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The duals of certain matrix domains of factorable triangles and some related visualisations

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Abstract. This paper presents a connection between some new results from the theory of sequence spaces in functional analysis and computer science, with an application to physical chemistry and crystallography. We determine the β–duals of the matrix domains of factorable triangles in the spaces of strongly C_1 summable and bounded sequences, with index p. Furthermore we apply our results to crystallography, in particular, to determine the shape of Wulff's crystals which, in some cases, can be considered as neighbourhoods in certain metrizable topologies. Finally we use our own software for the graphical representations of some of the crystals.

1. Introduction

This paper deals with two subjects in mathematics and computer graphics, namely the study of certain linear topological spaces and the graphical representation of neighbourhoods in the considered topologies. Many important topologies arise in the theory of sequence spaces, in particular, in *FK* and *BK* spaces and their dual spaces.

Factorable matrices, in particular factorable triangles, have recently been studied by various authors ([11–16]). Here we present some new results on the matrix domains of factorable triangles in spaces of sequences that are strongly summable or bounded, with index p, by the Cesàro method of order one, and determine their β–duals.

Our results have an interesting and important application in physical chemistry, in particular in crystallography concerning the growth of crystals. According to *Wulff's principle* [19], the shape of a crystal is uniquely determined by its surface energy function. A surface energy function is a real–valued function depending on a direction in space. We will see that if the surface energy function is given by a norm then the shape of the corresponding crystal is given by a neighbourhood in the dual norm.

Visualisation and animations are of vital importance in modern mathematics. For those purposes, we developed our own software package ([5, 8]) and some important extensions ([2, 3, 10]). We demonstrate

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how our software can be applied to the representations of certain surface energy functions related to the *FK* topologies of our new spaces and of the corresponding Wulff's crystals.

We emphasize that all our graphics in this paper were created with our software package. We solved the visibility and contour problems for various types of surfaces analytically. This also involves several numerical methods ([3, 5, 8]). We had to extend our software to represent the new surfaces related to the sequence spaces of our new mathematical results.

2. Notations and Known Results

Let ω denote the set of all complex sequences $x = (x_k)_{k=1}^{\infty}$, and ℓ_∞, c, c_0 and ϕ be the sets of all bounded, convergent, null and finite sequences; also let cs, bs and ℓ_1 denote the sets of all convergent, bounded and absolutely convergent series. Finally, let $\mathcal{U}$ be the set of all sequences $u = (u_k)_{k=1}^{\infty}$ with $u_k \neq 0$ for all k. We write e and $e^{(n)}$ $(n = 1, 2, \dots)$ for the sequences with $e_k = 1$ for all k, and $e_n^{(n)} = 1$ and $e_k^{(n)} = 0$ for $k \neq n$. Let $x, y \in \omega$. Then we write $x \cdot y = (x_k y_k)_{k=1}^{\infty}$; if $u \in \mathcal{U}$, then $x/u = (x_k/u_k)_{k=1}^{\infty}$, in particular, $1/u = e/u$.

An *FK* space X is a complete linear metric sequence space with continuous coordinates $P_n : X \to \mathbb{C}$ $(n = 1, 2, \dots)$ where $P_n(x) = x_n$ for all $x = (x_k)_{k=1}^{\infty} \in X$; a *BK* space is an *FK* space whose metric is given by a norm. An *FK* space X is said to have *AK* if $x = \sum_{k=1}^{\infty} x_k e^{(k)}$ for each sequence $x = (x_k)_{k=0}^{\infty} \in X$, that is, $x^{[m]} \to x$ $(m \to \infty)$ for the m–section $x^{[m]} = \sum_{k=1}^{m} x_k e^{(k)}$ of the sequence x.

Let X and Y be subsets of ω, and z be a sequence. Then we write $z^{-1} * X = \{x \in \omega : x \cdot z \in X\}$, and $M(X, Y) = \{a \in \omega : a \cdot x \in Y \text{ for all } x \in X\}$ for the multiplier space of X and Y, in particular, the multipliers $X^\beta = M(X, cs)$ and $X^\gamma = M(X, bs)$ are called the β– and γ–duals of X. We note that obviously

$$M\left(u^{-1} * X, Y\right) = (1/u)^{-1} * M(X, Y) \text{ for each } u \in \mathcal{U}. \tag{1}$$

Let $A = (a_{nk})_{n,k=0}^{\infty}$ be an infinite matrix of complex numbers, $x \in \omega$, and X and Y be subsets of ω. We write $A_n = (a_{nk})_{k=1}^{\infty}$ $(n = 1, 2, \dots)$ for the sequence in the n^{th} row of A, and $A_n x = \sum_{k=1}^{\infty} a_{nk} x_k$ and $Ax = (A_n x)_{n=1}^{\infty}$ provided the series converge for all n. The set $X_A = \{x \in \omega : Ax \in X\}$ is called the matrix domain of A in X. We write (X, Y) for the set of all infinite matrices A that map X into Y, that is, for which $X \subset Y_A$. We note that obviously for all $u, v \in \mathcal{U}$

$$A \in \left(u^{-1} * X, v^{-1} * Y\right) \text{ if and only if } B = (b_{nk})_{n,k=1}^{\infty} \in (X, Y), \text{ where } b_{nk} = a_{nk} v_k / u_n \text{ for all } n \text{ and } k. \tag{2}$$

Let (X, d) be a metric space and $x_0 \in X$. Then $B_X(x_0, r) = \{x \in X : d(x, x_0) < r\}$ and $S_X(x_0, r) = \{x \in X : d(x, x_0) = r\}$ denote the open ball and sphere of radius $r > 0$ and centre in x_0. Let a be a sequence and (X, d) be a linear metric sequence space. Then we write

$$\|a\|^*_{X,\delta} = \sup_{x \in B_X(0,\delta)} \left| \sum_{k=1}^{\infty} a_k x_k \right|$$

provided the expression on the righthand side exists and is finite which is the case whenever X is an *FK* space and $a \in X^\beta$ by [18, Theorem 7.2.9]; if X is a normed sequence space, then we write

$$\|a\|^*_X = \sup_{\|x\|=1} \left| \sum_{k=1}^{\infty} a_k x_k \right|.$$

Throughout, let $1 \leq p < \infty$ and q be the conjugate number of p, that is, $q = \infty$ for $p = 1$ and $q = p/(p-1)$ for $1 < p < \infty$. Let

$$w_0^p = \left\{ x \in \omega : \lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} |x_k|^p = 0 \right\}, \; w^p = \left\{ x \in \omega : x - \xi \in w_0^p \text{ for some } \xi \in \mathbb{C} \right\}$$

and

$$w_\infty^p = \left\{ x \in \omega : \sup_n \frac{1}{n} \sum_{k=1}^n |x_k|^p < \infty \right\}$$

denote the sets of sequences that are strongly summable to zero, strongly summable and strongly bounded, with index p, by the Cesàro method of order 1. We denote the set of all integers k with $2^\nu \le k \le 2^{\nu+1} - 1$ by $K^{<\nu>}$, and write $\sum_\nu$ and $\max_\nu$ for the sum and maximum taken over all $k \in K^{<\nu>}$. It is well known ([4, 6]) that the *strong limit* of a sequence $x \in w^p$, that is, $\xi \in \mathbb{C}$ such that $x - \xi e \in w_0^p$ is unique, w_0^p, w^p and w_∞^p are *BK* spaces with the equivalent sectional and block norms

$$\|x\|' = \sup_{n \ge 1} \left(\frac{1}{n} \sum_{k=1}^n |x_k|^p \right)^{1/p} \quad \text{and} \quad \|x\| = \sup_{\nu \ge 0} \left(\frac{1}{2^\nu} \textstyle\sum_\nu |x_k|^p \right)^{1/p},$$

w_0^p has *AK*, and every sequence $x \in w^p$ has a unique representation $x = \xi \cdot e + \sum_{k=1}^\infty (x_k - \xi) e^{(k)}$ where ξ is the strong limit of the sequence x. Furthermore, we write

$$\|a\|_{\mathcal{M}_p} = \begin{cases} \sum\limits_{\nu=0}^\infty 2^\nu \max_\nu |a_k| & (p = 1) \\ \sum\limits_{\nu=0}^\infty 2^{\nu/p} \left(\sum_\nu |a_k|^q \right) < \infty & (1 < p < \infty) \end{cases}$$

and $\mathcal{M}_p = \{a \in \omega : \|a\|_{\mathcal{M}_p} < \infty\}$. It is known that

$$(w_0^p)^\beta = (w^p)^\beta = (w_\infty^p)^\beta = \mathcal{M}_p \text{ ([10, Theorem 5.5 (a)])}, \tag{3}$$

$\mathcal{M}_p$ is a *BK* space with *AK* ([10, Theorem 5.7]), and w_∞^p is β–perfect, that is,

$$(\mathcal{M}_p)^\beta = \left((w_\infty^p)^\beta \right)^\beta = w_\infty^p \text{ and } \| \cdot \|^*_{\mathcal{M}_p} = \| \cdot \| \text{ on } (\mathcal{M}_p)^\beta \text{ ([10, Theorem 5.8])}. \tag{4}$$

A matrix $T = (t_{nk})_{n,k=1}^\infty$ is said to be a triangle if $t_{nk} = 0$ for $k > n$ and $t_{nn} \ne 0$ ($n = 1, 2, \dots$). Let $u, v \in \mathcal{U}$ be given and $A(u, v) = (a_{nk})_{n,k=1}^\infty$ be the factorable triangle with $a_{nk} = u_n v_k$ for $1 \le k \le n$ ($n = 1, 2, \dots$). The sets $W(u, v; X) = X_{A(u,v)}$ were defined in [7] for arbitrary subsets X of ω. Here we study the sets $w_0^p(u, v) = W(u, v; w_0^p)$, $w^p(u, v) = W(u, v; w^p)$ and $w_\infty^p(u, v) = W(u, v; w_\infty^p)$. We mention that although the matrix domains of arbitrary triangles T in w_0^p, w^p and w_∞^p were considered in [1], we improve the corresponding results concerning the β–duals there, and add some new ones related to the second β-duals.

Since the spaces w_0^p, w^p and w_∞^p are *BK* spaces with the block norm $\| \cdot \|$, it is clear from [18, Theorem 4.3.12] that $w_0^p(u, v)$, $w^p(u, v)$ and $w_\infty^p(u, v)$ are *BK* spaces with

$$\|x\|_{(u,v)} = \sup_\nu \left(\frac{1}{2^\nu} \sum_\nu \left| \sum_{j=1}^k u_k v_j x_j \right|^p \right)^{1/p}.$$

3. Some Dual Spaces

Let $\Sigma = (s_{nk})_{n,k=1}^\infty$ and $\Delta^+ = (\Delta^+_{nk})_{n,k=1}^\infty$ be the matrices with $s_{nk} = 1$ for $1 \le k \le n$ and $s_{nk} = 0$ for $k > n$ ($n = 1, 2, \dots$), and $\Delta_{nn} = 1$, $\Delta_{n,n+1} = -1$ and $\Delta_{nk} = 0$ otherwise. We will use the convention throughout that any term with a subscript less than one is equal to zero.

Here we determine the β–duals of the matrix domain of Δ^+, which is not a triangle, in arbitrary *FK* spaces with *AK*, and of the spaces $w_0^p(u, v)$, $w^p(u, v)$ and $w_\infty^p(u, v)$, and apply our results to determine the β–dual of a sequence space related to the β–dual of $w_\infty^p(u,v)$. This last result will be needed in the representations of some Wulff's crystals.

Throughout, we write $R = (r_{nk})_{n,k=1}^\infty$ for the matrix with $r_{nk} = 1$ for $k \geq n$ and $r_{nk} = 0$ for $1 \leq k \leq n-1$ $(n = 1, 2, \dots)$. Given $a \in \omega$, we define the matrix $B = (B^a) = (b_{nk}^a)_{n,k=1}^\infty$ by $B_n^a = a_n R_n$ $(n = 1, 2, \dots)$, that is, $b_{nk}^a = a_n$ for $k \geq n$ and $b_{nk}^a = 0$ for $1 \leq k \leq n-1$ $(n = 1, 2, \dots)$.

Lemma 3.1. *Let $X \subset cs$ be an FK space with AK. Then we have $(X_{\Delta^+})^\beta \subset (X^\beta)_R$.*

Proof. We put $Z = X_{\Delta^+}$. Since $X \subset cs$, the sequence Rx is defined. So we have $x \in X$ if and only if $z = Rx \in Z$, since $\Delta_k^+ z = R_k x - R_{k+1} x = x_k$ for all k, that is, $\Delta^+ z = x \in X$. We also observe that $a_n z_n = a_n R_n x = \sum_{k=n}^\infty a_n x_k = B_n^a x$ for all $a \in \omega$ and all n, hence $a \cdot z = B^a x$, and so $a \in Z^\beta$ if and only if $B^a \in (X, cs)$.
We assume $a \in Z^\beta$, and write $C = \Sigma B^a$, hence

$$c_{nk} = \sum_{j=1}^\infty s_{nj} b_{jk}^a = \begin{cases} \sum\limits_{j=k}^n a_j & (1 \leq k \leq n) \\ 0 & (k \geq n+1) \end{cases} \quad (n = 1, 2, \dots).$$

Then $B^a \in (X, cs)$ if and only if $C \in (X, c)$ by [9, Theorem 3.8]. Since X is an *FK* space with *AK*, it follows by [9, Theorem 1.23] and [18, 8.3.6] that

$$\lim_{n\to\infty} c_{nk} = \sum_{j=k}^\infty R_k a \text{ exists for each } k, \tag{5}$$

and $\sup_n \|C_n\|_{X,\delta}^* < \infty$ for some $\delta > 0$, that is, there is a constant K such that

$$|C_n x| = \left|\sum_{n=1}^\infty c_{nk} x_k\right| \leq K \text{ for all } x \in B_X(0, \delta). \tag{6}$$

Let $x \in X$ be given, and $\rho = \delta/2$. Since $B_X(0,\rho)$ is absorbing ([17, Chapter 4.1, Fact (ix)]), and X has *AK*, there are a positive real λ and a positive integer m_0 such that $y^{[m]} = \lambda x^{[m]} \in B_X(0,\rho)$ for all $m \geq m_0$. Let $m \geq m_0$ be given. Then we have for all $n \geq m$ by (6)

$$\left|\sum_{k=1}^m c_{nk} x_k\right| = \lambda \left|\sum_{k=1}^m c_{nk} y_k^{[m]}\right| = \lambda \left|C_n y^{[m]}\right| \leq \lambda \cdot K,$$

and so by (5)

$$\left|\sum_{k=1}^m (R_k a) x_k\right| = \lambda \cdot \lim_{n\to\infty} \left|C_n y^{[m]}\right| \leq \lambda \cdot K.$$

Since $m \geq m_0$ was arbitrary, it follows that $(Ra) \cdot x \in bs$, and since $x \in X$ was arbitrary, we conclude $Ra \in X^\gamma$. Finally, since X has *AK*, we have $X^\gamma = X^\beta$ by [18, Theorem 7.2.7], and so $a \in (X^\beta)_R$ □

Let L denote the matrix of the left shift operator on ω, that is $L_n x = x_{n-1}$ for all $x \in \omega$ and all $n \in \mathbb{N}$. We write X_- for the matric domain of L in X.

Theorem 3.2. *Let $X \subset cs$ be an FK space with AK, and $X_- \subset X$. Then we have $a \in (X_{\Delta^+})^\beta$ if and only if*

$$a \in (X^\beta)_R \text{ and } W \in (X, c_0) \tag{7}$$

where $W = (w_{mk})_{m,k=1}^\infty$ is the triangle with $w_{mk} = R_{m+1} a$ for $1 \leq k \leq m$ and $w_{mk} = 0$ for $k \geq m+1$ $(m = 1, 2, \dots)$; moreover, if $a \in (X_{\Delta^+})^\beta$ then

$$\sum_{k=1}^\infty a_k z_k = -\sum_{k=1}^\infty (R_{k+1} a) \Delta_k^+ z \text{ for all } z \in X_{\Delta^+}. \tag{8}$$

Proof. Again we write $Z = X_{\Delta^+}$.
First, we assume $a \in Z^\beta$. Then it follows by Lemma 3.1 that $Ra \in X^\beta$ and so $R_m a$ exists for all m and consequently the matrix W is defined. Let $z \in Z$ be given. Then $x = \Delta^+ z \in X$, and we obtain

$$\sum_{k=0}^{m}(R_{k+1}a)x_k - W_m x = \sum_{k=0}^{m}(R_{k+1}a)\Delta_k^+ z - (R_{m+1}a)\sum_{k=1}^{m}\Delta_k^+ z =$$

$$-\sum_{k=0}^{m}(R_{k+1}a)z_{k+1} + \sum_{k=1}^{m}(R_{k+1}a)z_k + (R_{m+1}a)z_{m+1} - (R_{m+1}a)z_1 =$$

$$\sum_{k=1}^{m}(R_{k+1}a - R_k a)z_k - (R_{m+1}a)z_{m+1} + (R_{m+1}a)z_{m+1} - (R_{m+1}a)z_1 = -\sum_{k=1}^{m} a_k z_k - (R_{m+1}a)z_1 \text{ for all } m,$$

that is,

$$\sum_{k=1}^{m} a_k z_k = -\sum_{k=1}^{m}(R_{k+1}a)x_k + W_m x - (R_{m+1}a)z_1 \text{ for all } m. \tag{9}$$

Let $x \in X$ be given. Then $z = Rx \in Z$, and so $a \cdot z \in cs$ and $Ra \in X^\beta$. It follows from $X_- \subset X$ that $X^\beta \subset X_-^\beta$ and so $(R_{k+1}a) \in X^\beta$. This, $a \in Z^\beta$ and $Ra \in c_0$ imply $W \in (X, c)$ by (9), and trivially $(X, c) \subset (X, \ell_\infty)$. Furthermore $\lim_{m\to\infty} w_{mk} = \lim_{m\to\infty} R_{m+1}a = 0$ for each k, and $W \in (X, \ell_\infty)$ together imply $W \in (X, c_0)$ by [18, 8.3.6].
Now if $a \in Z^\beta$ then the conditions in (7) hold and (8) follows from (9) and the fact that $Ra \in c_0$.
Conversely, we assume that the conditions in (7) are satisfied. Let $z \in Z$ be given. Then $x = \Delta^+ z \in X$, and so $a \cdot z \in cs$ by (9). Thus we have $a \in Z^\beta$. □

For each $m \in \mathbb{N}$, let $\nu(m)$ denote the unique integer such that $m \in N^{<\nu(m)>} = \{n \in \mathbb{N} : 2^{\nu(m)} \leq n \leq 2^{\nu(m)+1}-1\}$. We define the sequence $d(p) = (d_m(p))_{m=1}^\infty$ by $d_m(p) = 2^{\nu(m)/p}$ for $m \in N^{\nu(m)}$ and $1 \leq p < \infty$.
We need the following lemma.

Lemma 3.3. *We have*

(a) $M(w_0^p, c_0) = (d(p))^{-1} * \ell_\infty$, *(b)* $M(w^p, c) = (d(p))^{-1} * c$, *and*

(c) $M(w_\infty^p, c_0) = (d(p))^{-1} * c_0$.

Proof. Given any sequence a, we write $D(a) = (d_{nk}(a))_{n,k=1}^\infty$ for the diagonal matrix with $d_{nn}(a) = a_n$ $(n = 1, 2, \dots)$. Then we obviously have $a \in M(X, c_0)$ if and only if $D(a) \in (X, c_0)$ for any subset X of ω.
(a) If $X = w_0^p$ then, by [1, Theorem 2.4 **3.**], $D(a) \in (w_0^p, c_0)$ if and only if

$$\sup_m \|D_m(a)\|_{\mathcal{M}_p} = \sup_m \left(2^{\nu(m)/p} \cdot |a_m|\right) < \infty,$$

that is, $a \in (d(p))^{-1} * \ell_\infty$, and $\lim_{n\to\infty} d_{nk}(a) = 0$ for each $k \in \mathbb{N}$, which is redundant.
(b) First, we show $(d(p))^{-1} * c \subset M(w^p, c)$.
Let $b = d(p) \cdot a \in c$ and $x \in w^p$ be given with strong limit ξ. It follows that

$$|a_m x_m| \leq |a_m| \cdot |x_m - \xi| + |a_m| \cdot |\xi|$$

$$\leq |b_m| \cdot \left(\frac{1}{2^{\nu(m)}}\Sigma_{\nu(m)}|x_m - \xi|^p\right)^{1/p} + |b_m| \cdot \frac{|\xi|}{2^{\nu(m)/p}} \to 0 \ (m \to \infty),$$

hence $a \cdot x \in c$. This shows $(d(p))^{-1} * c \subset M(w^p, c)$.
Conversely, we assume $b = d(p) \cdot a \notin c$. Since $e \in w^p$ and $b \cdot e = b \notin c$, it follows that $b \notin M(w^p, c)$.
(c) If $X = w_\infty^p$ then, by [1, Theorem 2.4 **2.**], $D(a) \in (w_\infty^p, c_0)$ if and only if

$$\lim_{m\to\infty} \|D_m(a)\|_{\mathcal{M}_p} = \lim_{m\to\infty} \left(2^{\nu(m)/p} \cdot |a_m|\right) = 0,$$

that is, $a \in (d(p))^{-1} * c_0$. □

A subset X of ω is said to be normal, if $x \in X$ and $|y_k| \leq |x_k|$ for all k imply $y \in X$.

Let $u, v \in \mathcal{U}$. We write $\mathcal{M}_p(\Delta^+, u, v) = \{a \in \omega : (1/u)\,\Delta^+\,(a/v) \in \mathcal{M}_p\}$.

Theorem 3.4. *Let $u, v \in \mathcal{U}$ and $b = d(p)/(u \cdot v)$. Then we have*

(a) $(w_0^p(u,v))^\beta = \mathcal{M}_p(\Delta^+, u, v) \cap b^{-1} * \ell_\infty$;

(b) $(w^p(u,v))^\beta = \mathcal{M}_p(\Delta^+, u, v) \cap b^{-1} * c$;

(c) $(w_\infty^p(u,v))^\beta = \mathcal{M}_p(\Delta^+, u, v) \cap b^{-1} * c_0$.

Proof. (a) and (c). If $X = w_0^p$ or $X = w_\infty^p$, then X is normal, and it follows from [7, Theorem 2.4 (b)] and [10, Theorem 5.5 (a)] that $a \in (w_0^p(u,v))^\beta$ if and only if $(1/u)\,\Delta^+\,(a/v) \in \mathcal{M}_p$ that is, $a \in \mathcal{M}_p(\Delta^+, u, v)$, and $a/(u \cdot v) \in M(X, c_0)$. But by Lemma 3.3 (a) and (c), $a/(u \cdot v) \in M(w_0^p, c_0)$ if and only if $(d(p) \cdot a)/(u \cdot v) = ab \in \ell_\infty$ and $a/(u \cdot v) \in M(w_\infty^p, c_0)$ if and only if $(d(p) \cdot a)/(u \cdot v) = a \cdot b \in c_0$.

(b) First, we show $(w^p(u,v))^\beta \subset \mathcal{M}_p(\Delta^+, u, v) \cap b^{-1} * c$.

Let $a \in (w^p(u,v))^\beta$ be given. Since $w_0^p \subset w^p$ obviously implies $(w^p(u,v))^\beta \subset (w_0^p(u,v))^\beta$, we obtain $a \in \mathcal{M}_p(\Delta^+, u, v)$ by Part (a). Let $z \in w^p(u,v)$ be given. Then $x = u \cdot \Sigma(v \cdot z) \in w^p$ and

$$\sum_{k=1}^{n} a_k z_k = \sum_{k=1}^{n-1} \frac{1}{u_k} \Delta_k^+\,(a/v)\, x_k + \frac{a_n}{u_n v_n} \text{ for all } n, \tag{10}$$

and $a \cdot z \in cs$ and $a \in \mathcal{M}_p(\Delta^+, u, v)$ imply $a/(u \cdot v) \in M(w^p, c)$, that is, $a \in b^{-1} * c$ by Lemma 3.3 (b).
Conversely, we assume that $a \in \mathcal{M}_p(\Delta^+, u, v)$ and $a \cdot b = d(p) \cdot a/(u \cdot v)$, hence $a \in M(w^p, c)$ by Lemma 3.3 (b). Then it follows from (10) that $a \cdot z \in cs$ for all $z \in w^p(u,v)$, hence $a \in (w^p(u,v))^\beta$. □

We close this section with an application of our results.

Example 3.5. *Let $u, v \in \mathcal{U}$. Then we have $a \in (\mathcal{M}_p(\Delta^+, u, v))^\beta$ if and only if*

$$(u_k R_k(a \cdot v))_{k=1}^\infty \in w_\infty^p \text{ and } (u_m R_{m+1}(a \cdot v))_{m=1}^\infty \in c_0.$$

Proof. We write $Y = (1/u)^{-1} * \mathcal{M}_p$. Since $\mathcal{M}_p(\Delta^+, u, v) = (1/v)^{-1} * Y_{\Delta^+}$, we obtain from (1) that $a \in (\mathcal{M}_p(\Delta^+, u, v))^\beta$ if and only if $b = a \cdot v \in (Y_{\Delta^+})^\beta$. Furthermore $\mathcal{M}_p$ is a BK space with AK by [10, Theorem 5.7], hence Y is a BK space with AK by [18, Theorem 4.3.6]. Since also obviously $Y \subset cs$ and $Y_- \subset Y$, it follows from Theorem 3.2 that $b \in (Y_{\Delta^+})^\beta$ if and only if $Rb = R(a \cdot v) \in Y^\beta = u^{-1} * (\mathcal{M}_p)^\beta = u^{-1} * w_\infty^p$ by (1) and (4), and, by (2), $\tilde{W} \in (\mathcal{M}_p, c_0)$ where $\tilde{W} = (\tilde{w}_{mk})_{m,k=1}^\infty$ is the triangle with $\tilde{w}_{mk} = u_m R_{m+1}(a \cdot v)$ for $1 \leq k \leq m$ ($m = 1, 2, \dots$). Since $\mathcal{M}_p$ is a BK space with AK ([10, Theorem 5.7]) we have, by [18, 8.3.6], $\tilde{W} \in (\mathcal{M}_p, c_0)$ if and only if

$$\lim_{m \to \infty} (u_m R_{m+1}(a \cdot v)) = 0. \tag{11}$$

and

$$\tilde{W} \in (\mathcal{M}_p, \ell_\infty). \tag{12}$$

It follows from the equivalence of the block and sectional norms on w_∞^p, the fact that $\|\cdot\| = \|\cdot\|_{\mathcal{M}_p}^*$ by (4) and [9, Theorem 1.23 (b)] that the condition in (12) holds if and only if

$$\sup_m \left(\frac{1}{m} \sum_{k=1}^{m} |\tilde{w}_{mk}|^p \right)^{1/p} = \sup_m \left(\frac{1}{m} \sum_{k=1}^{m} |u_m R_{m+1}(a \cdot v)|^p \right)^{1/p} < \infty,$$

that is, $(u_m R_{m+1}(a \cdot v))_{m=1}^\infty \in \ell_\infty$, which is redundant by (11). □

4. Graphical Representations of Wulff's Crystals

Here we give the graphical representations of Wulff's crystals and their surface energy functions as *potential surfaces*.

According to *Wulff's principle* [19], the shape of a crystal is uniquely determined by its surface energy function. A surface energy function is a real–valued function depending on a direction in space.

Let S^n denote the unit sphere in euclidean $\mathbb{R}^{n+1}$, and let $F : S^n \to \mathbb{R}$ be a surface energy function. Then we may consider the set $PM = \{\vec{x} = F(\vec{e})\vec{e} \in \mathbb{R}^{n+1} : \vec{e} \in S^n\}$ as a natural representation of the function F.

If $n = 2$, then $\vec{e} = \vec{e}(u_1, u_2) = (\cos u_1 \cos u_2, \cos u_1 \sin u_2, \sin u_1)$ for $(u_1, u_2) \in R = (-\pi/2, \pi/2) \times (0, 2\pi)$ and we obtain a *potential surface* with a parametric representation

$$PS = \{\vec{x} = f(u_1, u_2)(\cos u_1 \cos u_2, \cos u_1 \sin u_2, \sin u_1) : (u_1, u_2) \in R\} \\ \text{where } f(u_1, u_2) = F(\vec{e}(u_1, u_2)). \tag{13}$$

It was shown in [2] that if F is equal to a norm, then the boundary of the corresponding Wulff's crystal is given by the dual norm. More precisely, we have

Corollary 4.1. ([2, Beispiel 6.15]) *Let $\|\cdot\|$ be a norm on $\mathbb{R}^3$, and for each $\vec{w} \in S^2$, let $\phi_{\vec{w}} : \mathbb{R}^3 \to \mathbb{R}$ be defined by $\phi_{\vec{w}}(x) = \vec{w} \bullet \vec{x} = \sum_{k=1}^{3} w_k x_k$ ($\vec{x} \in \mathbb{R}^3$). Then the boundary $\partial C_{\|\cdot\|}$ of Wulff's crystal corresponding to $\|\cdot\|$ is given by*

$$\partial C_{\|\cdot\|} = \left\{\vec{x} = \frac{1}{\|\phi_e\|^*} \cdot \vec{e} \in \mathbb{R}^3 : \vec{e} \in S^2\right\}, \tag{14}$$

where $\|\phi_{\vec{e}}\|^*$ is the norm of the functional $\phi_{\vec{e}}$, that is, the dual norm of $\|\cdot\|$.

Remark 4.2. *If the surface energy function F is given by a norm $\|\cdot\|$ in $\mathbb{R}^3$, then*

$$\vec{x}(u_1, u_2) = \left\|\vec{e}(u_1, u_2)\right\| \cdot \vec{e}(u_1, u_2)$$

is a parametric representation of the potential surface of $\|\cdot\|$ by (13), and

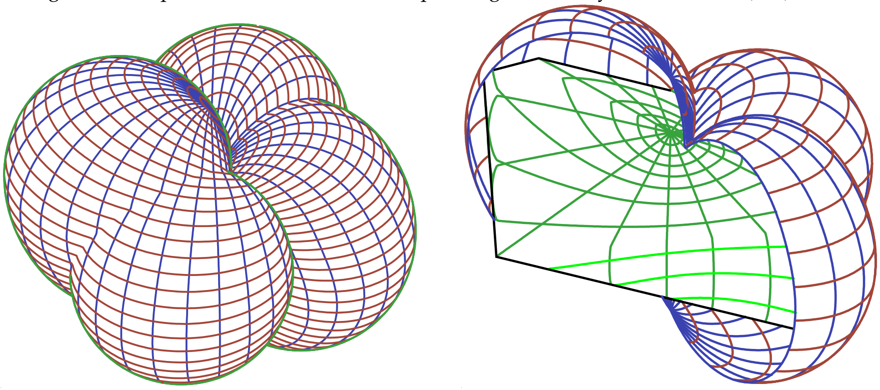
$$\vec{x}^*(u_1, u_2) = \frac{1}{\left\|\phi_{\vec{e}(u_1, u_2)}\right\|^*} \cdot \vec{e}(u_1, u_2)$$

is a parametric representation for the boundary of Wulff's crystal corresponding to $\|\cdot\|$ by (14).

Example 4.3. *We apply our results to the graphical representations of Wullf's crystals corresponding to the norms of $w_\infty^p(u, v)$ and $\mathcal{M}_p(\Delta^+, u, v)$. We consider $\mathbb{R}^3$ as a subset of ω by identifying every point $X = (x_1, x_2, x_3) \in \mathbb{R}^3$ with the sequence $y = \sum_{k=1}^{3} x_k e^{(k)}$.*
(a) We consider the case when the surface energy function F is given by the $w_\infty^p(u, v)$–norm. Then the dual norm is the natural norm of $\mathcal{M}_p(\Delta^+, u, v)$ by Theorem 3.4 and the identification just mentioned. Figure 1 shows the potential surface given by the $W(u, v, p)$–norm with $u = (-5.2, 3)$, $v = (2, -4, 3)$ and $p = 10$ on the lefthand side, and the same potential surface together with the corresponding Wulff's crystal on the righthand side.
(b) Now we consider dual case *of Part (a), that is, when the surface energy function F is given by the natural norm of $\mathcal{M}_p(\Delta^+, u, v)$. Then the dual norm is the natural norm of $v^{-1} * (u^{-1} * w_\infty^p)_R$ by Theorem 3.4 and the identification just mentioned. Figure 2 shows the potential surface given by the $\mathcal{M}_p(\Delta^+, u, v)$–norm with $u = (-5.2, 3)$, $v = (2, -4, 3)$ and $p = 10$ on the lefthand side, and the potential surface given by the $\mathcal{M}_p(\Delta^+, u, v)$–norm with $u = v(1, 1, 1)$ and $p = 1.1$ on the righthand side.*

We emphasize that Figures 1 and 2 were created by our own software for graphical representations.

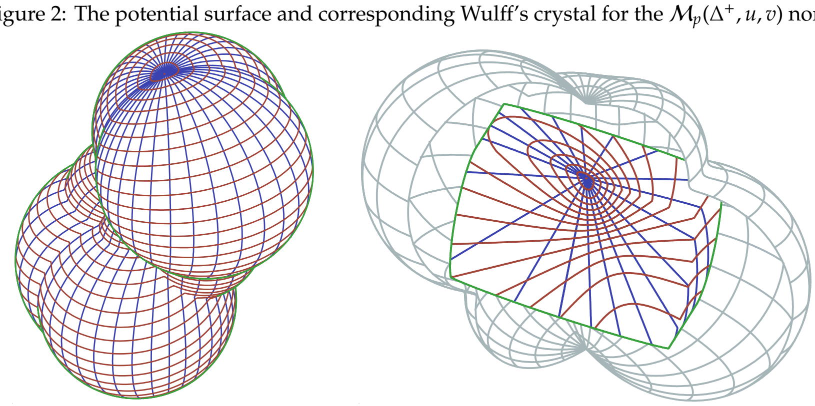
Figure 1: The potential surface and corresponding Wulff's crystal for the $w_\infty^p(u,v)$ norm



5. Conclusion

We achieved the aim of the paper, namely to present a connection between some new results from the theory of sequence spaces in functional analysis and computer sciences, with an application to physical chemistry and crystallography. More precisely, we introduced the *BK* spaces $w_0^p(u,v)$, $w^p(u,v)$ and $w_\infty^p(u,v)$ for $1 \leq p < \infty$ which are the matrix domains of factorable triangles $G(u,v)$ in w_0^p, w^p and w_∞^p. We were able to prove a new general result, Theorem 3.2, which reduces the determination of the β–dual of the matrix domain of Δ^+ in *FK* spaces X with *AK* to the determination of the β–dual of X and the characterisation of the class (X, c_0). Using this result, we were able to determine the β–duals and, essentially, the second β–duals of our new spaces in Theorem 3.4 and Example 3.5.

Figure 2: The potential surface and corresponding Wulff's crystal for the $\mathcal{M}_p(\Delta^+, u, v)$ norm



Furthermore, we gave an application of our results of Theorem 3.4 and Example 3.5 to the graphical representation of Wulff's crystals whose shape is uniquely determined by their surface energy function. If the surface energy function is a norm then the shape of the corresponding Wulff's crystal is given by a neighbourhood in the dual norm. We represented the potential surfaces given by the $w_\infty^p(u,v)$ norm and by its dual norm along with the corresponding Wulff's crystals in Figures 1 and 2, respectively. Our graphics were created with our own software.

It would be a challenging task to extend our results to the determination of the β–duals of the matrix domains of general triangles in spaces of strongly summable and bounded sequences, and the characterisations of classes of matrix transformations between those spaces. Furthermore, it would be interesting to obtain similar graphical representations.

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