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On some properties of solutions of the p-harmonic equation

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Abstract. A $2p$-times continuously differentiable complex-valued function $f = u+iv$ in a simply connected domain $\Omega \subseteq \mathbb{C}$ is *p-harmonic* if f satisfies the p-harmonic equation $\Delta^p f = 0$. In this paper, we investigate the properties of p-harmonic mappings in the unit disk $|z| < 1$. First, we discuss the convexity, the starlikeness and the region of variability of some classes of p-harmonic mappings. Then we prove the existence of Landau constant for the class of functions of the form $Df = zf_z - \overline{z}f_{\overline{z}}$, where f is p-harmonic in $|z| < 1$. Also, we discuss the region of variability for certain p-harmonic mappings. At the end, as a consequence of the earlier results of the authors, we present explicit upper estimates for Bloch norm for bi- and tri-harmonic mappings.

1. Introduction and Preliminaries

A complex-valued function $f = u + iv$ in a simply connected domain $\Omega \subseteq \mathbb{C}$ is called *p-harmonic* if u and v are p-harmonic in Ω, i.e. f satisfies the p-harmonic equation $\Delta^p f = 0$, where

$$\Delta^p f = \underbrace{\Delta \cdots \Delta}_{p} f,$$

where p is a positive integer and Δ represents the Laplacian operator

$$\Delta := 4\frac{\partial^2}{\partial z \partial \overline{z}} = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}.$$

Throughout this paper we consider p-harmonic mappings of the unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$. Obviously, when $p = 1$ (resp. $p = 2$), f is harmonic (resp. biharmonic). The properties of harmonic [11, 15] and biharmonic [1–3, 18, 19] mappings have been investigated by many authors. Concerning p-harmonic mappings, we easily have the following characterization.

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Proposition 1.1. *A mapping f is p-harmonic in $\mathbb{D}$ if and only if f has the following representation:*

$$f(z) = \sum_{k=1}^{p} |z|^{2(k-1)} G_{p-k+1}(z), \tag{1}$$

where G_{p-k+1} is harmonic for each $k \in \{1, \dots, p\}$.

Proof. We only need to prove the necessity since the proof for the sufficiency part is obvious. Again, as the cases $p = 1, 2$ are well-known, it suffices to prove the result for $p \geq 3$. We shall prove the proposition by the method of induction. So, we assume that the proposition is true for $p = n\,(\geq 3)$.

Let F be an $(n+1)$-harmonic mapping in $\mathbb{D}$. By assumption, ΔF is n-harmonic and so can be represented as

$$\Delta F(z) = \sum_{k=1}^{n} |z|^{2(k-1)} G_{n-k+1}(z),$$

where G_{n-k+1} $(1 \leq k \leq n)$ are harmonic functions with

$$G_{n-k+1}(z) = a_{0,n-k+1} + \sum_{j=1}^{\infty} a_{j,n-k+1} z^j + \sum_{j=1}^{\infty} \overline{b}_{j,n-k+1} \overline{z}^j \quad \text{for } k \in \{1, \dots, n\}.$$

Then

$$\int_0^z \int_0^{\overline{z}} \Delta F \, d\overline{z}\, dz \quad = \quad \sum_{k=1}^{n} |z|^{2k} T_{p-k+1}(z) + g(z),$$

where

$$T_{p-k+1}(z) = \sum_{k=1}^{n} \left(\frac{a_{0,n-k+1}}{k^2} + \sum_{j=1}^{\infty} \frac{a_{j,n-k+1}}{k(k+j)} z^j + \sum_{j=1}^{\infty} \frac{\overline{b}_{j,n-k+1}}{k(k+j)} \overline{z}^j \right)$$

and g is a harmonic function in $\mathbb{D}$. A rearrangement of the series in the sum shows that (1) holds for $p = n + 1$. $\square$

We remark that the representation (1) continues to hold even if f is p-harmonic in a simply connected domain Ω.

For a sense-preserving C^1-mapping (i.e. continuously differentiable), we let

$$\lambda_f = |f_z| - |f_{\overline{z}}| \text{ and } \Lambda_f = |f_z| + |f_{\overline{z}}|$$

so that the Jacobian J_f of f takes the form

$$J_f = \lambda_f \Lambda_f = |f_z|^2 - |f_{\overline{z}}|^2 > 0.$$

In [4], the authors obtained sufficient conditions for the univalence of C^1-functions. Now we introduce the concepts of starlikeness and convexity of C^1-functions.

Definition 1.2. *A C^1-mapping f with $f(0) = 0$ is called starlike if f maps $\mathbb{D}$ univalently onto a domain Ω that is starlike with respect to the origin, i.e. for every $w \in \Omega$ the line segment $[0, w]$ joining 0 and w is contained in Ω. It is known that f is starlike if it is sense-preserving, $f(0) = 0$, $f(z) \neq 0$ for all $z \in \mathbb{D} \setminus \{0\}$ and*

$$\frac{\partial}{\partial t}\Big(\arg f(re^{it}) \Big) := \mathrm{Re}\left(\frac{Df(z)}{f(z)} \right) > 0 \quad \textit{for all } z = re^{it} \in \mathbb{D} \setminus \{0\},$$

where $Df = zf_z - \overline{z} f_{\overline{z}}$ (cf. [23, Theorem 1]).

Definition 1.3. *Let f and Df belong to $C^1(\mathbb{D})$. Then we say that f is convex in $\mathbb{D}$ if it is sense-preserving, $f(0) = 0$, $f(z) \cdot Df(z) \neq 0$ for all $z \in \mathbb{D} \setminus \{0\}$ and*

$$\mathrm{Re}\left(\frac{D^2 f(z)}{Df(z)}\right) > 0 \quad \textit{for all } z \in \mathbb{D} \setminus \{0\}.$$

As $\arg Df(re^{it})$ represents the argument of the outer normal to the curve $C_r = \{f(re^{i\theta}) : 0 \leq \theta < 2\pi\}$ at the point $f(re^{it})$, the last condition gives that

$$\frac{\partial}{\partial t}\Big(\arg Df(re^{it})\Big) = \mathrm{Re}\left(\frac{D^2 f(z)}{Df(z)}\right) > 0 \quad \text{for all } z = re^{it} \in \mathbb{D} \setminus \{0\},$$

showing that the curve C_r is convex for each $r \in (0, 1)$ (see [23, Theorem 2]). Non-analytic starlike and convex functions were studied by Mocanu in [23]. Harmonic starlike and harmonic convex functions were systematically studied by Clunie and Sheil-Small [11], and these two classes of functions have been studied extensively by many authors. See for instance, the book by Duren [15] and the references therein.

The complex differential operator

$$D = z\frac{\partial}{\partial z} - \overline{z}\frac{\partial}{\partial \overline{z}}$$

defined by Mocanu [23] on the class of complex-valued C^1-functions satisfies the usual product rule:

$$D(af + bg) = aD(f) + bD(g) \ \text{ and } \ D(fg) = fD(g) + gD(f),$$

where a, b are complex constants, f and g are C^1-functions. The operator D possesses a number of interesting properties. For instance, the operator D preserves both harmonicity and biharmonicity (see also [3]). In the case of p-harmonic mappings, we also have the following property of the operator D.

Proposition 1.4. *D preserves p-harmonicity.*

Proof. Let f be a p-harmonic mapping with the form

$$f(z) = \sum_{k=1}^{p} |z|^{2(k-1)} G_{p-k+1}(z),$$

where each $G_{p-k+1}(z)$ is harmonic in $\mathbb{D}$ for $k \in \{1, \ldots, p\}$. As $D\Big(|z|^2\Big) = 0$, the product rule shows that $D\Big(|z|^{2(k-1)}\Big) = 0$ for each $k \in \{1, \ldots, p\}$. In view of this and the fact that D preserves harmonicity gives that

$$\begin{aligned} D(f(z)) &= \sum_{k=1}^{p} \Big[|z|^{2(k-1)} D(G_{p-k+1}(z)) + D(|z|^{2(k-1)}) G_{p-k+1}(z)\Big] \\ &= \sum_{k=1}^{p} |z|^{2(k-1)} D(G_{p-k+1}(z)). \end{aligned}$$

□

One of the aims of this paper is to generalize the main results of Abdulhadi, et. al. [3] to the case of p-harmonic mappings. The corresponding generalizations are Theorems 3.1 and 3.3.

The classical theorem of Landau for bounded analytic functions states that if f is analytic in $\mathbb{D}$ with $f(0) = f'(0) - 1 = 0$, and $|f(z)| < M$ for $z \in \mathbb{D}$, then f is univalent in the disk $\mathbb{D}_\rho := \{z \in \mathbb{C} : |z| < \rho\}$ and in addition, the range $f(\mathbb{D}_\rho)$ contains a disk of radius $M\rho^2$ (cf. [20]), where

$$\rho = \frac{1}{M + \sqrt{M^2 - 1}}.$$

Recently, many authors considered Landau's theorem for planar harmonic mappings (see for example, [6, 8, 9, 13, 16, 22, 28]) and biharmonic mappings (see [1, 7, 8, 21]). In Section 4, we consider Landau's theorem for p-harmonic mappings with the form $D(f)$ when f belongs to certain classes of p-harmonic mappings. Our results are Theorems 4.1 and 4.2.

In a series of papers the second author with Yanagihara and Vasudevarao (see [24, 25, 29, 30]) have discussed the regions of variability for certain classes of univalent analytic functions in $\mathbb{D}$. In Section 5 (see Theorem 5.2), we solve a related problem for certain p-harmonic mappings. Finally, in Section 6, we present explicit upper estimates for Bloch norm for bi- and tri-harmonic mappings (see Corollaries 6.2 and 6.3).

2. Lemmas

For the proofs of our main results we require a number of lemmas. We begin to recall the following version of Schwarz lemma due to Heinz ([17, Lemma]) and Colonna [12, Theorem 3], see also [6, 8, 9].

Lemma 2.1. *Let f be a harmonic mapping of $\mathbb{D}$ such that $f(0) = 0$ and $f(\mathbb{D}) \subset \mathbb{D}$. Then*

$$|f(z)| \leq \frac{4}{\pi} \arctan |z| \leq \frac{4}{\pi}|z| \ \ for \ \ z \in \mathbb{D}$$

and

$$\Lambda_f(z) \leq \frac{4}{\pi} \frac{1}{(1-|z|^2)} \ \ for \ \ z \in \mathbb{D}.$$

Lemma 2.2. ([22, Lemma 2.1]) *Suppose that $f(z) = h(z) + \overline{g(z)}$ is a harmonic mapping of $\mathbb{D}$ with $h(z) = \sum_{n=1}^{\infty} a_n z^n$ and $g(z) = \sum_{n=1}^{\infty} b_n z^n$ for $z \in \mathbb{D}$. If $J_f(0) = 1$ and $|f(z)| < M$, then*

$$|a_n|,\ |b_n| \leq \sqrt{M^2-1},\ n = 2, 3, \ldots,$$

$$|a_n| + |b_n| \leq \sqrt{2M^2-2},\ n = 2, 3, \ldots$$

and

$$\lambda_f(0) \geq \lambda_0(M) := \begin{cases} \dfrac{\sqrt{2}}{\sqrt{M^2-1}+\sqrt{M^2+1}} & if\ 1 \leq M \leq M_0, \\ \dfrac{\pi}{4M} & if\ M > M_0, \end{cases} \tag{2}$$

where $M_0 = \frac{\pi}{2\sqrt[4]{2\pi^2-16}} \approx 1.1296$.

The following lemma concerning coefficient estimates for harmonic mappings is crucial in the proofs of Theorems 3.1 and 3.3. This lemma has been proved by the authors in [10] with an additional assumption that $f(0) = 0$. However, for the sake of clarity, we present a slightly different proof than that in [10].

Lemma 2.3. *Let $f = h + \overline{g}$ be a harmonic mapping of $\mathbb{D}$ such that $|f(z)| < M$ with $h(z) = \sum_{n=0}^{\infty} a_n z^n$ and $g(z) = \sum_{n=1}^{\infty} b_n z^n$. Then $|a_0| \leq M$ and for any $n \geq 1$*

$$|a_n| + |b_n| \leq \frac{4M}{\pi}. \tag{3}$$

The estimate (3) *is sharp. The extremal functions are $f(z) \equiv M$ or*

$$f_n(z) = \frac{2M\alpha}{\pi} \arg\left(\frac{1+\beta z^n}{1-\beta z^n}\right),$$

where $|\alpha| = |\beta| = 1$.

Proof. Without loss of generality, we assume that $|f(z)| < 1$. For $\theta \in [0, 2\pi)$, let

$$v_\theta(z) = \mathrm{Im}\,(e^{i\theta} f(z))$$

and observe that

$$v_\theta(z) = \mathrm{Im}\,(e^{i\theta}h(z) + \overline{e^{-i\theta}g(z)}) = \mathrm{Im}\,(e^{i\theta}h(z) - e^{-i\theta}g(z)).$$

Because $|v_\theta(z)| < 1$, it follows that

$$e^{i\theta}h(z) - e^{-i\theta}g(z) \prec K(z) = \lambda + \frac{2}{\pi}\log\left(\frac{1+z\xi}{1-z}\right),$$

where $\xi = e^{-i\pi \mathrm{Im}(\lambda)}$ and $\lambda = e^{i\theta}h(0) - e^{-i\theta}g(0)$. The superordinate function $K(z)$ maps $\mathbb{D}$ onto a convex domain with $K(0) = \lambda$ and $K'(0) = \frac{2}{\pi}(1+\xi)$, and therefore, by a theorem of Rogosinski [26, Theorem 2.3] (see also [14, Theorem 6.4]), it follows that

$$|a_n - e^{-2i\theta}b_n| \le \frac{2}{\pi}|1+\xi| \le \frac{4}{\pi} \quad \text{for } n = 1, 2, \ldots$$

and the desired inequality (3), with $M = 1$, is a consequence of the arbitrariness of θ in $[0, 2\pi)$.

For the proof of sharpness part, consider the functions

$$f_n(z) = \frac{2M\alpha}{\pi}\mathrm{Im}\left(\log\frac{1+\beta z^n}{1-\beta z^n}\right), \quad |\alpha| = |\beta| = 1,$$

whose values are confined to a diametral segment of the disk $\mathbb{D}_M$. Also,

$$f_n(z) = \frac{2M\alpha}{i\pi}\left(\sum_{k=1}^{\infty}\frac{1}{2k-1}(\beta z^n)^{2k-1} - \sum_{k=1}^{\infty}\frac{1}{2k-1}(\overline{\beta z^n})^{2k-1}\right),$$

which gives

$$|a_n| + |b_n| = \frac{4M}{\pi}.$$

The proof of the lemma is complete. □

As an immediate consequence of Lemmas 2.2 and 2.3, we have

Corollary 2.4. *Let $f = h + \overline{g}$ be a harmonic mapping of $\mathbb{D}$ with $h(z) = \sum_{n=1}^{\infty} a_n z^n$, $g(z) = \sum_{n=1}^{\infty} b_n z^n$ and $|f(z)| \le M$. If $J_f(0) = 1$ and $M \ge \frac{\pi}{\sqrt{\pi^2-8}}$, then for any $n \ge 2$,*

$$|a_n| + |b_n| \le \frac{4M}{\pi} \le \sqrt{2M^2 - 2}.$$

3. The convexity and the starlikeness

The following simple result can be used to generate (harmonic) starlike and convex functions.

Theorem 3.1. *Let f be a univalent p-harmonic mapping with the form*

$$f(z) = G(z)\sum_{k=1}^{p}\lambda_k |z|^{2(k-1)},$$

where G is a locally univalent harmonic mapping and λ_k $(k = 1, \ldots, p)$ are complex constants. Then we have the following:

(a) $\dfrac{D(f)}{f} = \dfrac{D(G)}{G}$ *and* $\dfrac{D^2(f))}{D(f)} = \dfrac{D^2(G))}{D(G)}$.

(b) *f is convex (resp. starlike) if and only if G is convex (resp. starlike).*

Proof. **(a)** The two equalities are immediate consequences of the formula

$$D\Big(G(z)\sum_{k=1}^{p}\lambda_k|z|^{2(k-1)}\Big) = D(G(z))\sum_{k=1}^{p}\lambda_k|z|^{2(k-1)}.$$

So, we omit the details.

(b) It suffices to prove the case of convexity since the proof for the starlikeness is similar.

Let $z = re^{it}$, where $0 < r < 1$ and $0 \le t < 2\pi$. Then

$$f(z) = G(z)\sum_{k=1}^{p}\lambda_k|z|^{2(k-1)} = G(re^{i\theta})\sum_{k=1}^{p}\lambda_k r^{2(k-1)},$$

so that

$$\frac{\partial f(re^{it})}{\partial t} = \frac{\partial G(re^{it})}{\partial t}\sum_{k=1}^{p}\lambda_k r^{2(k-1)}$$

and

$$\frac{\partial^2 f(re^{it})}{\partial t^2} = \frac{\partial^2 G(re^{it})}{\partial t^2}\sum_{k=1}^{p}\lambda_k r^{2(k-1)}.$$

Therefore Part **(a)** yields

$$\frac{\partial}{\partial t}\Big(\arg\frac{\partial f(re^{it})}{\partial t}\Big) = \mathrm{Re}\Big(\frac{D^2(f)}{D(f)}\Big) = \mathrm{Re}\Big(\frac{D^2(G)}{D(G)}\Big) = \frac{\partial}{\partial t}\Big(\arg\frac{\partial G(re^{it})}{\partial t}\Big),$$

from which the proof of Part **(b)** of this theorem follows. □

As an immediate consequence of Theorem 3.1**(a)**, we easily have the following.

Corollary 3.2. *Let f be a univalent p-harmonic mapping defined as in Theorem 3.1. If f is convex and $D(f)$ is univalent, then $D(f)$ is starlike.*

Abdulhadi, et. al. [3, Theorem 1] discussed the univalence and the starlikeness of biharmonic mappings in $\mathbb{D}$. A natural question is whether [3, Theorem 1] holds for p-harmonic mappings. The following result gives a partial answer to this problem.

Theorem 3.3. *Let f be a p-harmonic mapping of $\mathbb{D}$ satisfying $f(z) = |z|^{2(p-1)}G(z)$, where G is harmonic, orientation preserving and starlike. Then f is starlike univalent.*

Proof. We see that the Jacobian J_f of f is

$$\begin{aligned} J_f &= |f_z|^2 - |f_{\overline{z}}|^2 \\ &= |z|^{4(p-1)}(|G_z|^2 - |G_{\overline{z}}|^2) + 2(p-1)|z|^{4p-6}|G|^2\mathrm{Re}\left(\frac{D(G)}{G}\right) \\ &\ge |z|^{4(p-1)}(|G_z|^2 - |G_{\overline{z}}|^2). \end{aligned}$$

Hence $J_f(z) > 0$ when $0 < |z| < 1$ and obviously, $J_f(0) = 0$. The univalence of f follows from a standard argument as in the proof of [3, Theorem 1]. Finally, Theorem 3.1 implies that f is starlike. □

4. The Landau theorem

We now discuss the existence of the Laudau constant for two classes of p-harmonic mappings.

Theorem 4.1. *Let $f(z) = \sum_{k=1}^{p} |z|^{2(k-1)} G_{p-k+1}(z)$ be a p-harmonic mapping of $\mathbb{D}$ satisfying $\Delta G_{p-k+1}(z) = f(0) = G_p(0) = J_f(0) - 1 = 0$ and for any $z \in \mathbb{D}$, $|G_{p-k+1}(z)| \leq M$, where $M \geq 1$. Then there is a constant ρ $(0 < \rho < 1)$ such that $D(f)$ is univalent in $\mathbb{D}_\rho$, where ρ satisfies the following equation:*

$$\lambda_0(M) - \frac{T(M)}{(1-\rho)^2} \sum_{k=2}^{p} (2k-1)\rho^{2(k-1)} - \sum_{k=1}^{p} \frac{2T(M)\rho^{2k-1}}{(1-\rho)^3} - \frac{16M}{\pi^2} s_0 \arctan \rho = 0$$

with

$$s_0 = \left(\frac{\sqrt{17}-1}{\sqrt{17}-3} \right) \sqrt{\frac{2}{5-\sqrt{17}}} \approx 4.1996,$$

$$T(M) = \begin{cases} \sqrt{2M^2-2} & \text{if } 1 \leq M \leq M_1 := \dfrac{\pi}{\sqrt{\pi^2-8}} \approx 2.2976 \\ \dfrac{4M}{\pi} & \text{if } M > M_1 \end{cases} \tag{4}$$

and $\lambda_0(M)$ is given by (2)*. Moreover, the range $D(f)(\mathbb{D}_\rho)$ contains a univalent disk $\mathbb{D}_R$, where*

$$R = \rho \Big[\lambda_0(M) - \sum_{k=2}^{p} \frac{T(M)\rho^{2(k-1)}}{(1-\rho)^2} - \frac{16M}{\pi^2} s_0 \arctan \rho \Big].$$

Proof. For each $k \in \{1, 2, \ldots, p\}$, let

$$G_{p-k+1}(z) = a_{0,p-k+1} + \sum_{j=1}^{\infty} a_{j,p-k+1} z^j + \sum_{j=1}^{\infty} \overline{b}_{j,p-k+1} \overline{z}^j,$$

where $a_{0,p} = 0$. We define the function H as

$$H = D\left(\sum_{k=1}^{p} |z|^{2(k-1)} G_{p-k+1} \right) = \sum_{k=1}^{p} |z|^{2(k-1)} D(G_{p-k+1}).$$

Using Lemmas 2.2, 2.3 and Corollary 2.4, we have

$$|a_{n,p}| + |b_{n,p}| \leq T(M),$$

where $T(M)$ is given by (4), and

$$|a_{j,p-k+1}| + |b_{j,p-k+1}| \leq \frac{4M}{\pi}$$

for $j \geq 1$, $n \geq 2$ and $2 \leq k \leq p$.

We observe that

$$J_f(0) = |(G_p)_z(0)|^2 - |(G_p)_{\overline{z}}(0)|^2 = J_{G_p}(0) = 1$$

and hence by Lemmas 2.1 and 2.2, we have

$$\lambda_f(0) \geq \lambda_0(M),$$

where $\lambda_0(M)$ is given by (2). Now, we define

$$q(x) = \frac{2-x^2}{(1-x^2)x} \ (0 < x < 1).$$

Then there is an $r_0 = \sqrt{\frac{5-\sqrt{17}}{2}} \approx 0.66$ such that

$$q(r_0) = \min_{0<x<1} q(x) = \left(\frac{\sqrt{17}-1}{\sqrt{17}-3}\right)\sqrt{\frac{2}{5-\sqrt{17}}} = s_0.$$

For each $\theta \in [0, 2\pi)$, the function

$$G_\theta(z) = (G_p)_z(z) - (G_p)(0) + ((G_p)_{\overline{z}}(z) - (G_p)_{\overline{z}}(0))e^{i(\pi-2\theta)}$$

is clearly a harmonic mapping of $\mathbb{D}$ and satisfies $G_\theta(0) = 0$. Moreover, it follows from Lemma 2.1 that

$$\Lambda_{G_p}(z) \leq \frac{4M}{\pi}\frac{1}{1-|z|^2} \ \text{ for } z \in \mathbb{D}.$$

In particular, this observation yields that

$$|G_\theta(z)| \leq \Lambda_{G_p}(z) + \Lambda_{G_p}(0) \leq \frac{4M}{\pi}\Big(1 + \frac{1}{1-|z|^2}\Big) = \frac{4M}{\pi}|z|q(|z|) \tag{5}$$

for all $z \in \mathbb{D}$.

Since $xq(x) - 1 = \frac{1}{1-x^2}$ is an increasing function in the interval $(0, 1)$, the inequality (5) shows that for any $z \in \mathbb{D}_{r_0}$,

$$|G_\theta(z)| \leq \frac{4M}{\pi} m_0,$$

where $m_0 = (2 - r_0^2)/(1 - r_0^2)$. Next, we consider the mapping F defined on $\mathbb{D}$ by

$$F(z) = \frac{\pi}{4Mm_0} G_\theta(r_0 z).$$

Applying Lemma 2.1 to the function $F(z)$ yields that for $z \in \mathbb{D}_{r_0}$,

$$|G_\theta(z)| \leq \frac{16M}{\pi^2} m_0 \arctan\left(\frac{|z|}{r_0}\right) \leq \frac{16M}{\pi^2} s_0 \arctan|z|,$$

where $s_0 = m_0/r_0$.

Now, we fix ρ with $\rho \in (0, 1)$. To prove the univalency of H, we choose two distinct points z_1, z_2 in $\mathbb{D}_\rho$. Let $\gamma = \{(z_2 - z_1)t + z_1 : \ 0 \leq t \leq 1\}$ and $z_2 - z_1 = |z_1 - z_2|e^{i\theta}$. We find that

$$
\begin{aligned}
&|H(z_1) - H(z_2)| \\
&= \Big|\int_\gamma H_z(z)\,dz + H_{\overline{z}}(z)\,d\overline{z}\Big| \\
&\geq \Big|\int_\gamma (G_p)_z(0)\,dz - (G_p)_{\overline{z}}(0)\,d\overline{z}\Big| \\
&\quad -\Big|\int_\gamma \sum_{k=2}^{p} |z|^{2(k-1)}[z(G_{p-k+1})_{z^2}(z)\,dz - \overline{z}(G_{p-k+1})_{\overline{z}^2}(z)\,d\overline{z}]\Big| \\
&\quad -\Big|\int_\gamma \sum_{k=2}^{p} (k-1)|z|^{2(k-2)}[z^2(G_{p-k+1})_z(z)\,d\overline{z} - \overline{z}^2(G_{p-k+1})_{\overline{z}}(z)\,dz]\Big| \\
&\quad -\Big|\int_\gamma \sum_{k=2}^{p} k|z|^{2(k-1)}[(G_{p-k+1})_z(z)\,dz - (G_{p-k+1})_{\overline{z}}(z)\,d\overline{z}]\Big| \\
&\quad -\Big|\int_\gamma [(G_p)_z(z) - (G_p)_z(0)]\,dz - [(G_p)_{\overline{z}}(z) - (G_p)_{\overline{z}}(0)]\,d\overline{z}\Big| \\
&\geq |z_1 - z_2|\Big\{\lambda_f(0) - |G_\theta(\rho)| \\
&\quad - \sum_{k=1}^{p} \rho^{2(k-1)} \sum_{n=2}^{\infty} n(n-1)(|a_{n,p-k+1}| + |b_{n,p-k+1}|)\rho^{n-1} \\
&\quad - \sum_{k=2}^{p} (2k-1)\rho^{2(k-2)} \sum_{n=1}^{\infty} n(|a_{n,p-k+1}| + |b_{n,p-k+1}|)\rho^{n+1}\Big\} \\
&> |z_1 - z_2|\Big[\lambda_0(M) - \frac{T(M)}{(1-\rho)^2}\sum_{k=2}^{p}(2k-1)\rho^{2(k-1)} \\
&\quad - \sum_{k=1}^{p} \frac{2T(M)\rho^{2k-1}}{(1-\rho)^3} - \frac{16M}{\pi^2}s_0 \arctan\rho\Big].
\end{aligned}
$$

Let

$$
P(\rho) = \lambda_0(M) - \frac{T(M)}{(1-\rho)^2}\sum_{k=2}^{p}(2k-1)\rho^{2(k-1)} - \sum_{k=1}^{p} \frac{2T(M)\rho^{2k-1}}{(1-\rho)^3} - \frac{16M}{\pi^2}s_0 \arctan\rho.
$$

Then it is easy to verify that $P(\rho)$ is a decreasing function on the interval $(0, 1)$,

$$
\lim_{\rho\to 0+} P(\rho) = \lambda_0(M) \ \text{ and } \ \lim_{\rho\to 1-} P(\rho) = -\infty.
$$

Hence there exists a unique ρ_0 in $(0, 1)$ satisfying $P(\rho_0) = 0$. This observation shows that $|H(z_1) - H(z_2)| > 0$ for arbitrary two distinct points z_1, z_2 in $|z| < \rho_0$ which proves the univalency of $D(F)$ in $\mathbb{D}_{\rho_0}$.

M	p	$\rho = \rho(M, p)$	$R = R(M, \rho(M, p))$	ρ'	R'
1.1296	2	0.0714741	0.0101601	0.0420157	0.00945379
2	2	0.0206783	0.00227639	0.0139439	0.00164502
2.2976	2	0.0155966	0.00151523	0.0106132	0.00108021
3	2	0.00922255	0.00067425	0.00626141	0.000482413
1.1296	3	0.071463	0.0101647	–	–
2	3	0.0206782	0.00227641	–	–
2.2976	3	0.0155966	0.00151523	–	–
3	3	0.00922254	0.000674251	–	–
1.1296	4	0.0714629	0.0101647	–	–
2	4	0.0206782	0.00227641	–	–
2.2976	4	0.0155966	0.00151523	–	–
3	4	0.00922254	0.000674251	–	–

Table 1: Values of ρ and R for Theorem 4.1 for $p = 2$, and the corresponding values of ρ' and R' of [7, Theorem 1.1] (for $p = 2$)

For any z with $|z| = \rho_0$, we have

$$\begin{aligned}
|H(z)| &= \Big|\sum_{k=1}^{p} |z|^{2(k-1)}[z(G_{p-k+1})_z(z) - \overline{z}(G_{p-k+1})_{\overline{z}}(z)]\Big| \\
&\geq \Big|z(G_p)_z(0) - \overline{z}(G_p)_{\overline{z}}(0)\Big| \\
&\quad - \Big|z[(G_p)_z(z) - (G_p)_z(0)] - \overline{z}[(G_p)_{\overline{z}}(z) - (G_p)_{\overline{z}}(0)]\Big| \\
&\quad - \Big|\sum_{k=2}^{p} |z|^{2(k-1)}[z(G_{p-k+1})_z(z) - \overline{z}(G_{p-k+1})_{\overline{z}}(z)]\Big| \\
&\geq \rho_0\Big[\lambda_0(M) - \sum_{k=2}^{p} \frac{T(M)\rho_0^{2(k-1)}}{(1-\rho_0)^2} - \frac{16M}{\pi^2} s_0 \arctan \rho_0\Big] \\
&= R
\end{aligned}$$

and the proof of the theorem is complete. □

From Table 1, we see that Theorem 4.1 improves Theorem 1.1 of [7] for the case $p = 2$, and the results for the rest of the values of p are new. In Table 1, third and fourth columns refer to values obtained from Theorem 4.1 for cases $p = 2, 3, 4$ for certain choices of M, while the right two columns correspond to the values obtained from [7, Theorem 1.1] for the case $p = 2$.

Theorem 4.2. *Let $f(z) = |z|^{2(p-1)}G(z)$ be a p-harmonic mapping of $\mathbb{D}$ satisfying $G(0) = J_G(0) - 1 = 0$ and $|G(z)| \leq M$, where $M \geq 1$ and G is harmonic. Then there is a constant ρ $(0 < \rho < 1)$ such that $D(f)$ is univalent in $\mathbb{D}_\rho$, where ρ satisfies the following equation:*

$$\lambda_0(M) - \frac{48M}{\pi^2} s_0 \arctan \rho - \frac{2T(M)\rho}{(1-\rho)^3} = 0,$$

where the constants s_0, $\lambda_0(M)$ and $T(M)$ are the same as in Theorem 4.1. Moreover, the range $D(f)(\mathbb{D}_\rho)$ contains a univalent disk $\mathbb{D}_R$, where

$$R = \rho^{2p-1}\Big[\lambda_0(M) - \frac{16M}{\pi^2} s_0 \arctan \rho\Big].$$

Especially, if $M = 1$, then $G(z) = z$, i.e. $f(z) = |z|^{2(p-1)}z$ which is univalent in $\mathbb{D}$.

Proof. Let $G(z) = \sum_{n=1}^{\infty} a_n z^n + \sum_{n=1}^{\infty} \overline{b}_n \overline{z}_n$. Using Lemmas 2.2, 2.3 and Corollary 2.4, we have

$$|a_n| + |b_n| \leq T(M) \ \text{ for } n \geq 2.$$

Note that

$$J_G(0) = |a_1|^2 - |b_1|^2 = 1$$

and hence, by Lemmas 2.1 and 2.2, we have

$$\lambda_G(0) \geq \lambda_0(M).$$

Next, we set $H = D(f) = |z|^{2(p-1)} D(G)$ and fix ρ with $\rho \in (0, 1)$. To prove the univalency of f, we choose two distinct points z_1, z_2 in $\mathbb{D}_\rho$. Let $\gamma = \{(z_2 - z_1)t + z_1 : \ 0 \leq t \leq 1\}$ and $z_2 - z_1 = |z_1 - z_2| e^{i\theta}$. Then

$$\begin{aligned}
|H(z_1) - H(z_2)| &= \left| \int_{[z_1, z_2]} H_z(z)\, dz + H_{\overline{z}}(z)\, d\overline{z} \right| \\
&= \Big| \int_{[z_1, z_2]} p|z|^{2(p-1)} (G_z(z)\, dz - G_{\overline{z}}(z)\, d\overline{z}) \\
&\quad + |z|^{2(p-1)} (z G_{z^2}(z)\, dz - \overline{z} G_{\overline{z}^2}(z)\, d\overline{z}) \\
&\quad + (p-1)|z|^{2(p-2)} (z^2 G_z(z)\, d\overline{z} - \overline{z}^2 G_{\overline{z}}(z)\, dz) \Big| \\
&\geq \Big| \int_{[z_1, z_2]} \Big[G_z(0)(p|z|^{2(p-1)}\, dz + (p-1)|z|^{2(p-2)} z^2\, d\overline{z}) \\
&\quad - G_{\overline{z}}(0)(p|z|^{2(p-1)}\, d\overline{z} - (p-1)|z|^{2(p-2)} \overline{z}^2\, dz) \Big] \Big| \\
&\quad - p \left| \int_{[z_1, z_2]} |z|^{2(p-1)} \Big[(G_z(z) - G_z(0))\, dz - (G_{\overline{z}}(z) - G_{\overline{z}}(0))\, d\overline{z} \Big] \right| \\
&\quad - \Big| (p-1) \int_{[z_1, z_2]} |z|^{2(p-1)} \Big[\frac{z}{\overline{z}} (G_z(z) - G_z(0))\, d\overline{z} \\
&\quad - \frac{\overline{z}}{z} (G_{\overline{z}}(z) - G_{\overline{z}}(0))\, dz \Big] \Big| \\
&\quad - \left| \int_{[z_1, z_2]} |z|^{2(p-1)} (z G_{z^2}(z)\, dz - \overline{z} G_{\overline{z}^2}(z)\, d\overline{z}) \right| \\
&\geq |z_1 - z_2| \Big(\int_0^1 |z|^{2(p-1)} dt \Big) \Big\{ \lambda_0(M) - \frac{48M}{\pi^2} s_0 \arctan \rho \\
&\quad - \sum_{n=2}^{\infty} n(n-1)(|a_n| + |b_n|) \rho^{n-1} \Big\} \\
&> |z_1 - z_2| \Big(\int_0^1 |z|^{2(p-1)} dt \Big) \Big[\lambda_0(M) - \frac{48M}{\pi^2} s_0 \arctan \rho - \frac{2T(M)\rho}{(1-\rho)^3} \Big].
\end{aligned}$$

Since there exists a unique ρ in $(0, 1)$ which satisfies the following equation:

$$\lambda_0(M) - \frac{48M}{\pi^2} s_0 \arctan \rho - \frac{2T(M)\rho}{(1-\rho)^3} = 0,$$

we see that $H(z_1) \neq H(z_2)$ and so, $H(z)$ is univalent for $|z| < \rho_0$.

Furthermore, we observe that for any z with $|z| = \rho_0$,

$$\begin{aligned}
|H(z)| &= \rho_0^{2(p-1)} \left| z G_z(0) - \overline{z} G_{\overline{z}}(0) + z(G_z(z) - G_z(0)) - \overline{z}(G_{\overline{z}}(z) - G_{\overline{z}}(0)) \right| \\
&\geq \rho_0^{2p-1} \Big[\lambda_0(M) - \frac{16M}{\pi^2} s_0 \arctan \rho_0 \Big] \\
&= R.
\end{aligned}$$

M	p	$\rho = \rho(M, p)$	$R = R(M, \rho(M, p))$	ρ'	R'
1.1296	2	0.0281673	0.0000106985	0.0194864	3.54498×10^{-6}
2	2	0.00856025	1.73218×10^{-7}	0.00623202	6.5415×10^{-8}
2.2976	2	0.00646284	6.4986×10^{-8}	0.0047235	2.47902×10^{-8}
3	2	0.0037942	1.00669×10^{-8}	0.00277162	3.83502×10^{-9}
1.1296	3	0.0281673	8.48819×10^{-9}	–	–
2	3	0.00856025	1.2693×10^{-11}	–	–
2.2976	3	0.00646284	2.71435×10^{-12}	–	–
3	3	0.0037942	1.44922×10^{-13}	–	–

Table 2: Values of ρ and R for Theorem 4.2 for $p = 2, 3$, and the corresponding values of ρ' and R' of [7, Theorem 1.2] (for $p = 2$)

The proof of the theorem is complete. □

We remark that Theorem 4.2 is an improved version of [7, Theorem 1.2] when $p = 2$. In order to be more explicit, we refer to Table 2 in which the third and fourth columns refer to values obtained from Theorem 4.2 for cases $p = 2, 3$ for certain choices of M, while the right two columns correspond to the values obtained from [7, Theorem 1.2] for the case $p = 2$.

5. The Region of Variability

Definition 5.1. *Let $\mathcal{H}_p$ denote the set of all p-harmonic mappings of the unit disk $\mathbb{D}$ with the normalization $f_{z^{p-1}}(0) = (p-1)!$ and $|f(z)| \leq 1$ for $|z| < 1$. Here we prescribe that $\mathcal{H}_0 = \emptyset$.*

For a fixed point $z_0 \in \mathbb{D}$, let

$$V_p(z_0) = \{f(z_0) : \ f \in \mathcal{H}_p \setminus \mathcal{H}_{p-1}\}.$$

Now, we have

Theorem 5.2. **(a)** *If $p = 1$, then $V_1(z_0) = \{1\}$;*

(b) *If $p \geq 2$, $V_p(z_0) = \overline{\mathbb{D}}$.*

Proof. We first prove **(a)**. Let $f \in \mathcal{H}_1$ and $f(z) = \sum_{n=0}^{\infty} a_n z^n + \sum_{n=1}^{\infty} \overline{b}_n \overline{z}^n$. By Parseval's identity and the hypotheses $|f(z)| \leq 1$ and $f(0) = 1$, we have

$$\begin{aligned}\lim_{r\to 1-} \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^2 \, d\theta &= \lim_{r\to 1-} \frac{1}{2\pi} \int_0^{2\pi} \left(|h(re^{i\theta})|^2 + |g(re^{i\theta})|^2\right) d\theta \\ &= |a_0|^2 + \sum_{n=1}^{\infty} \left(|a_n|^2 + |b_n|^2\right) \leq 1.\end{aligned}$$

This inequality implies that for any $n \geq 1$, $a_n = b_n = 0$ which gives that $f(z) \equiv 1$ for $z \in \mathbb{D}$. Thus, we have $V_1(z_0) = \{1\}$.

In order to prove **(b)**, we consider the function

$$\phi(z) = \frac{z^{p-1} - w}{1 - w\overline{z}^{p-1}} = |z|^{2(p-1)} \sum_{n=1}^{\infty} w^n \overline{z}^{(n-1)(p-1)} + z^{p-1} - w - \sum_{n=1}^{\infty} w^{n+1} \overline{z}^{(p-1)n},$$

where $w \in \overline{\mathbb{D}}$ and $p \geq 2$.

Then $\phi_{z^{p-1}}(0) = (p-1)!$, $\Delta^p \phi = 0$ and therefore, $\phi \in \mathcal{H}_p \setminus \mathcal{H}_{p-1}$. For each fixed $a \in \overline{\mathbb{D}}$, $z \mapsto f_a(z) = (z^{p-1} - a)/(1 - a\overline{z}^{p-1})$ is a p-harmonic mapping and $f_a(\mathbb{D}) \subset \mathbb{D}$.

Obviously, $a \mapsto f_a(z_0) = \frac{z_0^{p-1}-a}{1-a\overline{z_0}^{p-1}}$ is a conformal automorphism of $\mathbb{D}$ and the image of $\overline{\mathbb{D}}$ under $f_a(z_0)$ is $\overline{\mathbb{D}}$ itself. By hypotheses, we obtain that for any $g \in \mathcal{H}_p \setminus \mathcal{H}_{p-1}$, $g(z_0) \in \overline{\mathbb{D}}$. Hence $V_0(z_0)$ coincides with $\overline{\mathbb{D}}$. The proof of this theorem is complete. □

By the method of proof used in Theorem 5.2**(a)**, we obtain the following generalization of Cartan's uniqueness theorem (see [5] or [27, p. 23]) for harmonic mappings.

Theorem 5.3. *Let f be a harmonic mapping in $\mathbb{D}$ with $f(\mathbb{D}) \subseteq \mathbb{D}$ and $f_z(0) = 1$. Then $f(z) = z$ in $\mathbb{D}$.*

6. Estimates for Bloch norm for bi- and tri-harmonic mappings

In the case of p-harmonic Bloch mappings, the authors in [10] obtained the following result.

Theorem 6.1. *Let f be a p-harmonic mapping in $\mathbb{D}$ of the form* (1) *satisfying $B_f < \infty$, where*

$$B_f := \sup_{z,w\in\mathbb{D},\, z\neq w} \frac{|f(z)-f(w)|}{\rho(z,w)} < \infty \quad \text{with} \quad \rho(z,w) = \frac{1}{2}\log\left(\frac{1+|\frac{z-w}{1-\bar{z}w}|}{1-|\frac{z-w}{1-\bar{z}w}|}\right).$$

Then

$$\begin{aligned}
B_f \quad := \quad & \sup_{z\in\mathbb{D}}(1-|z|^2)\Bigg\{\Bigg|\sum_{k=1}^{p}|z|^{2(k-1)}(G_{p-k+1})_z(z) \\
& + \sum_{k=1}^{p}(k-1)\bar{z}|z|^{2(k-2)}G_{p-k+1}(z)\Bigg| + \Bigg|\sum_{k=1}^{p}|z|^{2(k-1)}(G_{p-k+1})_{\bar{z}}(z) \\
& + \sum_{k=1}^{p}(k-1)z|z|^{2(k-2)}G_{p-k+1}(z)\Bigg|\Bigg\} \\
\geq \quad & \sup_{z\in\mathbb{D}}(1-|z|^2)\Bigg|\Bigg|\sum_{k=1}^{p}|z|^{2(k-1)}(G_{p-k+1})_z(z)\Bigg| - \Bigg|\sum_{k=1}^{p}|z|^{2(k-1)}(G_{p-k+1})_{\bar{z}}(z)\Bigg|\Bigg| \qquad (6)
\end{aligned}$$

and (6) *is sharp. The equality sign in* (6) *occurs when f is analytic or anti-analytic.*

Furthermore, if for each $k \in \{1, 2, \ldots, p\}$, the harmonic functions G_{p-k+1} in (1) *are such that $|G_{p-k+1}(z)| \leq M$, then*

$$B_f \leq 2M\phi_p(y_0). \tag{7}$$

Here y_0 is the unique root in $(0,1)$ of the equation $\phi_p'(y) = 0$, where

$$\phi_p(y) = \frac{2}{\pi}\sum_{k=1}^{p} y^{2(k-1)} + y(1-y^2)\sum_{k=2}^{p}(k-1)y^{2(k-2)}. \tag{8}$$

The bound in (7) *is sharp when $p = 1$, where M is a positive constant. The extremal functions are*

$$f(z) = \frac{2M\alpha}{\pi}\mathrm{Im}\left(\log\frac{1+S(z)}{1-S(z)}\right),$$

where $|\alpha| = 1$ and $S(z)$ is a conformal automorphism of $\mathbb{D}$.

In order to emphasize the importance of this result, we recall that, when $p = 1$, (6) (resp. (7)) is a generalization of [12, Theorem 1] (resp. [12, Theorem 3]). In the case of $p = 2$ of Theorem 6.1, after some computation, one has the following simple formulation for biharmonic mappings.

Corollary 6.2. *Let $f = H + |z|^2 G$ be a biharmonic mapping of $\mathbb{D}$ such that $B_f < \infty$. Then, we have*

$$B_f \geq \sup_{z\in\mathbb{D}}(1-|z|^2)\left||H_z + |z|^2 G_z| - |H_{\bar{z}} + |z|^2 G_{\bar{z}}|\right| \tag{9}$$

and

$$B_f \leq \frac{4M}{27\pi^3}\left(8 + 36\pi^2 + \left(4+3\pi^2\right)^{3/2}\right) \approx 30.7682M. \tag{10}$$

Proof. According to our notation, (6) is equivalent to (9). In order to prove (10), we first observe that (7) is equivalent to

$$B_f \leq 2M \sup_{0<y<1} \phi_2(y),$$

where

$$\phi_2(y) = \frac{2}{\pi}(1+y^2) + y(1-y^2).$$

Now, to find $\sup_{0<y<1} \phi_2(y)$, we compute the derivative

$$\phi_2'(y) = 1 + \frac{4}{\pi}y - 3y^2 = -3\,(y-y_0)\left(y - \frac{2-\sqrt{4+3\pi^2}}{3\pi}\right)$$

so that $\phi_2'(y) \geq 0$ for $0 \leq y \leq y_0$ and $\phi_2'(y) \leq 0$ for $y_0 \leq y < 1$. Hence

$$y_0 = \frac{2+\sqrt{4+3\pi^2}}{3\pi} \approx 0.82732$$

is the critical point of $\phi_2(y)$. Consequently, $\phi_2(y) \leq \phi_2(y_0)$. A simple calculation shows that

$$\begin{aligned}
\phi_2(y_0) &= \frac{2}{\pi}(1+y_0^2) + y_0(1-y_0^2)\\
&= \frac{2}{\pi}\left(\frac{8+12\pi^2+4\sqrt{4+3\pi^2}}{9\pi^2}\right) + \left(\frac{2}{3\pi} + \frac{\sqrt{4+3\pi^2}}{3\pi}\right)\left(\frac{6\pi^2-8-4\sqrt{4+3\pi^2}}{9\pi^2}\right)\\
&= \frac{2}{27\pi^3}\left(16+42\pi^2+8\sqrt{4+3\pi^2} + \sqrt{4+3\pi^2}\left(3\pi^2-4-2\sqrt{3\pi^2+4}\right)\right)\\
&= \frac{2}{27\pi^3}\left(8+36\pi^2+\left(4+3\pi^2\right)^{3/2}\right) \approx 15.3841
\end{aligned}$$

and therefore, $B_f \leq 2M\phi_2(y_0)$ which is the desired inequality (10). The result follows. □

In the case of $p = 3$ of Theorem 6.1, we have

Corollary 6.3. *Let $f = H + |z|^2 G + |z|^4 K$ be a triharmonic (i.e. 3-harmonic) mapping of the unit disk $\mathbb{D}$ such that $B_f < \infty$, where H, G and K are harmonic in $\mathbb{D}$. Then we have*

$$B_f \geq \sup_{z\in\mathbb{D}}(1-|z|^2)\left||H_z + |z|^2 G_z + |z|^4 K_z| - |H_{\bar{z}} + |z|^2 G_{\bar{z}} + |z|^4 K_{\bar{z}}|\right| \tag{11}$$

and

$$B_f \leq 2M\phi_3(y_1) \approx 4.037006M, \tag{12}$$

where $\phi_3(y_1) = \sup_{0<y<1}\phi_3(y)$ and

$$\phi_3(y) = \frac{2}{\pi}(1+y^2+y^4) + y(1+y^2-2y^4).$$

Proof. Set $p = 3$ in Theorem 6.1. Then, (11) is equivalent to (6) and therefore, it suffices to prove (12). The choice $p = 3$ in (7) shows that

$$B_f \leq 2M \sup_{0<y<1} \phi_3(y),$$

where $\phi_3(y)$ is obtained from (8).

We see that $\phi_3(y)$ has a unique positive root in $(0, 1)$. Also,

$$\phi_3'(y) = \frac{4}{\pi}(y + 2y^3) + 1 + 3y^2 - 10y^4.$$

Computations show that $\phi_3'(y) \geq 0$ for $0 \leq y \leq y_1$ and $\phi_3'(y) \leq 0$ for $y_1 \leq y < 1$. Hence

$$y_1 \approx 0.891951$$

is the only critical point of $\phi_3(y)$ in the interval $(0, 1)$. It follows that

$$\phi_3(y) \leq \phi_3(y_1) \approx 2.018503.$$

Thus, $B_f \leq 2M\phi_3(y_1)$ which is the desired inequality (12). □

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