Filomat 27:5 (2013), 831–842
DOI 10.2298/FIL1305831S

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Coefficient Estimates for a General Subclass of Analytic and Bi-Univalent Functions

H. M. Srivastava[a], Serap Bulut[b], Murat Çağlar[c], Nihat Yağmur[d]

[a]*Department of Mathematics and Statistics, University of Victoria, Victoria, British Columbia V8W 3R4, Canada*
[b]*Kocaeli University, Civil Aviation College, Arslanbey Campus, TR-41285 İzmit-Kocaeli, Turkey*
[c]*Department of Mathematics, Faculty of Science, Atatürk University, TR-25240 Yakutiye-Erzurum, Turkey*
[d]*Department of Mathematics, Faculty of Science and Art, Erzincan University, TR-24000 Erzincan, Turkey*

Abstract. In this paper, we introduce and investigate an interesting subclass $\mathcal{N}_{\Sigma}^{h,p}(\lambda,\mu)$ of analytic and bi-univalent functions in the open unit disk $\mathbb{U}$. For functions belonging to the class $\mathcal{N}_{\Sigma}^{h,p}(\lambda,\mu)$, we obtain estimates on the first two Taylor-Maclaurin coefficients $|a_2|$ and $|a_3|$. The results presented in this paper would generalize and improve some recent works of Çağlar *et al.* [3], Xu *et al.* [10], and other authors.

1. Introduction

Let $\mathbb{R} = (-\infty, \infty)$ be the set of real numbers, $\mathbb{C}$ be the set of complex numbers and

$$\mathbb{N} := \{1, 2, 3, \ldots\} = \mathbb{N}_0 \setminus \{0\}$$

be the set of positive integers.

Let $\mathcal{A}$ denote the class of all functions of the form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1}$$

which are analytic in the open unit disk

$$\mathbb{U} = \{z : z \in \mathbb{C} \quad \text{and} \quad |z| < 1\}.$$

We also denote by $\mathcal{S}$ the class of all functions in the normalized analytic function class $\mathcal{A}$ which are univalent in $\mathbb{U}$.

It is well known that every function $f \in \mathcal{S}$ has an inverse f^{-1}, which is defined by

$$f^{-1}\big(f(z)\big) = z \qquad (z \in \mathbb{U})$$

2010 *Mathematics Subject Classification*. Primary 30C45; Secondary 30C50
Keywords. Analytic functions; Univalent functions; Bi-univalent functions; Taylor-Maclaurin series expansion; Coefficient bounds and coefficient estimates; Taylor-Maclaurin coefficients
Received: 26 September 2012; Accepted: 21 March 2013
Communicated by Miodrag Mateljević
Email addresses: harimsri@math.uvic.ca (H. M. Srivastava), serap.bulut@kocaeli.edu.tr (Serap Bulut), mcaglar25@gmail.com (Murat Çağlar), nhtyagmur@gmail.com (Nihat Yağmur)

and

$$f\left(f^{-1}(w)\right) = w \qquad \left(|w| < r_0(f);\ r_0(f) \geqq \frac{1}{4}\right).$$

In fact, the inverse function f^{-1} is given by

$$f^{-1}(w) = w - a_2 w^2 + \left(2a_2^2 - a_3\right) w^3 - \left(5a_2^3 - 5a_2 a_3 + a_4\right) w^4 + \cdots .$$

A function $f \in \mathcal{A}$ is said to be bi-univalent in $\mathbb{U}$ if both f and f^{-1} are univalent in $\mathbb{U}$. Let Σ denote the class of bi-univalent functions in $\mathbb{U}$ given by (1). For a brief history and interesting examples of functions in the class Σ, see [8] (see also [1]). In fact, the aforecited work of Srivastava *et al.* [8] essentially revived the investigation of various subclasses of the bi-univalent function class Σ in recent years; it was followed by such works as those by Frasin and Aouf [4], Xu *et al.* [9, 10], Hayami and Owa [6], and others (see, for example, [5], [7] and [11]).

Recently, Çağlar *et al.* [3] introduced the following two subclasses of the bi-univalent function class Σ and obtained non-sharp estimates on the first two Taylor-Maclaurin coefficients $|a_2|$ and $|a_3|$ of functions in each of these subclasses (see also [4] and [10]). It should be mentioned in passing that the functional expression used in the inequalities in (2) and (7) of Definitions 1 and 2 is precisely the same as that used by Zhu [12] for investigating various extensions, generalizations and improvements of the starlikeness criteria which were proven by earlier authors (see, for details, Remark 1 below).

Definition 1. (see [3]) A function $f(z)$ given by (1) is said to be in the class $\mathcal{N}_{\Sigma}^{\mu}(\alpha,\lambda)$ if the following conditions are satisfied:

$$f \in \Sigma \quad \text{and} \quad \left|\arg\left((1-\lambda)\left(\frac{f(z)}{z}\right)^{\mu} + \lambda f'(z)\left(\frac{f(z)}{z}\right)^{\mu-1}\right)\right| < \frac{\alpha\pi}{2} \tag{2}$$

$$(0 < \alpha \leqq 1;\ \lambda \geqq 1;\ \mu \geqq 0;\ z \in \mathbb{U})$$

and

$$\left|\arg\left((1-\lambda)\left(\frac{g(w)}{w}\right)^{\mu} + \lambda g'(w)\left(\frac{g(w)}{w}\right)^{\mu-1}\right)\right| < \frac{\alpha\pi}{2} \tag{3}$$

$$(0 < \alpha \leqq 1;\ \lambda \geqq 1;\ \mu \geqq 0;\ w \in \mathbb{U}),$$

where the function g is given by

$$g(w) = w - a_2 w^2 + \left(2a_2^2 - a_3\right) w^3 - \left(5a_2^3 - 5a_2 a_3 + a_4\right) w^4 + \cdots . \tag{4}$$

Theorem 1. (see [3]) *Let the function $f(z)$ given by the Taylor-Maclaurin series expansion* (1) *be in the class*

$$\mathcal{N}_{\Sigma}^{\mu}(\alpha,\lambda) \qquad (0 < \alpha \leqq 1;\ \lambda \geqq 1;\ \mu \geqq 0).$$

Then

$$|a_2| \leqq \frac{2\alpha}{\sqrt{(\lambda+\mu)^2 + \alpha(\mu + 2\lambda - \lambda^2)}} \tag{5}$$

and

$$|a_3| \leqq \frac{4\alpha^2}{(\lambda+\mu)^2} + \frac{2\alpha}{2\lambda+\mu}. \tag{6}$$

Definition 2. (see [3]) A function $f(z)$ given by (1) is said to be in the class $\mathcal{N}_{\Sigma}^{\mu}(\beta,\lambda)$ if the following conditions are satisfied:

$$f \in \Sigma \quad \text{and} \quad \Re\left((1-\lambda)\left(\frac{f(z)}{z}\right)^{\mu} + \lambda f'(z)\left(\frac{f(z)}{z}\right)^{\mu-1}\right) > \beta \tag{7}$$

$$(0 \leqq \beta < 1;\ \lambda \geqq 1;\ \mu \geqq 0;\ z \in \mathbb{U})$$

and

$$\Re\left((1-\lambda)\left(\frac{g(w)}{w}\right)^{\mu} + \lambda g'(w)\left(\frac{g(w)}{w}\right)^{\mu-1}\right) > \beta \tag{8}$$

$$(0 \leqq \beta < 1;\ \lambda \geqq 1;\ \mu \geqq 0;\ w \in \mathbb{U}),$$

where the function g is defined by (4).

Remark 1. For functions $f(z)$, which are analytic in $\mathbb{U}$ and normalized by

$$f(z) = z + \sum_{k=n+1}^{\infty} a_k z^k \qquad (n \in \mathbb{N}),$$

Zhu [12] determined the conditions on the parameters M, α, λ and μ such that the following inequality:

$$\left|(1-\lambda)\left(\frac{f(z)}{z}\right)^{\mu} + \lambda f'(z)\left(\frac{f(z)}{z}\right)^{\mu-1} - 1\right| < M$$

implies that the so-normalized function $f(z)$ is in the corresponding class of starlike functions of order α $(0 \leqq \alpha < 1)$. Interestingly, the functional expression used by Zhu [12] is precisely the same as that used in the inequalities in (2) and (7) above. The work of Zhu [12] provided extensions, generalizations and improvements of the various starlikeness criteria which were proven by a number of earlier authors (see, for details, [12]).

Theorem 2. (see [3]) *Let the function $f(z)$ given by the Taylor-Maclaurin series expansion* (1) *be in the class*

$$\mathcal{N}_{\Sigma}^{\mu}(\beta,\lambda) \qquad (0 \leqq \beta < 1;\ \lambda \geqq 1;\ \mu \geqq 0).$$

Then

$$|a_2| \leqq \min\left\{\sqrt{\frac{4(1-\beta)}{(\mu+1)(2\lambda+\mu)}},\ \frac{2(1-\beta)}{\lambda+\mu}\right\} \tag{9}$$

and

$$|a_3| \leqq \begin{cases} \min\left\{\dfrac{4(1-\beta)}{(\mu+1)(2\lambda+\mu)},\ \dfrac{4(1-\beta)^2}{(\lambda+\mu)^2} + \dfrac{2(1-\beta)}{2\lambda+\mu}\right\} & (0 \leqq \mu < 1) \\[2ex] \dfrac{2(1-\beta)}{2\lambda+\mu} & (\mu \geqq 1). \end{cases} \tag{10}$$

Remark 2. The following special cases of Definitions 1 and 2 are worthy of note:
(i) For $\mu = 1$, we obtain the bi-univalent function classes

$$\mathcal{N}_{\Sigma}^{1}(\alpha,\lambda) = \mathcal{B}_{\Sigma}(\alpha,\lambda) \qquad \text{and} \qquad \mathcal{N}_{\Sigma}^{1}(\beta,\lambda) = \mathcal{B}_{\Sigma}(\beta,\lambda)$$

introduced by Frasin and Aouf [4].

(ii) For $\mu = 1$ and $\lambda = 1$, we have the bi-univalent function classes

$$\mathcal{N}_{\Sigma}^{1}(\alpha, 1) = \mathcal{H}_{\Sigma}^{\alpha} \qquad \text{and} \qquad \mathcal{N}_{\Sigma}^{1}(\beta, 1) = \mathcal{H}_{\Sigma}(\beta)$$

introduced by Srivastava *et al.* [8].

(iii) For $\mu = 0$ and $\lambda = 1$, we get the well-known classes

$$\mathcal{N}_{\Sigma}^{0}(\alpha, 1) = \mathcal{S}_{\Sigma}^{*}[\alpha] \qquad \text{and} \qquad \mathcal{N}_{\Sigma}^{0}(\beta, 1) = \mathcal{S}_{\Sigma}^{*}(\beta)$$

of strongly bi-starlike functions of order α and of bi-starlike functions of order β, respectively.

This paper is essentially a sequel to some of the aforecited works (especially see [3] and [10]). Here we introduce and investigate the general subclass $\mathcal{N}_{\Sigma}^{h,p}(\lambda, \mu)$ $(\lambda \geqq 1;\ \mu \geqq 0)$ of the analytic function class $\mathcal{A}$, which is given by Definition 3 below.

Definition 3. Let the functions $h, p : \mathbb{U} \to \mathbb{C}$ be so constrained that

$$\min\left\{\Re\big(h(z)\big), \Re\big(p(z)\big)\right\} > 0 \quad (z \in \mathbb{U}) \quad \text{and} \quad h(0) = p(0) = 1.$$

Also let the function f, defined by (1), be in the analytic function class $\mathcal{A}$. We say that

$$f \in \mathcal{N}_{\Sigma}^{h,p}(\lambda, \mu) \qquad (\lambda \geqq 1;\ \mu \geqq 0)$$

if the following conditions are satisfied:

$$f \in \Sigma \quad \text{and} \quad (1-\lambda)\left(\frac{f(z)}{z}\right)^{\mu} + \lambda f'(z)\left(\frac{f(z)}{z}\right)^{\mu-1} \in h(\mathbb{U}) \quad (z \in \mathbb{U}) \tag{11}$$

and

$$(1-\lambda)\left(\frac{g(w)}{w}\right)^{\mu} + \lambda g'(w)\left(\frac{g(w)}{w}\right)^{\mu-1} \in p(\mathbb{U}) \quad (w \in \mathbb{U}), \tag{12}$$

where the function g is defined by (4).

We note that the class $\mathcal{N}_{\Sigma}^{h,p}(\lambda, \mu)$ reduces to the function classes $\mathcal{B}_{\Sigma}^{h,p}(\lambda)$ and $\mathcal{H}_{\Sigma}^{h,p}$ given by

$$\mathcal{B}_{\Sigma}^{h,p}(\lambda) = \mathcal{N}_{\Sigma}^{h,p}(\lambda, 1),$$

$$\mathcal{B}_{\Sigma}^{h,p} = \mathcal{N}_{\Sigma}^{h,p}(1, 0)$$

and

$$\mathcal{H}_{\Sigma}^{h,p} = \mathcal{N}_{\Sigma}^{h,p}(1, 1),$$

respectively, each of which was introduced and studied recently by Xu *et al.* [10], Bulut [2] and Xu *et al.* [9], respectively.

Remark 3. There are many choices of the functions $h(z)$ and $p(z)$ which would provide interesting subclasses of the analytic function class $\mathcal{A}$. For example, if we let

$$h(z) = p(z) = \left(\frac{1+z}{1-z}\right)^{\alpha} \qquad (0 < \alpha \leqq 1;\ z \in \mathbb{U}) \tag{13}$$

or

$$h(z) = p(z) = \frac{1 + (1 - 2\beta) z}{1 - z} \qquad (0 \leqq \beta < 1;\ z \in \mathbb{U}), \tag{14}$$

it is easy to verify that the functions $h(z)$ and $p(z)$ satisfy the hypotheses of Definition 3. If $f \in \mathcal{N}_{\Sigma}^{h,p}(\lambda, \mu)$, then

$$f \in \Sigma \quad \text{and} \quad \left| \arg \left((1 - \lambda) \left(\frac{f(z)}{z} \right)^{\mu} + \lambda f'(z) \left(\frac{f(z)}{z} \right)^{\mu - 1} \right) \right| < \frac{\alpha \pi}{2} \tag{15}$$

$$(0 < \alpha \leqq 1;\ \lambda \geqq 1;\ \mu \geqq 0;\ z \in \mathbb{U})$$

and

$$\left| \arg \left((1 - \lambda) \left(\frac{g(w)}{w} \right)^{\mu} + \lambda g'(w) \left(\frac{g(w)}{w} \right)^{\mu - 1} \right) \right| < \frac{\alpha \pi}{2} \tag{16}$$

$$(0 < \alpha \leqq 1;\ \lambda \geqq 1;\ \mu \geqq 0;\ w \in \mathbb{U})$$

or

$$f \in \Sigma \qquad \text{and} \qquad \Re \left((1 - \lambda) \left(\frac{f(z)}{z} \right)^{\mu} + \lambda f'(z) \left(\frac{f(z)}{z} \right)^{\mu - 1} \right) > \beta \tag{17}$$

$$(0 \leqq \beta < 1;\ \lambda \geqq 1;\ \mu \geqq 0;\ z \in \mathbb{U})$$

and

$$\Re \left((1 - \lambda) \left(\frac{g(w)}{w} \right)^{\mu} + \lambda g'(w) \left(\frac{g(w)}{w} \right)^{\mu - 1} \right) > \beta \tag{18}$$

$$(0 \leqq \beta < 1;\ \lambda \geqq 1;\ \mu \geqq 0;\ w \in \mathbb{U}),$$

where the function g is defined by (4). This means that

$$f \in \mathcal{N}_{\Sigma}^{\mu}(\alpha, \lambda) \qquad (0 < \alpha \leqq 1;\ \lambda \geqq 1;\ \mu \geqq 0)$$

or

$$f \in \mathcal{N}_{\Sigma}^{\mu}(\beta, \lambda) \qquad (0 \leqq \beta < 1;\ \lambda \geqq 1;\ \mu \geqq 0).$$

Our paper is motivated and stimulated especially by the works of Çağlar *et al.* [3] and Xu *et al.* [10]. Here we propose to investigate the bi-univalent function class $\mathcal{N}_{\Sigma}^{h,p}(\lambda, \mu)$ introduced in Definition 3 and derive coefficient estimates on the first two Taylor-Maclaurin coefficients $|a_2|$ and $|a_3|$ for a function $f \in \mathcal{N}_{\Sigma}^{h,p}(\lambda, \mu)$ given by (1). Our results for the bi-univalent function class $\mathcal{N}_{\Sigma}^{h,p}(\lambda, \mu)$ would generalize and improve the related works of Çağlar *et al.* [3] and Xu *et al.* [10] (see also [4] and [8]).

2. A Set of General Coefficient Estimates

In this section, we state and prove our general results involving the bi-univalent function class $\mathcal{N}_{\Sigma}^{h,p}(\lambda, \mu)$ given by Definition 3.

Theorem 3. *Let the function* $f(z)$ *given by the Taylor-Maclaurin series expansion* (1) *be in the function class* $\mathcal{N}_{\Sigma}^{h,p}(\lambda,\mu)$. *Then*

$$|a_2| \leqq \min\left\{\sqrt{\frac{|h'(0)|^2+|p'(0)|^2}{2(\lambda+\mu)^2}},\ \sqrt{\frac{|h''(0)|+|p''(0)|}{2(\mu+1)(2\lambda+\mu)}}\right\} \tag{19}$$

and

$$|a_3| \leqq \min\left\{\frac{|h'(0)|^2+|p'(0)|^2}{2(\lambda+\mu)^2}+\frac{|h''(0)|+|p''(0)|}{4(2\lambda+\mu)},\right.$$
$$\left.\frac{(3+\mu)|h''(0)|+|1-\mu|\,|p''(0)|}{4(\mu+1)(2\lambda+\mu)}\right\}. \tag{20}$$

Proof. First of all, we write the argument inequalities in (11) and (12) in their equivalent forms as follows:

$$(1-\lambda)\left(\frac{f(z)}{z}\right)^{\mu}+\lambda f'(z)\left(\frac{f(z)}{z}\right)^{\mu-1}=h(z)\quad(z\in\mathbb{U}),$$

and

$$(1-\lambda)\left(\frac{g(w)}{w}\right)^{\mu}+\lambda g'(w)\left(\frac{g(w)}{w}\right)^{\mu-1}=p(w)\quad(w\in\mathbb{U}),$$

respectively, where $h(z)$ and $p(w)$ satisfy the conditions of Definition 3. Furthermore, the functions $h(z)$ and $p(w)$ have the following Taylor-Maclaurin series expensions:

$$h(z)=1+h_1z+h_2z^2+\cdots$$

and

$$p(w)=1+p_1w+p_2w^2+\cdots,$$

respectively. Now, upon equating the coefficients of

$$(1-\lambda)\left(\frac{f(z)}{z}\right)^{\mu}+\lambda f'(z)\left(\frac{f(z)}{z}\right)^{\mu-1}$$

with those of $h(z)$ and the coefficients of

$$(1-\lambda)\left(\frac{g(w)}{w}\right)^{\mu}+\lambda g'(w)\left(\frac{g(w)}{w}\right)^{\mu-1}$$

with those of $p(w)$, we get

$$(\lambda+\mu)a_2=h_1, \tag{21}$$

$$(2\lambda+\mu)a_3+(\mu-1)\left(\lambda+\frac{\mu}{2}\right)a_2^2=h_2, \tag{22}$$

$$-(\lambda+\mu)a_2=p_1 \tag{23}$$

and

$$-(2\lambda+\mu)\,a_3+(\mu+3)\left(\lambda+\frac{\mu}{2}\right)a_2^2=p_2. \tag{24}$$

From (21) and (23), we obtain

$$h_1=-p_1 \tag{25}$$

and

$$2(\lambda+\mu)^2\,a_2^2=h_1^2+p_1^2. \tag{26}$$

Also, from (22) and (24), we find that

$$(\mu+1)(2\lambda+\mu)\,a_2^2=h_2+p_2. \tag{27}$$

Therefore, we find from the equations (26) and (27) that

$$|a_2|^2 \leqq \frac{|h'(0)|^2+|p'(0)|^2}{2(\lambda+\mu)^2}$$

and

$$|a_2|^2 \leqq \frac{|h''(0)|+|p''(0)|}{2(\mu+1)(2\lambda+\mu)},$$

respectively. So we get the desired estimate on the coefficient $|a_2|$ as asserted in (19).

Next, in order to find the bound on the coefficient $|a_3|$, we subtract (24) from (22). We thus get

$$2(2\lambda+\mu)\,a_3-2(2\lambda+\mu)\,a_2^2=h_2-p_2. \tag{28}$$

Upon substituting the value of a_2^2 from (26) into (28), it follows that

$$a_3=\frac{h_1^2+p_1^2}{2(\lambda+\mu)^2}+\frac{h_2-p_2}{2(2\lambda+\mu)}.$$

We thus find that

$$|a_3| \leqq \frac{|h'(0)|^2+|p'(0)|^2}{2(\lambda+\mu)^2}+\frac{|h''(0)|+|p''(0)|}{4(2\lambda+\mu)}. \tag{29}$$

On the other hand, upon substituting the value of a_2^2 from (27) into (28), it follows that

$$a_3=\frac{(3+\mu)\,h_2+(1-\mu)\,p_2}{2(\mu+1)(2\lambda+\mu)}.$$

Consequently, we have

$$|a_3| \leqq \frac{(3+\mu)\,|h''(0)|+|1-\mu|\,|p''(0)|}{4(\mu+1)(2\lambda+\mu)}. \tag{30}$$

This evidently completes the proof of Theorem 3. □

3. Corollaries and Consequences

By setting $\mu = 1$ in Theorem 3, we get Corollary 1 below.

Corollary 1. *Let the function $f(z)$ given by the Taylor-Maclaurin series expansion* (1) *be in the bi-univalent function class $\mathcal{B}_{\Sigma}^{h,p}(\lambda)\ (\lambda \geqq 1)$. Then*

$$|a_2| \leqq \min\left\{\sqrt{\frac{|h'(0)|^2 + |p'(0)|^2}{2(1+\lambda)^2}},\ \sqrt{\frac{|h''(0)| + |p''(0)|}{4(1+2\lambda)}}\right\} \tag{31}$$

and

$$|a_3| \leqq \min\left\{\frac{|h'(0)|^2 + |p'(0)|^2}{2(1+\lambda)^2} + \frac{|h''(0)| + |p''(0)|}{4(1+2\lambda)},\ \frac{|h''(0)|}{2(1+2\lambda)}\right\}. \tag{32}$$

Remark 4. Corollary 1 is an improvement of the following estimates obtained by Xu *et al.* [10].

Corollary 2. (see [10]) *Suppose that $f(z)$ given by its Taylor-Maclaurin series expansion* (1) *is in the function class $\mathcal{B}_{\Sigma}^{h,p}(\lambda)\ (\lambda \geqq 1)$. Then*

$$|a_2| \leqq \sqrt{\frac{|h''(0)| + |p''(0)|}{4(1+2\lambda)}} \tag{33}$$

and

$$|a_3| \leqq \frac{|h''(0)|}{2(1+2\lambda)}. \tag{34}$$

By setting $\mu = 1$ and $\lambda = 1$ in Theorem 3, we get the following consequence.

Corollary 3. *Let the function $f(z)$ given by the Taylor-Maclaurin series expansion* (1) *be in the function class $\mathcal{H}_{\Sigma}^{h,p}$. Then*

$$|a_2| \leqq \min\left\{\sqrt{\frac{|h'(0)|^2 + |p'(0)|^2}{8}},\ \sqrt{\frac{|h''(0)| + |p''(0)|}{12}}\right\} \tag{35}$$

and

$$|a_3| \leqq \min\left\{\frac{|h'(0)|^2 + |p'(0)|^2}{8} + \frac{|h''(0)| + |p''(0)|}{12},\ \frac{|h''(0)|}{6}\right\}. \tag{36}$$

Remark 5. Corollary 3 is an improvement of the following estimates obtained by Xu *et al.* [9].

Corollary 4. (see [9]) *Let the function $f(z)$ given by the Taylor-Maclaurin series expansion* (1) *be in the function class $\mathcal{H}_{\Sigma}^{h,p}$. Then*

$$|a_2| \leqq \sqrt{\frac{|h''(0)| + |p''(0)|}{12}} \tag{37}$$

and

$$|a_3| \leqq \frac{|h''(0)|}{6}. \tag{38}$$

Remark 6. By setting $\mu = 0$ and $\lambda = 1$ in Theorem 3, we get [2, Theorem 2.1].

If we set

$$h(z) = p(z) = \left(\frac{1+z}{1-z}\right)^{\alpha} \qquad (0 < \alpha \leqq 1;\ z \in \mathbb{U})$$

in Theorem 3, we can readily deduce Corollary 5.

Corollary 5. *Let the function $f(z)$ given by the Taylor-Maclaurin series expansion* (1) *be in the bi-univalent function class*

$$\mathcal{N}_{\Sigma}^{\mu}(\alpha, \lambda)\ (0 < \alpha \leqq 1;\ \lambda \geqq 1;\ \mu \geqq 0).$$

Then

$$|a_2| \leqq \begin{cases} \dfrac{2\alpha}{\lambda + \mu} & \left(\lambda \geqq 1 + \sqrt{1+\mu}\right) \\[2ex] \dfrac{2\alpha}{\sqrt{(\mu+1)(2\lambda+\mu)}} & \left(1 \leqq \lambda < 1 + \sqrt{1+\mu}\right). \end{cases} \tag{39}$$

and

$$|a_3| \leqq \begin{cases} \min\left\{\dfrac{4\alpha^2}{(\lambda+\mu)^2} + \dfrac{2\alpha^2}{2\lambda+\mu}, \dfrac{4\alpha^2}{(\mu+1)(2\lambda+\mu)}\right\} & (0 \leqq \mu < 1) \\[2ex] \dfrac{2\alpha^2}{2\lambda+\mu} & (\mu \geqq 1). \end{cases} \tag{40}$$

Remark 7. It is easy to see, for the coefficient $|a_2|$, that

$$\frac{2\alpha}{\lambda+\mu} \leqq \frac{2\alpha}{\sqrt{(\lambda+\mu)^2 + \alpha(\mu + 2\lambda - \lambda^2)}}$$

$$\left(0 < \alpha \leqq 1;\ \lambda \geqq 1 + \sqrt{1+\mu};\ \mu \geqq 0\right)$$

and

$$\frac{2\alpha}{\sqrt{(\mu+1)(2\lambda+\mu)}} \leqq \frac{2\alpha}{\sqrt{(\lambda+\mu)^2 + \alpha(\mu + 2\lambda - \lambda^2)}}$$

$$\left(0 < \alpha \leqq 1;\ 1 \leqq \lambda < 1 + \sqrt{1+\mu};\ \mu \geqq 0\right).$$

On the other hand, for the coefficient $|a_3|$, we make the following observations:

(i) If $0 \leqq \mu < 1$ and

$$\min\left\{\frac{4\alpha^2}{(\lambda+\mu)^2} + \frac{2\alpha^2}{2\lambda+\mu}, \frac{4\alpha^2}{(\mu+1)(2\lambda+\mu)}\right\} = \frac{4\alpha^2}{(\lambda+\mu)^2} + \frac{2\alpha^2}{2\lambda+\mu},$$

then

$$\frac{4\alpha^2}{(\lambda+\mu)^2}+\frac{2\alpha^2}{2\lambda+\mu} \leqq \frac{4\alpha^2}{(\lambda+\mu)^2}+\frac{2\alpha}{2\lambda+\mu} \quad (0<\alpha \leqq 1;\ \lambda \geqq 1)\,;$$

(ii) If $0 \leqq \mu < 1$ and

$$\min\left\{\frac{4\alpha^2}{(\lambda+\mu)^2}+\frac{2\alpha^2}{2\lambda+\mu},\ \frac{4\alpha^2}{(\mu+1)(2\lambda+\mu)}\right\}=\frac{4\alpha^2}{(\mu+1)(2\lambda+\mu)},$$

then

$$\begin{aligned}\frac{4\alpha^2}{(\mu+1)(2\lambda+\mu)} &\leqq \frac{4\alpha^2}{(\lambda+\mu)^2}+\frac{2\alpha^2}{2\lambda+\mu}\\ &\leqq \frac{4\alpha^2}{(\lambda+\mu)^2}+\frac{2\alpha}{2\lambda+\mu} \quad (0<\alpha \leqq 1;\ \lambda \geqq 1)\,;\end{aligned}$$

(iii) If $\mu \geqq 1$, then

$$\begin{aligned}\frac{2\alpha^2}{2\lambda+\mu} &\leqq \frac{2\alpha}{2\lambda+\mu}\\ &\leqq \frac{4\alpha^2}{(\lambda+\mu)^2}+\frac{2\alpha}{2\lambda+\mu} \quad (0<\alpha \leqq 1;\ \lambda \geqq 1)\,.\end{aligned}$$

Thus, clearly, Corollary 5 is an improvement of Theorem 1.

By setting $\mu = 1$ in Corollary 5, we obtain the following consequence.

Corollary 6. *Let the function $f(z)$ given by the Taylor-Maclaurin series expansion* (1) *be in the bi-univalent function class*

$$\mathcal{B}_\Sigma(\alpha,\lambda)\ (0<\alpha \leqq 1;\ \lambda \geqq 1)\,.$$

Then

$$|a_2| \leqq \begin{cases} \dfrac{2\alpha}{\lambda+1} & \left(\lambda \geqq 1+\sqrt{2}\right) \\[2ex] \sqrt{\dfrac{2}{2\lambda+1}}\,\alpha & \left(1 \leqq \lambda < 1+\sqrt{2}\right). \end{cases} \tag{41}$$

and

$$|a_3| \leqq \frac{2\alpha^2}{2\lambda+1}. \tag{42}$$

Remark 8. Corollary 6 provides an improvement of the following estimates obtained by Frasin and Aouf [4].

Corollary 7. (see [4]) *Let the function $f(z)$ given by the Taylor-Maclaurin series expansion* (1) *be in the bi-univalent function class*

$$\mathcal{B}_\Sigma(\alpha,\lambda) \quad (0<\alpha \leqq 1;\ \lambda \geqq 1)\,.$$

Then

$$|a_2| \leqq \frac{2\alpha}{\sqrt{(\lambda+1)^2 + \alpha\,(1 + 2\lambda - \lambda^2)}} \tag{43}$$

and

$$|a_3| \leqq \frac{4\alpha^2}{(\lambda+1)^2} + \frac{2\alpha}{2\lambda+1}. \tag{44}$$

By setting $\mu = 1$ and $\lambda = 1$ in Corollary 5, we get the following consequence.

Corollary 8. *Let the function* $f(z)$ *given by the Taylor-Maclaurin series expansion* (1) *be in the bi-univalent function class* $\mathcal{H}_{\Sigma}^{\alpha}$ $(0 < \alpha \leqq 1)$. *Then*

$$|a_2| \leqq \sqrt{\frac{2}{3}}\,\alpha \tag{45}$$

and

$$|a_3| \leqq \frac{2\alpha^2}{3}. \tag{46}$$

Remark 9. Corollary 8 is an improvement of the following estimates which were given by Srivastava *et al.* [8].

Corollary 9. (see [8]) *Let the function* $f(z)$ *given by the Taylor-Maclaurin series expansion* (1) *be in the bi-univalent function class* $\mathcal{H}_{\Sigma}^{\alpha}$ $(0 < \alpha \leqq 1)$. *Then*

$$|a_2| \leqq \sqrt{\frac{2}{\alpha+2}}\,\alpha \tag{47}$$

and

$$|a_3| \leqq \frac{\alpha\,(3\alpha+2)}{3}. \tag{48}$$

By setting $\mu = 0$ and $\lambda = 1$ in Corollary 5, we get the following consequence.

Corollary 10. *Let the function* $f(z)$ *given by the Taylor-Maclaurin series expansion* (1) *be in the bi-univalent function class* $\mathcal{S}_{\Sigma}^{*}[\alpha]$ $(0 < \alpha \leqq 1)$. *Then*

$$|a_2| \leqq \sqrt{2}\,\alpha \tag{49}$$

and

$$|a_3| \leqq 2\alpha^2. \tag{50}$$

Remark 10. If we set

$$h(z) = p(z) = \frac{1 + (1-2\beta)\,z}{1-z} \qquad (0 \leqq \beta < 1;\ \lambda \geqq 1;\ z \in \mathbb{U}) \tag{51}$$

in Theorem 3, we can readily deduce Theorem 2.

Remark 11. The aforecited work by Çağlar *et al.* [3] contains several interesting *further* special cases and consequences of Theorem 2, which we have generalized here by means of Theorem 3 (see Remark 3). The reader will find each of these *further* special cases and consequences of Theorem 2, too, to be motivatingly interesting.

4. Concluding Remarks and Observations

In our present investigation, we have considered an interesting subclass $\mathcal{N}_{\Sigma}^{h,p}(\lambda,\mu)$ of analytic and bi-univalent functions in the open unit disk $\mathbb{U}$. We have derived estimates on the first two Taylor-Maclaurin coefficients $|a_2|$ and $|a_3|$ for functions belonging to the class $\mathcal{N}_{\Sigma}^{h,p}(\lambda,\mu)$. By means of corollaries and consequences which we discussed in the preceding section by suitably specializing the functions $h(z)$ and $p(z)$ (*and* also the parameters λ and μ), we have also shown already that the results presented in this paper would generalize and improve some recent works of Çağlar *et al.* [3], Xu *et al.* [10], and other authors.

Finally, our motivation for introducing the subclass $\mathcal{N}_{\Sigma}^{h,p}(\lambda,\mu)$ of analytic and bi-univalent functions in the open unit disk $\mathbb{U}$ in Definition 3 is motivated *at least partially* by the work of Zhu [12] who provided extensions, generalizations and improvements of the various starlikeness criteria which were proven by a number of earlier authors (see, for details, Remark 1).

References

[1] D. A. Brannan and T. S. Taha, On some classes of bi-univalent functions, in *Mathematical Analysis and Its Applications* (S. M. Mazhar, A. Hamoui and N. S. Faour, Editors) (Kuwait; February 18–21, 1985), KFAS Proceedings Series, Vol. **3**, Pergamon Press (Elsevier Science Limited), Oxford, 1988, pp. 53–60; see also *Studia Univ. Babeş-Bolyai Math.* **31** (2) (1986), 70–77.

[2] S. Bulut, Coefficient estimates for a class of analytic and bi-univalent functions, *Novi Sad J. Math.* **43** (2) (2013) (in press).

[3] M. Çağlar, H. Orhan and N. Yağmur, Coefficient bounds for new subclasses of bi-univalent functions, *Filomat* **27:7** (2013), 1165-1171.

[4] B. A. Frasin and M. K. Aouf, New subclasses of bi-univalent functions, *Appl. Math. Lett.* **24** (2011), 1569–1573.

[5] S. P. Goyal and P. Goswami, Estimate for initial Maclaurin coefficients of bi-univalent functions for a class defined by fractional derivatives, *J. Egyptian Math. Soc.* **20** (2012), 179–182.

[6] T. Hayami and S. Owa, Coefficient bounds for bi-univalent functions, *Pan Amer. Math. J.* **22** (4) (2012), 15–26.

[7] X.-F. Li and A.-P. Wang, Two new subclasses of bi-univalent functions, *Internat. Math. Forum* **7** (2012), 1495–1504.

[8] H. M. Srivastava, A. K. Mishra and P. Gochhayat, Certain subclasses of analytic and bi-univalent functions, *Appl. Math. Lett.* **23** (2010), 1188–1192.

[9] Q.-H. Xu, Y.-C. Gui and H. M. Srivastava, Coefficient estimates for a certain subclass of analytic and bi-univalent functions, *Appl. Math. Lett.* **25** (2012), 990–994.

[10] Q.-H. Xu, H.-G. Xiao and H. M. Srivastava, A certain general subclass of analytic and bi-univalent functions and associated coefficient estimate problems, *Appl. Math. Comput.* **218** (2012), 11461–11465.

[11] H. M. Srivastava, G. Murugusundaramoorthy and N. Mangesh, Certain subclasses of bi-univalent functions associated with the Hohlov operator, *Global J. Math. Anal.* **1** (2) (2013), 67–73.

[12] Y. Zhu, Some starlikeness criterions for analytic functions, *J. Math. Anal. Appl.* **335** (2007), 1452–1459.