| + \omega(r - |z|)\right) \\ &= \|f\|_{\Lambda_\omega(\mathbb{D})}\,\omega\left(r\Big|\zeta - \frac{z}{|z|}\Big| + \omega(r - |z|)\right). \end{aligned}$$

Then we infer from (8), Lemma 6 and [7, Lemma 2] that

$$\begin{aligned} |f_z(z)| &\le \|f\|_{\Lambda_\omega(\mathbb{D})}\left(\int_T \frac{\omega(r|\zeta - \frac{z}{|z|}|)}{r^2|\zeta - \frac{z}{r}|^2}dm(\zeta) + \frac{\omega(r - |z|)}{r^2 - |z|^2}\right) \\ &\le \|f\|_{\Lambda_\omega(\mathbb{D})}\left(C\frac{\omega(r(1-|z|))}{r(1-|z|)} + \frac{\omega(r - |z|)}{r(r - |z|)}\right). \end{aligned}$$

Letting $r \to 1$ shows

$$|f_z(z)| \le C\frac{\omega(1-|z|)}{1-|z|}.$$

Note that $f \in \Lambda_\omega(\mathbb{D})$ if and only if $\overline{f} \in \Lambda_\omega(\mathbb{D})$. A similar argument as above implies

$$|f_{\bar z}(z)| = |\overline{f}_z(z)| \le C\frac{\omega(1-|z|)}{1-|z|}.$$

Hence the proof of the necessity is complete.

For the proof of the sufficiency, we assume that $f = h + \overline{g}$. Then

$$|h'(z)| = |f_z(z)| \le C\frac{\omega(1-|z|)}{1-|z|} \quad \text{and} \quad |g'(z)| = |f_{\bar z}(z)| \le C\frac{\omega(1-|z|)}{1-|z|}$$

for some constant C depending only on ω and f. By [19, Lemma 1], we know that $h \in \Lambda_\omega(\mathbb{D})$ and $g \in \Lambda_\omega(\mathbb{D})$, which implies $f \in \Lambda_\omega(\mathbb{D})$. $\square$

3.1. **The proof of Theorem 3**

It follows from Lemma 3 that $f \in \Lambda_\omega(\mathbb{B}^n)$ if and only if $h \in \Lambda_\omega(\mathbb{B}^n)$ and $g \in \Lambda_\omega(\mathbb{B}^n)$. By [8, Theorem 1] and [8, Proposition 1], the proof of the theorem is finished. □

3.2. **The proof of Theorem 4**

Assume that $f \in \Lambda_\omega(\mathbb{B}^n)$. For $z = (z_1, \cdots, z_n) \in \mathbb{B}^n$, let $z_1 = \sqrt{1 - |z_2|^2 - \cdots - |z_n|^2}\lambda_0$, where $\lambda_0 \in \mathbb{D}$, and let

$$f_0(\lambda) = f(\sqrt{1 - |z_2|^2 - \cdots - |z_n|^2}\lambda, z_2, \cdots, z_n)$$

in $\mathbb{D}$. For $\lambda_1, \lambda_2 \in \mathbb{D}$,

$$\begin{aligned} \left|f_0(\lambda_1) - f_0(\lambda_2)\right| &= |f(\sqrt{1 - |z_2|^2 - \cdots - |z_n|^2}\lambda_1, z_2, \cdots, z_n) \\ &\quad - f(\sqrt{1 - |z_2|^2 - \cdots - |z_n|^2}\lambda_2, z_2, \cdots, z_n)| \\ &\leq C\omega(\sqrt{1 - |z_2|^2 - \cdots - |z_n|^2}|\lambda_1 - \lambda_2|). \end{aligned}$$

Let $\omega_1(t) = \omega(\sqrt{1 - |z_2|^2 - \cdots - |z_n|^2}t)$. Then, by Lemma 6, ω_1 is a regular majorant and $f_0 \in \Lambda_{\omega_1}(\mathbb{D})$. By Lemma 7,

$$\left|\frac{\partial f_0}{\partial \lambda}(\lambda)\right| \leq C\frac{\omega_1(1 - |\lambda|)}{1 - |\lambda|} \text{ and } \left|\frac{\partial f_0}{\partial \bar{\lambda}}(\lambda)\right| \leq C\frac{\omega_1(1 - |\lambda|)}{1 - |\lambda|},$$

and then

$$\begin{aligned} &\left|\frac{\partial f}{\partial z_1}\left(\sqrt{1 - |z_2|^2 - \cdots - |z_n|^2}\lambda_0, z_2, \cdots, z_n\right)\right| \\ &= \frac{\left|\frac{\partial f_0}{\partial \lambda}(\lambda_0)\right|}{\sqrt{1 - |z_2|^2 - \cdots - |z_n|^2}} \\ &\leq C\frac{\omega(\sqrt{1 - |z_2|^2 - \cdots - |z_n|^2}(1 - |\lambda_0|))}{\sqrt{1 - |z_2|^2 - \cdots - |z_n|^2}(1 - |\lambda_0|)} \\ &\leq C\frac{\omega((1 - |z_2|^2 - \cdots - |z_n|^2)(1 - |\lambda_0|))}{(1 - |z_2|^2 - \cdots - |z_n|^2)(1 - |\lambda_0|)} \\ &\leq C\frac{\omega((1 - |z|)/2)}{(1 - |z|)/2} \\ &\leq C\frac{\omega(1 - |z|)}{1 - |z|}. \end{aligned}$$

Similarly,

$$\left|\frac{\partial f}{\partial \bar{z}_1}(\sqrt{1 - |z_2|^2 - \cdots - |z_n|^2}\lambda_0, z_2, \cdots, z_n)\right| \leq C\frac{\omega(1 - |z|)}{1 - |z|}.$$

Similar arguments as above also show that for each $i \in \{1, \cdots, n\}$ $(i \neq 1)$ and all $z \in \mathbb{B}^n$,

$$\left|\frac{\partial f}{\partial z_i}(z)\right| \leq C\frac{\omega(1 - |z|)}{1 - |z|} \text{ and } \left|\frac{\partial f}{\partial \bar{z}_i}(z)\right| \leq C\frac{\omega(1 - |z|)}{1 - |z|}.$$

Conversely, suppose that for each $i \in \{1, \cdots, n\}$ and all $z \in \mathbb{B}^n$,

$$\left|\frac{\partial f}{\partial z_i}(z)\right| \leq C\frac{\omega(1 - |z|)}{1 - |z|} \text{ and } \left|\frac{\partial f}{\partial \bar{z}_i}(z)\right| \leq C\frac{\omega(1 - |z|)}{1 - |z|}$$

for some constant C. Let $z, w \in \mathbb{B}^n$. Since $\mathbb{B}^n$ is a Λ_ω-extension domain, it follows that z, w can be joined by a rectifiable curve $\gamma \subset \Omega$ satisfying

$$\int_\gamma \frac{\omega(1 - |z|)}{1 - |z|}\, ds(z) \leq C\omega(|z - w|).$$

Hence

$$\begin{aligned} |f(z) - f(w)| &\leq \int_{\gamma} \Big(|\nabla h(z)| + |\nabla g(z)|\Big) ds(z) \\ &\leq C \int_{\gamma} \frac{\omega(1 - |z|)}{1 - |z|} ds(z) \\ &\leq C\omega(|z - w|), \end{aligned}$$

which shows that $f \in \Lambda_{\omega}(\mathbb{B}^n)$.

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